Representation Theory

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Contents

Chapter 1

Representations and Characters

Throughout this chapter, G denotes a finite group and all vector spaces are finite-dimensional over $\mathbb{C}$.

1.1 Linear Representations and Invariant Subspaces

Definition 1.1. A *linear representation* of G on a vector space V is a group homomorphism

$$\rho\colon G \longrightarrow \mathrm{GL}(V).$$

When ρ is understood, we also say that V is a representation space of G.

Definition 1.2. Let

$$\rho\colon G \to \mathrm{GL}(V), \qquad \rho'\colon G \to \mathrm{GL}(V')$$

be two representations. They are said to be *equivalent*, or *isomorphic*, if there exists a linear isomorphism $T\colon V \to V'$ such that

$$T\rho(g) = \rho'(g)T$$

for every $g \in G$. Such a map is called an *intertwining operator*.

Definition 1.3. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. A vector subspace $W \subseteq V$ is called *G-invariant* if

$$\rho(g)W \subseteq W$$

for every $g \in G$. Since each $\rho(g)$ is invertible, this is equivalent to $\rho(g)W = W$. The restricted maps

$$\rho|_W(g) = \rho(g)|_W$$

define a representation of G on W, called a *subrepresentation* of V.

Theorem 1.4 (Maschke's theorem)**.** *Let $W \subseteq V$ be a G-invariant subspace. Then W admits a G-invariant complement: there exists a G-invariant subspace W' such that*

$$V = W \oplus W'.$$

Proof. Choose an arbitrary vector-space complement U of W and let $p\colon V \to W$ be the projection with kernel U. Define the averaged map

$$p^0 = \frac{1}{|G|} \sum_{g \in G} \rho(g) p \rho(g)^{-1}.$$

Since W is G-invariant, every summand maps V into W, hence so does p^0. Moreover, if $w \in W$, then

$$\rho(g) p \rho(g)^{-1} w = w$$

for every $g \in G$, and therefore $p^0|_W = \mathrm{id}_W$. It follows that $(p^0)^2 = p^0$ and

$$V = W \oplus \ker(p^0).$$

For $s \in G$ we have

$$\begin{aligned} \rho(s) p^0 \rho(s)^{-1} &= \frac{1}{|G|} \sum_{g \in G} \rho(sg) p \rho(sg)^{-1} \\ &= p^0, \end{aligned}$$

so p^0 commutes with the action of G. Hence $\ker(p^0)$ is G-invariant, and it is the required complement. □

Definition 1.5. A nonzero representation V is called *irreducible* if its only G-invariant subspaces are 0 and V. Equivalently, V cannot be written as a direct sum of two nonzero subrepresentations.

Theorem 1.6 (Complete reducibility)**.** *Every representation of G is a direct sum of irreducible representations.*

Proof. We argue by induction on $\dim V$. There is nothing to prove when $V = 0$ or when V is irreducible. Otherwise, choose a nonzero proper invariant subspace $W \subsetneq V$. By Maschke's theorem,

$$V = W \oplus W'$$

for some invariant complement W'. Both W and W' have smaller dimension than V, so the induction hypothesis decomposes each of them into irreducible representations. Combining the two decompositions gives the result. □

1.2 Constructions of Representations

1.2.1 Direct sums

Let $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_2\colon G \to \mathrm{GL}(V_2)$ be representations. Their *direct sum* is the representation

$$\rho_1 \oplus \rho_2 \colon G \longrightarrow \mathrm{GL}(V_1 \oplus V_2)$$

defined by

$$(\rho_1 \oplus \rho_2)(g)(v_1, v_2) = \big(\rho_1(g)v_1, \rho_2(g)v_2\big).$$

With respect to bases of V_1 and V_2, its matrix is block diagonal:

$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}.$$

1.2.2 Tensor products

The *tensor product representation* on $V_1 \otimes V_2$ is defined by

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)v_1 \otimes \rho_2(g)v_2.$$

This indeed gives a representation because

$$(\rho_1 \otimes \rho_2)(gh) = (\rho_1 \otimes \rho_2)(g)(\rho_1 \otimes \rho_2)(h).$$

Let (e_i) be a basis of V_1 and (f_j) a basis of V_2, and write

$$\rho_1(g)e_j = \sum_i r_{ij}^{(1)}(g)e_i, \qquad \rho_2(g)f_\ell = \sum_k r_{k\ell}^{(2)}(g)f_k.$$

Then

$$(\rho_1 \otimes \rho_2)(g)(e_j \otimes f_\ell) = \sum_{i,k} r_{ij}^{(1)}(g)r_{k\ell}^{(2)}(g)e_i \otimes f_k.$$

Thus the matrix of the tensor product representation is the Kronecker product of the matrices of $\rho_1(g)$ and $\rho_2(g)$.

1.2.3 Symmetric and alternating squares

Suppose now that $V_1 = V_2 = V$. Define

$$\tau\colon V \otimes V \longrightarrow V \otimes V, \qquad \tau(v \otimes w) = w \otimes v.$$

Since $\tau^2 = \mathrm{id}$, its eigenvalues are 1 and -1. We set

$$\mathrm{Sym}^2(V) = \{z \in V \otimes V \mid \tau(z) = z\},$$
$$\bigwedge^2 V = \{z \in V \otimes V \mid \tau(z) = -z\}.$$

Because the ground field has characteristic different from 2,

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \bigwedge^2 V.$$

The flip map τ commutes with $\rho(g) \otimes \rho(g)$, so both summands are G-invariant.

If $(e_1, \dots, e_n)$ is a basis of V, then a basis of $\mathrm{Sym}^2(V)$ is

$$e_i \otimes e_i \quad (1 \le i \le n), \qquad e_i \otimes e_j + e_j \otimes e_i \quad (1 \le i < j \le n),$$

and a basis of $\bigwedge^2 V$ is

$$e_i \otimes e_j - e_j \otimes e_i \quad (1 \le i < j \le n).$$

In particular,

$$\dim \mathrm{Sym}^2(V) = \frac{n(n+1)}{2}, \qquad \dim \bigwedge^2 V = \frac{n(n-1)}{2}.$$

1.2.4 Dual representations

Let $V^* = \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$. The *dual representation* ρ^* is defined by

$$\rho^*(g) = \big(\rho(g)^{-1}\big)^*.$$

Equivalently,

$$\big(\rho^*(g)\lambda\big)(v) = \lambda\big(\rho(g)^{-1}v\big)$$

for $\lambda \in V^*$ and $v \in V$. It is characterized by the identity

$$\langle \rho(g)v, \rho^*(g)\lambda \rangle = \langle v, \lambda \rangle.$$

The homomorphism property follows from

$$\rho^*(gh) = \big(\rho(h)^{-1}\rho(g)^{-1}\big)^* = \rho^*(g)\rho^*(h).$$

1.3 Characters of Representations

Definition 1.7. Let $A\colon V \to V$ be linear. If (a_{ij}) is its matrix in a basis of V, its *trace* is

$$\operatorname{Tr}(A) = \sum_i a_{ii}.$$

The trace is independent of the chosen basis, since

$$\operatorname{Tr}(PAP^{-1}) = \operatorname{Tr}(A)$$

for every invertible matrix P.

Definition 1.8. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. Its *character* is the function

$$\chi_\rho\colon G \longrightarrow \mathbb{C}, \qquad \chi_\rho(g) = \operatorname{Tr}(\rho(g)).$$

The integer $\dim V$ is also called the *degree* of the character.

Proposition 1.9 (Basic properties of characters)**.** *Let χ be the character of a representation V of degree n. Then:*

1. *$\chi(1) = n$;*
2. *$\chi(g^{-1}) = \overline{\chi(g)}$ for every $g \in G$;*
3. *$\chi(hgh^{-1}) = \chi(g)$ for every $g, h \in G$.*

Proof. The first assertion is immediate. For the second, let m be the order of g. Then $\rho(g)^m = \mathrm{id}$, so the minimal polynomial of $\rho(g)$ divides $x^m - 1$. Since $x^m - 1$ has distinct roots over $\mathbb{C}$, the operator $\rho(g)$ is diagonalizable. If its eigenvalues are $\lambda_1, \dots, \lambda_n$, then $\lambda_i^m = 1$, and hence $\lambda_i^{-1} = \overline{\lambda_i}$. Therefore

$$\chi(g^{-1}) = \sum_i \lambda_i^{-1} = \overline{\sum_i \lambda_i} = \overline{\chi(g)}.$$

Finally,

$$\begin{aligned}\chi(hgh^{-1}) &= \operatorname{Tr}\big(\rho(h)\rho(g)\rho(h)^{-1}\big)\\ &= \operatorname{Tr}(\rho(g)) = \chi(g).\end{aligned}$$

□

Proposition 1.10. *Let V_1, V_2 have characters χ_1, χ_2. Then*

$$\chi_{V_1 \oplus V_2} = \chi_1 + \chi_2, \qquad \chi_{V_1 \otimes V_2} = \chi_1 \chi_2.$$

Proof. The first formula follows from the trace of a block diagonal matrix. For the second, use the notation of the tensor-product construction. The diagonal entries of the matrix of $\rho_1(g) \otimes \rho_2(g)$ are $r_{ii}^{(1)}(g) r_{jj}^{(2)}(g)$, and hence

$$\begin{aligned}
\chi_{V_1 \otimes V_2}(g) &= \sum_{i,j} r_{ii}^{(1)}(g) r_{jj}^{(2)}(g) \\
&= \left(\sum_i r_{ii}^{(1)}(g) \right) \left(\sum_j r_{jj}^{(2)}(g) \right) \\
&= \chi_1(g) \chi_2(g).
\end{aligned}$$

□

Proposition 1.11 (Characters of the symmetric and alternating squares)**.** *Let V have character χ. Then*

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{1}{2}\big(\chi(g)^2 + \chi(g^2)\big)$$

and

$$\chi_{\bigwedge^2 V}(g) = \frac{1}{2}\big(\chi(g)^2 - \chi(g^2)\big).$$

In particular,

$$\chi_{\mathrm{Sym}^2(V)} + \chi_{\bigwedge^2 V} = \chi^2.$$

Proof. Fix $g \in G$. Since $\rho(g)$ has finite order, it is diagonalizable. Choose an eigenbasis $e_1, \dots, e_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. On the basis vectors of $\mathrm{Sym}^2(V)$, the eigenvalues are

$$\lambda_i^2 \quad \text{and} \quad \lambda_i \lambda_j \quad (i < j),$$

whereas on $\bigwedge^2 V$ they are $\lambda_i \lambda_j$ for $i < j$. Therefore

$$\begin{aligned}
\chi_{\mathrm{Sym}^2(V)}(g) &= \sum_i \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j \\
&= \frac{1}{2}\left(\left(\sum_i \lambda_i \right)^2 + \sum_i \lambda_i^2 \right) \\
&= \frac{1}{2}\big(\chi(g)^2 + \chi(g^2)\big),
\end{aligned}$$

and similarly

$$\begin{aligned}
\chi_{\bigwedge^2 V}(g) &= \sum_{i<j} \lambda_i \lambda_j \\
&= \frac{1}{2}\left(\left(\sum_i \lambda_i \right)^2 - \sum_i \lambda_i^2 \right) \\
&= \frac{1}{2}\big(\chi(g)^2 - \chi(g^2)\big).
\end{aligned}$$

□

Corollary 1.12. *Let V and W have characters χ and ψ. Then*

$$\chi_{\mathrm{Sym}^2(V\oplus W)} = \chi_{\mathrm{Sym}^2 V} + \chi\psi + \chi_{\mathrm{Sym}^2 W},$$
$$\chi_{\bigwedge^2(V\oplus W)} = \chi_{\bigwedge^2 V} + \chi\psi + \chi_{\bigwedge^2 W}.$$

Proof. Use the G-equivariant decompositions

$$\mathrm{Sym}^2(V\oplus W) \cong \mathrm{Sym}^2 V \oplus (V\otimes W) \oplus \mathrm{Sym}^2 W,$$
$$\bigwedge^2(V\oplus W) \cong \bigwedge^2 V \oplus (V\otimes W) \oplus \bigwedge^2 W,$$

and apply Proposition 1.10. □

Proposition 1.13. *If V has character χ, then its dual representation has character*

$$\chi_{V^*}(g) = \chi(g^{-1}) = \overline{\chi(g)}.$$

Proof. The eigenvalues of $\rho^*(g)$ are the inverses of the eigenvalues of $\rho(g)$. Hence

$$\chi_{V^*}(g) = \chi(g^{-1}),$$

and Proposition 1.9 gives the second equality. □

1.4 Schur's Lemma and Averaging Operators

Proposition 1.14 (Schur's lemma). *Let V_1, V_2 be irreducible representations and let $T\colon V_1 \to V_2$ satisfy*

$$T\rho_1(g) = \rho_2(g)T$$

for all $g \in G$. Then:

1. *if $V_1 \ncong V_2$, then $T = 0$;*
2. *if $V_1 = V_2$ and $\rho_1 = \rho_2$, then $T = \lambda\,\mathrm{id}_{V_1}$ for some $\lambda \in \mathbb{C}$.*

Proof. The kernel of T is a G-invariant subspace of V_1, and the image of T is a G-invariant subspace of V_2. If $T \neq 0$, irreducibility gives

$$\ker T = 0, \qquad \operatorname{im} T = V_2,$$

so T is an isomorphism. This proves the first assertion.

For the second, choose an eigenvalue λ of T, which exists because the field is $\mathbb{C}$. The map $T - \lambda\,\mathrm{id}_{V_1}$ is again an intertwining operator and has nonzero kernel. By irreducibility its kernel is all of V_1, so $T = \lambda\,\mathrm{id}_{V_1}$. □

Definition 1.15. For a linear map $h\colon V_1 \to V_2$, define its *average* by

$$h^0 = \frac{1}{|G|}\sum_{g\in G} \rho_2(g)^{-1} h \rho_1(g).$$

Proposition 1.16. *The averaged map h^0 is an intertwining operator. Consequently, if V_1, V_2 are irreducible, then:*

1. *if* $V_1 \not\cong V_2$, *then* $h^0 = 0$;
2. *if* $V_1 = V_2 = V$ *and* $\rho_1 = \rho_2$, *then*

$$h^0 = \frac{\mathrm{Tr}(h)}{\dim V}\,\mathrm{id}_V\,.$$

Proof. For $s \in G$,

$$\begin{aligned}\rho_2(s)^{-1}h^0\rho_1(s) &= \frac{1}{|G|}\sum_{g\in G}\rho_2(gs)^{-1}h\rho_1(gs)\\ &= h^0.\end{aligned}$$

Thus $h^0\rho_1(s) = \rho_2(s)h^0$, so h^0 is an intertwiner. The first conclusion follows from Schur's lemma. In the second case, Schur's lemma gives $h^0 = \lambda\,\mathrm{id}_V$. Since trace is invariant under conjugation,

$$\mathrm{Tr}(h^0) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(h) = \mathrm{Tr}(h).$$

Hence $\lambda \dim V = \mathrm{Tr}(h)$. □

Lemma 1.17 (Averaging onto invariant vectors)**.** *Let* E *be a representation of* G *with character* χ_E*. Then*

$$\dim E^G = \frac{1}{|G|}\sum_{g\in G}\chi_E(g),$$

where

$$E^G = \{v \in E \mid \rho(g)v = v \text{ for every } g \in G\}.$$

Proof. Set

$$P = \frac{1}{|G|}\sum_{g\in G}\rho(g).$$

For every $s \in G$ we have $\rho(s)P = P$, so $\mathrm{im}\,P \subseteq E^G$. Conversely, P acts as the identity on E^G, and therefore P is the projection of E onto E^G. Thus

$$\dim E^G = \mathrm{Tr}(P) = \frac{1}{|G|}\sum_{g\in G}\chi_E(g).$$

□

1.5 Orthogonality Relations

For complex-valued functions φ, ψ on G, define

$$\langle \varphi, \psi\rangle = \frac{1}{|G|}\sum_{g\in G}\varphi(g)\overline{\psi(g)}.$$

This is the standard Hermitian inner product on the space of functions $G \to \mathbb{C}$.

1.5.1 Orthogonality of matrix coefficients

Let V_1, V_2 be irreducible representations. Choose bases and write

$$\rho_\alpha(g)e_j^{(\alpha)} = \sum_i r_{ij}^{(\alpha)}(g)e_i^{(\alpha)} \qquad (\alpha = 1, 2).$$

Theorem 1.18 (Orthogonality of matrix coefficients)**.** *If $V_1 \ncong V_2$, then*

$$\frac{1}{|G|}\sum_{g\in G} r_{i_2j_2}^{(2)}(g^{-1})r_{j_1i_1}^{(1)}(g) = 0$$

for all indices. If $V_1 = V_2 = V$ and the same basis is used on both sides, then, with $n = \dim V$,

$$\frac{1}{|G|}\sum_{g\in G} r_{i_2j_2}(g^{-1})r_{j_1i_1}(g) = \frac{1}{n}\delta_{i_1i_2}\delta_{j_1j_2}.$$

Proof. Let $h\colon V_1 \to V_2$ have matrix $(x_{j_2j_1})$. By direct matrix multiplication, the (i_2, i_1)-entry of its average is

$$(h^0)_{i_2i_1} = \frac{1}{|G|}\sum_{g\in G}\sum_{j_1,j_2} r_{i_2j_2}^{(2)}(g^{-1})x_{j_2j_1}r_{j_1i_1}^{(1)}(g).$$

If $V_1 \ncong V_2$, Proposition 1.16 gives $h^0 = 0$ for every choice of the matrix $(x_{j_2j_1})$, so every coefficient of every $x_{j_2j_1}$ is zero. This proves the first formula.

In the second case,

$$h^0 = \frac{\operatorname{Tr}(h)}{n}\,\mathrm{id}_V\,.$$

The coefficient of $x_{j_2j_1}$ in the (i_2, i_1)-entry of the right-hand side is

$$\frac{1}{n}\delta_{j_1j_2}\delta_{i_1i_2},$$

which gives the stated identity. □

1.5.2 Orthogonality of irreducible characters

Theorem 1.19 (Character orthogonality)**.** *Let χ and ψ be irreducible characters. Then*

$$\langle\chi, \psi\rangle = \begin{cases} 1, & \chi = \psi, \\ 0, & \chi \neq \psi. \end{cases}$$

Thus the irreducible characters form an orthonormal set.

Proof. Using $\overline{\psi(g)} = \psi(g^{-1})$ and expanding both traces in terms of matrix coefficients, we obtain

$$\begin{aligned}\langle\chi, \psi\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1}) \\ &= \sum_{i,j}\frac{1}{|G|}\sum_{g\in G} r_{ii}^{(1)}(g)r_{jj}^{(2)}(g^{-1}).\end{aligned}$$

If the representations are nonisomorphic, every summand is zero by Theorem 1.18. If they are the same irreducible representation of dimension n, the same theorem gives

$$\langle \chi, \chi \rangle = \sum_{i,j} \frac{1}{n} \delta_{ij} = 1.$$

□

Remark 1.20. There is also a conceptual proof. The space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is a representation under

$$(g \cdot T) = \rho_W(g) T \rho_V(g)^{-1},$$

and its character is $\chi_W \overline{\chi_V}$. By Lemma 1.17,

$$\langle \chi_W, \chi_V \rangle = \dim \mathrm{Hom}_G(V, W).$$

Schur's lemma then gives the orthogonality relations.

Theorem 1.21 (Multiplicity formula)**.** *Suppose*

$$V = U_1 \oplus \cdots \oplus U_m$$

is a decomposition into irreducible representations, and let φ be the character of V. If W is irreducible with character χ, then the number of summands U_j isomorphic to W is

$$\langle \varphi, \chi \rangle.$$

Proof. Write χ_j for the character of U_j. Then

$$\varphi = \chi_1 + \cdots + \chi_m.$$

By the orthogonality theorem,

$$\langle \varphi, \chi \rangle = \sum_{j=1}^{m} \langle \chi_j, \chi \rangle,$$

and each summand is 1 precisely when $U_j \cong W$, and 0 otherwise. □

Corollary 1.22. *Two representations with the same character are isomorphic.*

Proof. For every irreducible representation W, the multiplicity formula shows that its multiplicity is determined by the character. Hence the two representations have the same irreducible summands with the same multiplicities, and complete reducibility gives an isomorphism. □

Theorem 1.23 (Irreducibility criterion)**.** *Let φ be the character of a representation V. Then*

$$\langle \varphi, \varphi \rangle \in \mathbb{Z}_{\geq 0},$$

and

$$\langle \varphi, \varphi \rangle = 1$$

if and only if V is irreducible.

Proof. Let $\chi_1, \ldots, \chi_r$ be the distinct irreducible characters occurring in V, with multiplicities $m_1, \ldots, m_r$. Then

$$\varphi = \sum_{i=1}^{r} m_i \chi_i.$$

By orthogonality,

$$\langle \varphi, \varphi \rangle = \sum_{i=1}^{r} m_i^2.$$

This is a nonnegative integer, and it equals 1 exactly when one multiplicity is 1 and all the others are zero, which is equivalent to irreducibility. □

1.6 Canonical Decomposition into Isotypic Components

Let $W_1, \ldots, W_r$ be pairwise nonisomorphic irreducible representations of G, with characters $\chi_1, \ldots, \chi_r$ and degrees $n_i = \dim W_i$. Suppose that a representation V has an irreducible decomposition

$$V = U_1 \oplus \cdots \oplus U_m.$$

For each i, let V_i be the direct sum of those U_j which are isomorphic to W_i. Then

$$V = V_1 \oplus \cdots \oplus V_r.$$

At first sight the subspaces V_i appear to depend on the chosen irreducible decomposition. The following theorem shows that they do not.

Theorem 1.24 (Canonical isotypic decomposition)**.** *For each i, define*

$$p_i = \frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho(g) = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g^{-1}) \rho(g).$$

Then p_i is the G-equivariant projection of V onto V_i. In particular,

$$p_i^2 = p_i, \qquad p_i p_j = 0 \quad (i \neq j), \qquad \sum_{i=1}^{r} p_i = \mathrm{id}_V,$$

and the decomposition

$$V = V_1 \oplus \cdots \oplus V_r$$

is independent of the chosen decomposition of V into irreducible summands.

Proof. Since χ_i is constant on conjugacy classes, for every $h \in G$ we have

$$\begin{aligned} \rho(h) p_i \rho(h)^{-1} &= \frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho(hgh^{-1}) \\ &= p_i. \end{aligned}$$

Thus p_i is G-equivariant.

Restrict p_i to an irreducible subrepresentation $U \subseteq V$ with character χ_j. By Schur's lemma,

$$p_i|_U = \lambda \, \mathrm{id}_U$$

for some $\lambda \in \mathbb{C}$. Taking traces gives

$$\begin{aligned}\lambda n_j &= \operatorname{Tr}(p_i|_U) \\ &= \frac{n_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_j(g) \\ &= n_i \langle \chi_j, \chi_i \rangle.\end{aligned}$$

By character orthogonality, this is 0 if $i \neq j$, and it is $n_i = n_j$ if $i = j$. Hence

$$p_i|_U = \begin{cases} \mathrm{id}_U, & U \cong W_i, \\ 0, & U \not\cong W_i. \end{cases}$$

Therefore p_i is exactly the projection onto the sum of all irreducible summands isomorphic to W_i. The identities for the p_i follow by checking them on every irreducible summand of V. Since each p_i is defined directly from ρ and χ_i, its image is independent of any chosen irreducible decomposition. □

Definition 1.25. The subspace

$$V_i = \operatorname{im}(p_i)$$

is called the W_i*-isotypic component* of V. Equivalently, it is the sum of all subrepresentations of V isomorphic to W_i.

Remark 1.26. The decomposition into individual irreducible subspaces need not be unique. For example, if

$$V \cong W \oplus W,$$

there may be many different subspaces of V isomorphic to W which can be used as summands. What is canonical is their total isotypic component, which in this example is all of V.

1.6.1 Multiplicity spaces

Proposition 1.27. *Let*

$$M_i = \operatorname{Hom}_G(W_i, V).$$

Then

$$\dim M_i = \langle \chi_V, \chi_i \rangle,$$

and the evaluation map

$$\begin{aligned} \mathrm{ev}_i \colon W_i \otimes M_i &\longrightarrow V_i, \\ w \otimes f &\longmapsto f(w) \end{aligned}$$

is a G-equivariant isomorphism. Consequently,

$$V \cong \bigoplus_{i=1}^{r} W_i \otimes \operatorname{Hom}_G(W_i, V).$$

Proof. By Schur's lemma and complete reducibility, $\dim \operatorname{Hom}_G(W_i, V)$ is exactly the number of copies of W_i in an irreducible decomposition of V. By the multiplicity formula this number is $\langle \chi_V, \chi_i \rangle$.

The evaluation map is G-equivariant because, for $g \in G$ and $f \in \mathrm{Hom}_G(W_i, V)$,

$$\mathrm{ev}_i(\rho_i(g)w \otimes f) = f(\rho_i(g)w) = \rho(g)f(w).$$

Its image is contained in V_i. Conversely, every irreducible copy of W_i inside V is the image of an intertwining map $W_i \to V$, so the image of ev_i contains every such copy and hence equals V_i. Finally,

$$\dim(W_i \otimes M_i) = n_i \dim M_i = \dim V_i,$$

so the surjective evaluation map is an isomorphism. □

1.7 Permutation and Regular Representations

1.7.1 Permutation representations

Definition 1.28. Suppose that G acts on a finite set X. Let V_X be the vector space with basis $(e_x)_{x\in X}$. The rule

$$\rho(g)e_x = e_{gx}$$

defines a representation of G, called the *permutation representation* associated with X.

Proposition 1.29. *The character χ_X of the permutation representation satisfies*

$$\chi_X(g) = |X^g|,$$

where

$$X^g = \{x \in X \mid gx = x\}.$$

Proof. The matrix of $\rho(g)$ in the basis $(e_x)_{x\in X}$ is a permutation matrix. Its diagonal entry corresponding to x is 1 exactly when $gx = x$, and is 0 otherwise. Hence its trace is the number of fixed points of g. □

Proposition 1.30 (Orbit-counting formula)**.** *The multiplicity of the trivial representation in V_X is the number of G-orbits on X. Equivalently,*

$$\#(G\backslash X) = \frac{1}{|G|}\sum_{g\in G} |X^g| = \langle \chi_X, \mathbf{1}_G\rangle = \dim V_X^G.$$

Proof. An invariant vector

$$v = \sum_{x\in X} a_x e_x$$

has coefficients which are constant on every orbit. Therefore the orbit sums

$$\sum_{x\in\mathcal{O}} e_x, \qquad \mathcal{O} \in G\backslash X,$$

form a basis of V_X^G, so $\dim V_X^G = \#(G\backslash X)$. Lemma 1.17 and Proposition 1.29 give the remaining equalities. □

If the action is transitive, the trivial representation occurs exactly once. In this case

$$V_X = \mathbb{C}\left(\sum_{x\in X} e_x\right) \oplus V_0,$$

where

$$V_0 = \left\{ \sum_{x\in X} a_x e_x \;\middle|\; \sum_{x\in X} a_x = 0 \right\}.$$

The representation V_0 contains no copy of the trivial representation.

Definition 1.31. Assume that G acts transitively on X and $|X| \geq 2$. The action is called *doubly transitive* if for any $x \neq y$ and $x' \neq y'$ in X, there exists $g \in G$ such that

$$gx = x', \qquad gy = y'.$$

Proposition 1.32. *Let G act transitively on X, let χ_X be the permutation character, and write*

$$\chi_X = \mathbf{1}_G + \theta,$$

where θ is the character of V_0. The following are equivalent:

1. *the action of G on X is doubly transitive;*
2. *the diagonal action of G on $X \times X$ has exactly two orbits;*
3. *$\langle \chi_X, \chi_X \rangle = 2$;*
4. *V_0 is irreducible.*

Proof. The diagonal

$$\Delta = \{(x, x) \mid x \in X\}$$

is one orbit because the action on X is transitive. The action is doubly transitive precisely when all pairs (x, y) with $x \neq y$ form one further orbit. Thus (1) and (2) are equivalent.

The character of the permutation representation on $X \times X$ is χ_X^2, because

$$|(X \times X)^g| = |X^g|^2.$$

By Proposition 1.30, the number of orbits on $X \times X$ is

$$\langle \chi_X^2, \mathbf{1}_G \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_X(g)^2 = \langle \chi_X, \chi_X \rangle,$$

since χ_X is real-valued. Hence (2) and (3) are equivalent.

Finally, $\langle \mathbf{1}_G, \theta \rangle = 0$, so

$$\langle \chi_X, \chi_X \rangle = 1 + \langle \theta, \theta \rangle.$$

By the irreducibility criterion, this equals 2 precisely when V_0 is irreducible. Thus (3) and (4) are equivalent. □

1.7.2 The regular representation

Definition 1.33. Let V have basis $(e_t)_{t\in G}$. For $g \in G$, define

$$\rho_{\mathrm{reg}}(g)e_t = e_{gt}.$$

This is the *left regular representation* of G.

Its character χ_{reg} is

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G|, & g = 1, \\ 0, & g \neq 1. \end{cases}$$

Indeed, if $g \neq 1$, left multiplication by g fixes no element of G.

Theorem 1.34. *Let $W_1, \dots, W_r$ be all irreducible representations of G, and put $n_i = \dim W_i$. Then*

$$V_{\mathrm{reg}} \cong \bigoplus_{i=1}^{r} W_i^{\oplus n_i}.$$

Thus every irreducible representation occurs in the regular representation, with multiplicity equal to its degree.

Proof. Let χ_i be the character of W_i. By the multiplicity formula, its multiplicity in the regular representation is

$$\begin{aligned}\langle \chi_{\mathrm{reg}}, \chi_i \rangle &= \frac{1}{|G|} \sum_{g\in G} \chi_{\mathrm{reg}}(g)\overline{\chi_i(g)} \\ &= \chi_i(1) = n_i.\end{aligned}$$

□

Corollary 1.35. *The degrees of the irreducible representations satisfy*

$$\sum_{i=1}^{r} n_i^2 = |G|.$$

Moreover, for every $g \neq 1$,

$$\sum_{i=1}^{r} n_i \chi_i(g) = 0.$$

Proof. Taking dimensions in Theorem 1.34 gives the first identity. Taking characters gives

$$\chi_{\mathrm{reg}} = \sum_{i=1}^{r} n_i \chi_i,$$

and evaluating at $g \neq 1$ gives the second. □

1.8 Class Functions and Character Tables

Definition 1.36. A function $f\colon G \to \mathbb{C}$ is called a *class function* if

$$f(hgh^{-1}) = f(g)$$

for all $g, h \in G$. Thus a class function is constant on every conjugacy class. Every character is a class function.

Let $\mathcal{H}$ denote the vector space of class functions on G.

Proposition 1.37. *Let f be a class function and let V be a representation with character χ. Define*

$$T_f = \sum_{g \in G} f(g)\rho(g).$$

Then T_f commutes with the action of G. If V is irreducible of degree n, then

$$T_f = \lambda \, \mathrm{id}_V,$$

where

$$\lambda = \frac{1}{n}\sum_{g \in G} f(g)\chi(g) = \frac{|G|}{n}\langle f, \chi_{V^*}\rangle.$$

Proof. For $h \in G$,

$$\begin{aligned} \rho(h)^{-1}T_f\rho(h) &= \sum_{g \in G} f(g)\rho(h^{-1}gh) \\ &= \sum_{u \in G} f(huh^{-1})\rho(u) \\ &= T_f. \end{aligned}$$

Thus T_f is an intertwining operator. If V is irreducible, Schur's lemma gives $T_f = \lambda \, \mathrm{id}_V$. Taking traces yields

$$\lambda n = \sum_{g \in G} f(g)\chi(g).$$

Since $\chi_{V^*}(g) = \overline{\chi(g)}$, we have

$$\langle f, \chi_{V^*}\rangle = \frac{1}{|G|}\sum_{g \in G} f(g)\chi(g),$$

which proves the formula. □

Theorem 1.38. *The irreducible characters $\chi_1, \dots, \chi_r$ form an orthonormal basis of $\mathcal{H}$.*

Proof. They are orthonormal by Theorem 1.19. It remains to prove that they span $\mathcal{H}$.

Suppose that $f \in \mathcal{H}$ is orthogonal to every irreducible character. The dual of an irreducible representation is again irreducible, so f is also orthogonal to every dual character. By Proposition 1.37, the operator T_f acts as zero on every irreducible representation. Complete reducibility therefore implies that $T_f = 0$ on the regular representation.

Let e_1 be the basis vector of the regular representation corresponding to the identity. Then

$$0 = T_f e_1 = \sum_{g \in G} f(g)e_g.$$

Since the e_g form a basis, $f(g) = 0$ for every $g \in G$. Hence the orthogonal complement of the span of the irreducible characters is zero, so they form a basis. □

Theorem 1.39. *The number of isomorphism classes of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. A class function is determined uniquely by its value on each conjugacy class. Therefore the dimension of $\mathcal{H}$ is the number of conjugacy classes. By Theorem 1.38, the dimension of $\mathcal{H}$ is also the number of irreducible characters. □

1.8.1 Column orthogonality

For $s \in G$, denote its conjugacy class by $\mathrm{Cl}(s)$ and its centralizer by

$$C_G(s) = \{g \in G \mid gs = sg\}.$$

Recall that

$$|\mathrm{Cl}(s)| = \frac{|G|}{|C_G(s)|}.$$

Theorem 1.40 (Column orthogonality)**.** *Let $\chi_1, \dots, \chi_r$ be all irreducible characters. For $s, t \in G$,*

$$\sum_{i=1}^{r} \overline{\chi_i(s)}\chi_i(t) = \begin{cases} |C_G(s)|, & t \text{ is conjugate to } s, \\ 0, & t \text{ is not conjugate to } s. \end{cases}$$

In particular,

$$\sum_{i=1}^{r} |\chi_i(s)|^2 = |C_G(s)| = \frac{|G|}{|\mathrm{Cl}(s)|}.$$

Proof. Let δ_s be the indicator function of the conjugacy class of s:

$$\delta_s(t) = \begin{cases} 1, & t \in \mathrm{Cl}(s), \\ 0, & t \notin \mathrm{Cl}(s). \end{cases}$$

Since the irreducible characters form an orthonormal basis of $\mathcal{H}$,

$$\delta_s = \sum_{i=1}^{r} \langle \delta_s, \chi_i \rangle \chi_i.$$

Now

$$\begin{aligned} \langle \delta_s, \chi_i \rangle &= \frac{1}{|G|} \sum_{g \in \mathrm{Cl}(s)} \overline{\chi_i(g)} \\ &= \frac{|\mathrm{Cl}(s)|}{|G|} \overline{\chi_i(s)} \\ &= \frac{1}{|C_G(s)|} \overline{\chi_i(s)}. \end{aligned}$$

Hence

$$|C_G(s)| \delta_s(t) = \sum_{i=1}^{r} \overline{\chi_i(s)} \chi_i(t),$$

which is the desired relation. □

Definition 1.41. Choose representatives $c_1, \dots, c_r$ of the conjugacy classes and list the irreducible characters $\chi_1, \dots, \chi_r$. The square matrix

$$\big(\chi_i(c_j)\big)_{1 \le i,j \le r}$$

is called the *character table* of G.

The row orthogonality relations are

$$\sum_{j=1}^{r} |\operatorname{Cl}(c_j)| \chi_i(c_j)\overline{\chi_k(c_j)} = |G|\delta_{ik},$$

while Theorem 1.40 gives the column orthogonality relations.

1.9 Induced Representations

Let $H \le G$ and let

$$\theta \colon H \to \operatorname{GL}(W)$$

be a representation of H.

Definition 1.42. Suppose $\rho\colon G \to \operatorname{GL}(V)$ is a representation such that W is an H-invariant subspace of V and the restricted action on W is θ. For $g \in G$, set

$$W_{gH} = \rho(g)W.$$

This depends only on the left coset gH, because

$$\rho(gh)W = \rho(g)\rho(h)W = \rho(g)W$$

for $h \in H$. We say that V is *induced by* W, or that ρ is *induced by* θ, if

$$V = \bigoplus_{gH \in G/H} W_{gH}.$$

We write

$$V \cong \operatorname{Ind}_H^G W.$$

Equivalently, if R is a set of representatives of G/H, then

$$V = \bigoplus_{r \in R} \rho(r)W.$$

It follows immediately that

$$\dim \operatorname{Ind}_H^G W = [G : H] \dim W.$$

Example 1.43. The permutation representation of G on G/H is induced by the trivial representation of H:

$$\mathbb{C}[G/H] \cong \operatorname{Ind}_H^G \mathbf{1}_H.$$

Indeed, the one-dimensional subspace corresponding to the coset H is H-invariant and affords the trivial representation, and its translates are the basis lines corresponding to all cosets.

Example 1.44. The regular representation of G is induced by the regular representation of H. The subspace spanned by $(e_h)_{h\in H}$ is the regular representation of H, and its translates are the subspaces spanned by the basis vectors belonging to the left cosets of H.

Proposition 1.45. *Suppose that V is induced by W, and let $W_1 \subseteq W$ be an H-invariant subspace. Then*

$$V_1 = \bigoplus_{gH\in G/H} \rho(g)W_1$$

is a G-invariant subspace of V, and the representation on V_1 is induced by the representation of H on W_1.

Proof. The sum is direct because it is contained term by term in the direct sum

$$V = \bigoplus_{gH\in G/H} \rho(g)W.$$

For $s \in G$, left multiplication sends the coset gH to sgH, and hence

$$\rho(s)\rho(g)W_1 = \rho(sg)W_1.$$

Thus $\rho(s)$ permutes the summands of V_1, so V_1 is G-invariant. The defining direct-sum condition for induction is then automatic. □

Lemma 1.46 (Universal property of induction)**.** *Suppose that (V, ρ) is induced by (W, θ), and let $\rho' \colon G \to \mathrm{GL}(V')$ be another representation. If $f \colon W \to V'$ is H-equivariant, then there exists a unique G-equivariant linear map*

$$F \colon V \to V'$$

whose restriction to W is f.

Proof. Choose a set R of representatives of G/H. Since

$$V = \bigoplus_{r\in R} \rho(r)W,$$

it is enough to define F on each summand. For $r \in R$ and $w \in W$, set

$$F(\rho(r)w) = \rho'(r)f(w).$$

This is well defined on the chosen direct-sum decomposition and clearly extends f when the representative of H is chosen to be 1.

We check G-equivariance. Let $g \in G$ and write

$$gr = r'h$$

with $r' \in R$ and $h \in H$. Then

$$\begin{aligned} F\big(\rho(g)\rho(r)w\big) &= F\big(\rho(r')\theta(h)w\big) \\ &= \rho'(r')f(\theta(h)w) \\ &= \rho'(r')\rho'(h)f(w) \\ &= \rho'(g)\rho'(r)f(w) \\ &= \rho'(g)F(\rho(r)w). \end{aligned}$$

Thus F is G-equivariant. Uniqueness follows because every element of V is a sum of vectors of the form $\rho(r)w$, and equivariance forces

$$F(\rho(r)w) = \rho'(r)F(w) = \rho'(r)f(w).$$

□

Theorem 1.47 (Existence and uniqueness of induced representations)**.** *For every representation $\theta\colon H \to \mathrm{GL}(W)$, there exists a representation of G induced by θ. It is unique up to a unique G-equivariant isomorphism which restricts to the identity on W.*

Proof. Choose a set R of representatives of the left cosets G/H, with $1 \in R$, and let

$$V = \bigoplus_{r\in R} W_r,$$

where each W_r is a copy of W. For $g \in G$ and $r \in R$, write uniquely

$$gr = r'h$$

with $r' \in R$ and $h \in H$. Define

$$\rho(g)(w \in W_r) = \theta(h)w \in W_{r'}.$$

This is linear and invertible. To check the homomorphism property, suppose

$$kr = r_1h_1, \qquad gr_1 = r_2h_2.$$

Then

$$gkr = r_2h_2h_1,$$

and therefore

$$\begin{aligned}\rho(g)\rho(k)(w \in W_r) &= \theta(h_2)\theta(h_1)w \in W_{r_2}\\ &= \theta(h_2h_1)w \in W_{r_2}\\ &= \rho(gk)(w \in W_r).\end{aligned}$$

Thus ρ is a representation. The summand W_1 is H-invariant and the action of H on it is θ, while its G-translates are precisely the summands W_r. Hence V is induced by W.

For uniqueness, let V and V' be two representations induced by the same H-representation W. Applying Lemma 1.46 to the inclusion $W \hookrightarrow V'$ gives a G-equivariant map $F\colon V \to V'$ extending the identity on W. Similarly, there is a G-equivariant map $F'\colon V' \to V$ extending the identity on W. Both $F'F$ and FF' extend the identity on W; by the uniqueness part of the universal property, they are the identity maps. Thus F is the required isomorphism, and its uniqueness follows from the same universal property. □

Theorem 1.48 (Character of an induced representation)**.** *Let R be a set of representatives of G/H, and let χ_θ be the character of W. If χ_{Ind} is the character of $\mathrm{Ind}_H^G W$, then for every $u \in G$,*

$$\chi_{\mathrm{Ind}}(u) = \sum_{\substack{r\in R\\ r^{-1}ur\in H}} \chi_\theta(r^{-1}ur).$$

Equivalently,

$$\chi_{\mathrm{Ind}}(u) = \frac{1}{|H|}\sum_{\substack{s\in G\\ s^{-1}us\in H}} \chi_\theta(s^{-1}us).$$

Proof. With respect to the decomposition

$$\operatorname{Ind}_H^G W = \bigoplus_{r \in R} W_r,$$

the operator $\rho(u)$ sends W_r to the summand corresponding to the coset urH. A summand contributes to the trace only when it is fixed, that is, when

$$urH = rH,$$

or equivalently $r^{-1}ur \in H$. On such a summand, the action of u is, after identifying W_r with W, the operator $\theta(r^{-1}ur)$. Therefore its contribution to the trace is $\chi_\theta(r^{-1}ur)$, proving the first formula.

For the second formula, every left coset rH has $|H|$ elements. If $s = rh$, then

$$s^{-1}us = h^{-1}(r^{-1}ur)h,$$

and χ_θ is constant on conjugacy classes in H. Hence all $|H|$ elements of the coset give the same contribution, so summing over all $s \in G$ multiplies the first sum by $|H|$. □

Chapter 2

Group Algebra

Throughout this chapter, unless otherwise stated, all representations are finite-dimensional complex representations of finite groups. If

$$u = \sum_{s \in G} u(s)s \in \mathbb{C}[G],$$

we shall identify the element u with its coefficient function $s \mapsto u(s)$ whenever no confusion can arise.

2.1 Semisimple Modules and Semisimple Algebras

Definition 2.1. Let A be a ring and let M be a left A-module. We say that M is *semisimple* if every submodule of M has a complement. We say that A is a *semisimple ring* if ${}_AA$ is a semisimple A-module. Equivalently, every left A-module is semisimple.

Lemma 2.2. *If S is a simple A-module, then*

$$D = \mathrm{End}_A(S)$$

is a skew field.

Proof. This is exactly the same proof as Schur's lemma. Let $0 \neq f \in \mathrm{End}_A(S)$. Then $\ker f$ and $\operatorname{im} f$ are submodules of S. Since S is simple, $\ker f = 0$ and $\operatorname{im} f = S$. Thus f is an isomorphism. Hence every nonzero element of $\mathrm{End}_A(S)$ is invertible, so $\mathrm{End}_A(S)$ is a skew field. □

Theorem 2.3 (Artin–Wedderburn)**.** *Let A be a finite-dimensional semisimple K-algebra, where K is a field. Suppose that, as a left A-module,*

$$A \simeq S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r},$$

where $S_1, \ldots, S_r$ are pairwise non-isomorphic simple A-modules. Put

$$D_i = \mathrm{End}_A(S_i)^{\mathrm{op}}.$$

Then

$$A \simeq \prod_{i=1}^{r} M_{n_i}(D_i).$$

Proof. We use the regular module. Since A acts on itself on the left, right multiplication gives

$$A^{\mathrm{op}} \simeq \mathrm{End}_A(A).$$

Using the decomposition of A as a left A-module, we get

$$\mathrm{End}_A(A) \simeq \mathrm{End}_A(S_1^{\oplus n_1} \oplus \cdots \oplus S_r^{\oplus n_r}).$$

By Schur's lemma, $\mathrm{Hom}_A(S_i, S_j) = 0$ if $i \neq j$, and hence

$$\mathrm{End}_A(A) \simeq \prod_{i=1}^{r} \mathrm{End}_A(S_i^{\oplus n_i}).$$

Moreover,

$$\mathrm{End}_A(S_i^{\oplus n_i}) \simeq M_{n_i}(\mathrm{End}_A(S_i)).$$

Therefore

$$A^{\mathrm{op}} \simeq \prod_{i=1}^{r} M_{n_i}(\mathrm{End}_A(S_i)).$$

Taking opposite algebras gives

$$A \simeq \prod_{i=1}^{r} M_{n_i}(\mathrm{End}_A(S_i)^{\mathrm{op}}) = \prod_{i=1}^{r} M_{n_i}(D_i).$$

□

2.2 The Group Algebra

Definition 2.4. Let G be a finite group and let K be a commutative ring. We denote by $K[G]$ the algebra of G over K, whose elements can be uniquely written as

$$f = \sum_{s \in G} a_s s, \qquad a_s \in K.$$

Addition is coefficientwise, and multiplication is induced by the multiplication in G:

$$\left(\sum_{s \in G} a_s s\right)\left(\sum_{t \in G} b_t t\right) = \sum_{s,t \in G} a_s b_t (st).$$

Let V be a K-module and let $\rho : G \to \mathrm{GL}(V)$ be a linear representation. Then V is naturally a left $K[G]$-module by

$$\left(\sum_{s \in G} a_s s\right) v = \sum_{s \in G} a_s \rho(s) v.$$

Conversely, every left $K[G]$-module gives a representation of G by restricting the action to the basis elements $s \in G$. Thus representations of G over K are the same thing as left $K[G]$-modules.

Proposition 2.5 (Maschke)**.** *If K is a field with $\mathrm{char}\, K \nmid |G|$, then $K[G]$ is semisimple. In particular, if $\mathrm{char}\, K = 0$, then $K[G]$ is semisimple.*

Proof. It is enough to show that every submodule has a complement. Let V be a finite-dimensional $K[G]$-module and let $W \subseteq V$ be a $K[G]$-submodule. Choose an arbitrary K-linear projection

$$p : V \to W, \qquad p|_W = \mathrm{id}_W.$$

Define the averaged projection

$$p^0 = \frac{1}{|G|} \sum_{s \in G} sps^{-1}.$$

Then for each $g \in G$,

$$gp^0g^{-1} = \frac{1}{|G|} \sum_{s \in G} gsps^{-1}g^{-1} = \frac{1}{|G|} \sum_{t \in G} tpt^{-1} = p^0.$$

Thus p^0 is G-equivariant. If $w \in W$, then $s^{-1}w \in W$, hence

$$sps^{-1}w = w.$$

Therefore $p^0|_W = \mathrm{id}_W$, and

$$V = W \oplus \ker(p^0)$$

as $K[G]$-modules. Hence every submodule has a complement, so $K[G]$ is semisimple. □

Corollary 2.6. *If K is a field of characteristic zero, then $K[G]$ is a product of matrix algebras over skew fields of finite degree over K.*

Proof. By Maschke's theorem, $K[G]$ is semisimple. Applying Artin–Wedderburn to the finite-dimensional semisimple K-algebra $K[G]$, we get

$$K[G] \simeq \prod_i M_{n_i}(D_i),$$

where the D_i are skew fields finite-dimensional over K. □

From now on, we take $K = \mathbb{C}$, or more generally an algebraically closed field of characteristic zero. Then every skew field of finite degree over K is equal to K. Hence

$$\mathbb{C}[G] \simeq \prod_{i=1}^{h} M_{n_i}(\mathbb{C}).$$

Let

$$\rho_i : G \to \mathrm{GL}(W_i), \qquad 1 \leq i \leq h,$$

be the irreducible complex representations of G. Each ρ_i extends to an algebra homomorphism

$$\widehat{\rho}_i : \mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(W_i).$$

Thus the family $(\widehat{\rho}_i)_i$ defines a homomorphism

$$\widehat{\rho} : \mathbb{C}[G] \to \prod_{i=1}^{h} \mathrm{End}_{\mathbb{C}}(W_i) \simeq \prod_{i=1}^{h} M_{n_i}(\mathbb{C}).$$

By the above decomposition, this homomorphism is an isomorphism.

2.3 Fourier Inversion and Plancherel Formula

For

$$u = \sum_{s \in G} u(s)s \in \mathbb{C}[G],$$

write

$$\widehat{u}_i = \widehat{\rho}_i(u) = \sum_{s \in G} u(s)\rho_i(s) \in \operatorname{End}(W_i).$$

Proposition 2.7 (Fourier inversion). *Let* $(u_i)_{1 \le i \le h} \in \prod_i \operatorname{End}(W_i)$, *and let*

$$u = \sum_{s \in G} u(s)s \in \mathbb{C}[G]$$

be the element such that $\widehat{\rho}_i(u) = u_i$ *for each* i. *Then*

$$u(s) = \frac{1}{|G|} \sum_{i=1}^{h} n_i \operatorname{Tr}\big(\rho_i(s^{-1})u_i\big), \qquad n_i = \dim W_i.$$

Proof. By linearity, it suffices to check the formula when $u = t \in G$. Then $u(s) = \delta_{s,t}$ and $u_i = \rho_i(t)$. The formula becomes

$$\delta_{s,t} = \frac{1}{|G|} \sum_{i=1}^{h} n_i \operatorname{Tr}\big(\rho_i(s^{-1})\rho_i(t)\big) = \frac{1}{|G|} \sum_{i=1}^{h} n_i \chi_i(s^{-1}t).$$

This is exactly the character formula for the regular representation, since

$$\chi_{\text{reg}} = \sum_{i=1}^{h} n_i \chi_i$$

and $\chi_{\text{reg}}(g) = |G|$ if $g = 1$, while $\chi_{\text{reg}}(g) = 0$ if $g \neq 1$. □

Proposition 2.8 (Plancherel formula). *Let*

$$u = \sum_{s \in G} u(s)s, \qquad v = \sum_{s \in G} v(s)s$$

be two elements of $\mathbb{C}[G]$, *and put*

$$\langle u, v \rangle = \sum_{s \in G} u(s^{-1})v(s).$$

Then

$$\langle u, v \rangle = \frac{1}{|G|} \sum_{i=1}^{h} n_i \operatorname{Tr}\big(\widehat{\rho}_i(u)\widehat{\rho}_i(v)\big).$$

Proof. By bilinearity, it is enough to check the formula when $u = a$ and $v = b$ are elements of G. Then

$$\langle a, b \rangle = \delta_{a^{-1},b} = \delta_{1,ab}.$$

On the other hand,

$$\frac{1}{|G|} \sum_{i} n_i \operatorname{Tr}(\rho_i(a)\rho_i(b)) = \frac{1}{|G|} \sum_{i} n_i \chi_i(ab),$$

which is again $|G|^{-1}\chi_{\text{reg}}(ab)$. This equals 1 if $ab = 1$, and 0 otherwise. □

2.4 The Center of the Group Algebra

Let C be a conjugacy class of G, and put

$$e_C = \sum_{s \in C} s.$$

Proposition 2.9. *The elements e_C, where C runs through the conjugacy classes of G, form a basis of $Z(\mathbb{C}[G])$. In particular,*

$$\dim_{\mathbb{C}} Z(\mathbb{C}[G]) = h,$$

where h is the number of irreducible characters of G.

Proof. Let

$$u = \sum_{s \in G} u(s) s \in \mathbb{C}[G].$$

Then $u \in Z(\mathbb{C}[G])$ if and only if for each $t \in G$,

$$tut^{-1} = u.$$

This condition is equivalent to

$$u(tst^{-1}) = u(s) \qquad \text{for all } s, t \in G.$$

Thus the coefficient function of u is constant on conjugacy classes. Hence u is uniquely a linear combination of the class sums e_C. The final statement follows from the fact that the number of conjugacy classes equals the number of irreducible characters. $\square$

Let $\rho_i : G \to \mathrm{GL}(W_i)$ be irreducible with character χ_i, and let

$$\widehat{\rho}_i : \mathbb{C}[G] \to \mathrm{End}(W_i)$$

be the corresponding algebra homomorphism.

Proposition 2.10. *The homomorphism $\widehat{\rho}_i$ maps $Z(\mathbb{C}[G])$ into the scalar transformations of W_i. Hence it defines an algebra homomorphism*

$$\omega_i : Z(\mathbb{C}[G]) \to \mathbb{C}.$$

If $u = \sum_{s \in G} u(s)s$, then

$$\omega_i(u) = \frac{1}{n_i} \mathrm{Tr}(\widehat{\rho}_i(u)) = \frac{1}{n_i} \sum_{s \in G} u(s) \chi_i(s), \qquad n_i = \dim W_i.$$

Proof. If $u \in Z(\mathbb{C}[G])$, then $\widehat{\rho}_i(u)$ commutes with every $\rho_i(g)$. By Schur's lemma, $\widehat{\rho}_i(u)$ is a scalar transformation. Write this scalar as $\omega_i(u)$. Taking trace gives

$$n_i \omega_i(u) = \mathrm{Tr}(\widehat{\rho}_i(u)) = \sum_{s \in G} u(s) \chi_i(s).$$

Since $\widehat{\rho}_i$ is an algebra homomorphism and scalar transformations identify with $\mathbb{C}$, ω_i is an algebra homomorphism. $\square$

Proposition 2.11. *The family* $(\omega_i)_{1\leq i\leq h}$ *defines an isomorphism*

$$Z(\mathbb{C}[G]) \xrightarrow{\sim} \mathbb{C}^h.$$

Proof. Under the isomorphism

$$\mathbb{C}[G] \simeq \prod_{i=1}^{h} \operatorname{End}(W_i),$$

the center of $\mathbb{C}[G]$ is identified with the center of the product. But the center of $\operatorname{End}(W_i)$ consists of scalar transformations. Hence

$$Z(\mathbb{C}[G]) \simeq \prod_{i=1}^{h} \mathbb{C} = \mathbb{C}^h,$$

and the coordinate maps are precisely the homomorphisms ω_i. □

Proposition 2.12. *Each algebra homomorphism*

$$\omega : Z(\mathbb{C}[G]) \to \mathbb{C}$$

is equal to one of the ω_i.

Proof. Using the preceding proposition, identify $Z(\mathbb{C}[G])$ with $\mathbb{C}^h$. The algebra homomorphisms $\mathbb{C}^h \to \mathbb{C}$ are precisely the coordinate projections. Indeed, if $e_1, \ldots, e_h$ are the standard primitive idempotents of $\mathbb{C}^h$, then

$$1 = \omega(1) = \omega(e_1) + \cdots + \omega(e_h).$$

Each $\omega(e_i)$ is an idempotent in $\mathbb{C}$, hence is either 0 or 1. Since the sum is 1, exactly one of them is 1. Thus ω is a coordinate projection, hence one of the ω_i. □

2.5 Algebraic Integers and Applications to Degrees

Proposition 2.13. *Let* χ *be the character of a representation of the finite group* G*. Then* $\chi(s)$ *is an algebraic integer for every* $s \in G$.

Proof. The number $\chi(s)$ is the sum of the eigenvalues of $\rho(s)$. Since s has finite order, every eigenvalue of $\rho(s)$ is a root of unity, hence is an algebraic integer. Therefore their sum $\chi(s)$ is an algebraic integer. □

Proposition 2.14. *Let*

$$u = \sum_{s\in G} u(s)s \in Z(\mathbb{C}[G])$$

be such that each coefficient $u(s)$ *is an algebraic integer. Then* u *is integral over* $\mathbb{Z}$.

Proof. Let $C_1, \ldots, C_h$ be the conjugacy classes of G, and put

$$e_i = \sum_{s\in C_i} s.$$

Choose $s_i \in C_i$. Since u is central, its coefficient function is constant on conjugacy classes, so

$$u = \sum_{i=1}^{h} u(s_i) e_i.$$

It suffices to show that the e_i are integral over $\mathbb{Z}$. The product $e_i e_j$ is a linear combination of the e_k with integer coefficients, because multiplication in the group algebra counts products of group elements. Hence

$$R = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_h$$

is a subring of $Z(\mathbb{C}[G])$, finitely generated as a $\mathbb{Z}$-module. Therefore each element of R, in particular each e_i, is integral over $\mathbb{Z}$. Since the coefficients $u(s_i)$ are algebraic integers, the element u is integral over $\mathbb{Z}$. □

Corollary 2.15. *Let ρ be an irreducible representation of G of degree n and character χ. If u is as in the preceding proposition, then*

$$\frac{1}{n} \sum_{s \in G} u(s) \chi(s)$$

is an algebraic integer.

Proof. The displayed number is the image of u under the algebra homomorphism

$$\omega : Z(\mathbb{C}[G]) \to \mathbb{C}$$

associated with ρ. Since u is integral over $\mathbb{Z}$, its image under ω is also integral over $\mathbb{Z}$, hence is an algebraic integer. □

Corollary 2.16. *The degree of an irreducible representation of G divides $|G|$.*

Proof. Let ρ be irreducible of degree n and character χ. Take

$$u = \sum_{s \in G} \chi(s^{-1}) s.$$

This element is central because χ is a class function, and its coefficients are algebraic integers. By the previous corollary,

$$\frac{1}{n} \sum_{s \in G} \chi(s^{-1}) \chi(s) = \frac{|G|}{n} \langle \chi, \chi \rangle = \frac{|G|}{n}$$

is an algebraic integer. But $|G|/n \in \mathbb{Q}$, and a rational algebraic integer is an integer. Hence $n \mid |G|$. □

Proposition 2.17. *Let ρ be an irreducible representation of G of degree n and character χ. Then*

$$|\chi(s)| \leq n$$

for all $s \in G$, and equality holds if and only if $\rho(s)$ is a homothety.

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\rho(s)$. Since s has finite order, the λ_i are roots of unity. Therefore

$$|\chi(s)| = |\lambda_1 + \cdots + \lambda_n| \leq n.$$

Equality in the triangle inequality occurs if and only if all λ_i have the same argument, i.e. if and only if $\lambda_1 = \cdots = \lambda_n$. Since $\rho(s)$ has finite order, it is diagonalizable, and so this condition is equivalent to $\rho(s)$ being a scalar transformation. □

Lemma 2.18. *Let $\lambda_1, \ldots, \lambda_n$ be roots of unity and put*

$$a = \frac{1}{n} \sum_{i=1}^{n} \lambda_i.$$

If a is an algebraic integer, then either $a = 0$, or

$$\lambda_1 = \cdots = \lambda_n = a.$$

Proof. Let A be the product of all Galois conjugates of a over $\mathbb{Q}$. Since a is an algebraic integer, $A \in \mathbb{Z}$. For any Galois conjugate $\sigma(a)$, we have

$$\sigma(a) = \frac{1}{n} \sum_{i=1}^{n} \sigma(\lambda_i).$$

Each $\sigma(\lambda_i)$ is again a root of unity, so $|\sigma(a)| \leq 1$. Hence $|A| \leq 1$. Thus $A = 0$ or $A = \pm 1$.

If $A = 0$, then some conjugate of a is zero, so $a = 0$. If $|A| = 1$, then every conjugate of a has absolute value 1, in particular $|a| = 1$. But a is the average of complex numbers of absolute value 1. The average has absolute value 1 only when all the λ_i are equal. Hence $\lambda_1 = \cdots = \lambda_n = a$. □

Proposition 2.19. *Let ρ be an irreducible representation of degree n and character χ. Let $s \in G$, and let $c(s)$ be the number of elements in the conjugacy class of s. Then*

$$\frac{c(s)}{n} \chi(s)$$

is an algebraic integer. In particular, if $\gcd(c(s), n) = 1$ *and $\chi(s) \neq 0$, then $\rho(s)$ is a homothety.*

Proof. Let $C(s)$ be the conjugacy class of s, and take

$$u = \sum_{t \in C(s)} t \in Z(\mathbb{C}[G]).$$

All coefficients of u are 0 or 1, so the preceding integrality corollary gives

$$\frac{1}{n} \sum_{t \in G} u(t)\chi(t) = \frac{1}{n} \sum_{t \in C(s)} \chi(t) = \frac{c(s)}{n} \chi(s)$$

as an algebraic integer.

Now assume $\gcd(c(s), n) = 1$ and $\chi(s) \neq 0$. Choose $a, b \in \mathbb{Z}$ such that

$$ac(s) + bn = 1.$$

Then
$$\frac{\chi(s)}{n} = a\frac{c(s)\chi(s)}{n} + b\chi(s).$$
Both terms on the right are algebraic integers, so $\chi(s)/n$ is an algebraic integer. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\rho(s)$, then
$$\frac{\chi(s)}{n} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i.$$
By the lemma, since $\chi(s) \neq 0$, all λ_i are equal. Therefore $\rho(s)$ is a homothety. □

Proposition 2.20. *Let $s \in G$, $s \neq 1$. Suppose that the conjugacy class of s has cardinality*
$$c(s) = p^r$$
for some prime p. Then there exists an irreducible character χ, not equal to the unit character, such that
$$\chi(s) \neq 0, \qquad \chi(1) \not\equiv 0 \pmod{p}.$$
If ρ is a representation with character χ, then $\rho(s)$ is a homothety.

Proof. Let $\chi_1 = 1, \chi_2, \ldots, \chi_h$ be the irreducible characters of G, with $n_i = \chi_i(1)$. Since $s \neq 1$, the regular character gives
$$0 = \sum_{i=1}^{h} n_i\chi_i(s) = 1 + \sum_{i\neq 1} n_i\chi_i(s).$$
Suppose that no irreducible character with the desired properties exists. Then for every $i \neq 1$, either $\chi_i(s) = 0$, or $p \mid n_i$. Hence
$$-1 = \sum_{i\neq 1} n_i\chi_i(s) = p\alpha$$
for some algebraic integer α. Thus $1/p = -\alpha$ would be an algebraic integer, which is impossible because the only rational algebraic integers are the ordinary integers. Therefore such a character χ exists.

For this character, $n = \chi(1)$ is prime to $c(s) = p^r$. Since $\chi(s) \neq 0$, the previous proposition implies that $\rho(s)$ is a homothety. □

2.6 Induced Modules and Frobenius Reciprocity

Let $H \leq G$, and let R be a system of left coset representatives for H in G. Recall that a representation V of G is induced by a representation W of H if
$$V = \bigoplus_{r\in R} rW.$$

This property can be formulated by means of group algebras. Put
$$W' = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$
The inclusion $W \to V$ extends to a $\mathbb{C}[G]$-homomorphism
$$i : W' \to V, \qquad i(g \otimes w) = gw.$$

Proposition 2.21. *The representation V is induced by W if and only if*

$$i : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to V$$

is an isomorphism.

Proof. The elements of R form a basis of $\mathbb{C}[G]$ as a right $\mathbb{C}[H]$-module, since every $g \in G$ can be uniquely written as $g = rh$, with $r \in R$ and $h \in H$. Therefore

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \simeq \left(\bigoplus_{r \in R} r\mathbb{C}[H] \right) \otimes_{\mathbb{C}[H]} W \simeq \bigoplus_{r \in R} rW.$$

Under this identification, the map i is exactly the natural map

$$\bigoplus_{r \in R} rW \to V.$$

Thus it is an isomorphism precisely when V is induced by W. □

It follows from this perspective that the induced representation exists and is unique up to isomorphism. We denote it by

$$\mathrm{Ind}_H^G(W).$$

Proposition 2.22. *The following properties hold.*

1. *If $V = \mathrm{Ind}_H^G(W)$ and E is a $\mathbb{C}[G]$-module, then*

$$\mathrm{Hom}_{\mathbb{C}[H]}(W, E) \simeq \mathrm{Hom}_{\mathbb{C}[G]}(V, E).$$

   Here E is regarded as a $\mathbb{C}[H]$-module by restriction.

2. *If $H \le G \le K$, then*

$$\mathrm{Ind}_G^K(\mathrm{Ind}_H^G W) \simeq \mathrm{Ind}_H^K(W).$$

Proof. For (1), using $V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$, a $\mathbb{C}[G]$-homomorphism $V \to E$ is uniquely determined by its restriction to $1 \otimes W$, and this restriction is $\mathbb{C}[H]$-linear. Conversely, any $\mathbb{C}[H]$-homomorphism $W \to E$ extends uniquely by

$$g \otimes w \longmapsto gf(w).$$

This gives the desired isomorphism.

For (2), compute

$$\mathbb{C}[K] \otimes_{\mathbb{C}[G]} (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W) \simeq \mathbb{C}[K] \otimes_{\mathbb{C}[H]} W.$$

This is exactly the transitivity of induction. □

Proposition 2.23. *Let V be a $\mathbb{C}[G]$-module which is a direct sum*

$$V = \bigoplus_{i \in I} W_i$$

of vector spaces permuted transitively by G. Let $i_0 \in I$, put $W = W_{i_0}$, and let H be the stabilizer of W in G. Then W is H-invariant and V is induced by W:

$$V \simeq \mathrm{Ind}_H^G(W).$$

Proof. Since H stabilizes W, the subspace W is H-invariant. By the transitivity of the action of G on the summands, the translates gW are exactly the summands W_i. If R is a system of representatives of G/H, then these translates are precisely the rW, $r \in R$, and the direct sum decomposition becomes

$$V = \bigoplus_{r \in R} rW.$$

Hence $V \simeq \mathrm{Ind}_H^G(W)$. □

Let f be a class function on H. Define a function f' on G by

$$f'(s) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}st \in H}} f(t^{-1}st).$$

We say that f' is induced by f, and we denote it by

$$\mathrm{Ind}_H^G(f).$$

Proposition 2.24. *Let f be a class function on H. Then:*

1. *$\mathrm{Ind}_H^G(f)$ is a class function on G.*
2. *If f is the character of a representation W of H, then $\mathrm{Ind}_H^G(f)$ is the character of $\mathrm{Ind}_H^G(W)$.*

Proof. The first assertion follows directly from the formula: replacing s by a conjugate only changes the summation variable. The second assertion is the character formula for the induced representation. Indeed, in the decomposition

$$\mathrm{Ind}_H^G(W) = \bigoplus_{r \in R} rW,$$

the element $s \in G$ permutes the summands. The trace receives contributions precisely from those summands rW such that $r^{-1}sr \in H$, and on such a summand the trace is $f(r^{-1}sr)$. This gives exactly the displayed formula. □

For two class functions φ_1, φ_2 on G, put

$$\langle \varphi_1, \varphi_2 \rangle_G = \frac{1}{|G|} \sum_{s \in G} \varphi_1(s)\overline{\varphi_2(s)}.$$

If V_1, V_2 are $\mathbb{C}[G]$-modules, put

$$\langle V_1, V_2 \rangle_G = \dim \mathrm{Hom}_{\mathbb{C}[G]}(V_1, V_2).$$

For characters these two definitions agree.

Lemma 2.25. *If ψ_1, ψ_2 are characters of V_1, V_2, then*

$$\langle \psi_1, \psi_2 \rangle_G = \langle V_1, V_2 \rangle_G.$$

Proof. By complete reducibility, we may reduce to the case where V_1, V_2 are irreducible. Then the equality follows from Schur's lemma and the orthogonality relations of irreducible characters. □

Theorem 2.26 (Frobenius reciprocity). *Let ψ be a class function on H and let φ be a class function on G. Then*

$$\langle \psi, \operatorname{Res}_H^G \varphi \rangle_H = \langle \operatorname{Ind}_H^G \psi, \varphi \rangle_G.$$

Proof. Since every class function is a linear combination of irreducible characters, we may assume that ψ is the character of a $\mathbb{C}[H]$-module W, and that φ is the character of a $\mathbb{C}[G]$-module E. It is then enough to prove

$$\langle W, \operatorname{Res}_H^G E \rangle_H = \langle \operatorname{Ind}_H^G W, E \rangle_G.$$

But this is precisely the isomorphism

$$\operatorname{Hom}_{\mathbb{C}[H]}(W, E) \simeq \operatorname{Hom}_{\mathbb{C}[G]}(\operatorname{Ind}_H^G W, E)$$

from the adjunction property of induction. □

Remark 2.27. In fact, one does not need H to be a subgroup of G. It is enough to have a group homomorphism $\alpha : H \to G$, which gives $\mathbb{C}[G]$ the structure of a $\mathbb{C}[H]$-algebra. In this case one may define

$$\operatorname{Ind}_\alpha E = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} E$$

for a $\mathbb{C}[H]$-module E, and if φ is the character of a $\mathbb{C}[G]$-module, one defines

$$\operatorname{Res}_\alpha \varphi = \varphi \circ \alpha.$$

Then the same reciprocity formula holds:

$$\langle \psi, \operatorname{Res}_\alpha \varphi \rangle_H = \langle \operatorname{Ind}_\alpha \psi, \varphi \rangle_G.$$

2.7 Mackey Decomposition and Irreducibility Criterion

Let H and K be two subgroups of G, let $\rho : H \to \mathrm{GL}(W)$ be a representation of H, and put

$$V = \operatorname{Ind}_H^G(W).$$

We want to determine the restriction $\operatorname{Res}_K^G V$.

Choose a set S of representatives for the double cosets

$$K \backslash G / H,$$

so that

$$G = \bigsqcup_{s \in S} KsH.$$

For $s \in S$, put

$$H_s = sHs^{-1} \cap K.$$

This is a subgroup of K. Define a representation ρ^s of H_s on the same underlying vector space W by

$$\rho^s(x) = \rho(s^{-1}xs), \qquad x \in H_s.$$

We denote this H_s-module by W_s.

Theorem 2.28 (Mackey decomposition)**.** *There is an isomorphism of* $\mathbb{C}[K]$*-modules*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G(W) \simeq \bigoplus_{s\in S} \operatorname{Ind}_{H_s}^K(W_s).$$

Proof. We know that $V = \operatorname{Ind}_H^G(W)$ is a direct sum of the subspaces xW, indexed by left cosets $xH \in G/H$. Fix $s \in S$, and let V_s be the subspace generated by the xW with $xH \subset KsH$. Since the double cosets KsH are disjoint, we have a direct sum

$$V = \bigoplus_{s\in S} V_s.$$

Each V_s is stable under K, because left multiplication by K does not leave the double coset KsH.

It remains to identify V_s. The subgroup of K consisting of the elements x such that

$$x(sW) = sW$$

is exactly

$$H_s = sHs^{-1} \cap K.$$

Indeed, $x(sW) = sW$ if and only if $s^{-1}xs \in H$. Thus V_s is the direct sum of the translates $x(sW)$, with x running through representatives of K/H_s. By the transitive-summand criterion for induction,

$$V_s \simeq \operatorname{Ind}_{H_s}^K(sW).$$

Finally, the map

$$W_s \to sW, \qquad w \mapsto sw,$$

is H_s-linear, since for $x \in H_s$,

$$x(sw) = s(s^{-1}xs)w.$$

Hence $sW \simeq W_s$ as H_s-modules, and therefore

$$V_s \simeq \operatorname{Ind}_{H_s}^K(W_s).$$

Summing over $s \in S$ gives the result. □

Remark 2.29. If $t \in KsH$, say $t = ksh$ with $k \in K$ and $h \in H$, then

$$H_t = tHt^{-1} \cap K = kH_sk^{-1}.$$

Thus the induced representation obtained from a double coset is independent, up to isomorphism, of the representative chosen.

We now focus on the case $K = H$. For $s \in G$, write

$$H_s = sHs^{-1} \cap H.$$

Theorem 2.30 (Mackey's irreducibility criterion)**.** *Let* $\rho : H \to \mathrm{GL}(W)$ *be a representation of* H*. Then* $\operatorname{Ind}_H^G(W)$ *is irreducible if and only if:*

1. W *is irreducible;*

2. *for each representative $s \notin H$ of a nontrivial double coset in $H\backslash G/H$, the two H_s-representations*

$$\operatorname{Res}^H_{H_s}(W) \qquad \text{and} \qquad W_s$$

are disjoint, i.e.

$$\left\langle \operatorname{Res}^H_{H_s}(W), W_s \right\rangle_{H_s} = 0.$$

Proof. Let

$$V = \operatorname{Ind}^G_H(W).$$

Then V is irreducible if and only if

$$\langle V, V\rangle_G = 1.$$

By Frobenius reciprocity,

$$\langle V, V\rangle_G = \langle W, \operatorname{Res}^G_H V\rangle_H.$$

Applying Mackey decomposition with $K = H$, we obtain

$$\operatorname{Res}^G_H V \simeq \bigoplus_{s\in H\backslash G/H} \operatorname{Ind}^H_{H_s}(W_s).$$

Thus, again using Frobenius reciprocity,

$$\langle V, V\rangle_G = \sum_{s\in H\backslash G/H} \left\langle \operatorname{Res}^H_{H_s}(W), W_s \right\rangle_{H_s}.$$

For the double coset represented by $s = 1$, we have $H_s = H$ and $W_s = W$, so the corresponding term is

$$\langle W, W\rangle_H.$$

Therefore the whole sum equals 1 if and only if W is irreducible and all the remaining nonnegative integer terms vanish. This is exactly the stated criterion. ☐

Corollary 2.31. *If moreover H is normal in G, then $\operatorname{Ind}^G_H(\rho)$ is irreducible if and only if ρ is irreducible and ρ is not isomorphic to any of its conjugates ρ^s for $s \notin H$.*

Proof. If $H \triangleleft G$, then for every $s \in G$,

$$H_s = sHs^{-1} \cap H = H.$$

Thus the Mackey criterion says that $\operatorname{Ind}^G_H(\rho)$ is irreducible if and only if ρ is irreducible and, for each $s \notin H$, the two H-representations ρ and ρ^s are disjoint. Since both are irreducible, this is equivalent to saying that they are not isomorphic. ☐

Corollary 2.32. *Let k be a finite field, let $G = \mathrm{SL}_2(k)$, and let H be the subgroup of matrices*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c = 0$. Let $\omega : k^\times \to \mathbb{C}^\times$ be a homomorphism, and define a degree-one character of H by

$$\chi_\omega \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \omega(a).$$

Then

$$\operatorname{Ind}^G_H(\chi_\omega)$$

is irreducible if $\omega^2 \neq 1$.

Proof. We use Mackey's criterion. By the Bruhat decomposition for $\mathrm{SL}_2(k)$,

$$G = H \sqcup HwH, \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus only the nontrivial double coset represented by w needs to be checked. We have

$$H_w = H \cap wHw^{-1} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in k^\times \right\}.$$

On this subgroup,

$$\chi_\omega \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \omega(a).$$

The conjugate character is

$$\chi_\omega^w(x) = \chi_\omega(w^{-1}xw).$$

For

$$x = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

we have

$$w^{-1}xw = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

so

$$\chi_\omega^w(x) = \omega(a^{-1}) = \omega(a)^{-1}.$$

Thus $\chi_\omega|_{H_w}$ and χ_ω^w are disjoint precisely when $\omega \neq \omega^{-1}$, i.e. when $\omega^2 \neq 1$. Since χ_ω is one-dimensional, it is irreducible. Mackey's criterion gives the claim. □

2.8 Virtual Characters and the Representation Ring

Let $\chi_1, \dots, \chi_h$ be the distinct irreducible characters of G. A class function on G is a character if and only if it can be written as a linear combination of the χ_i's with nonnegative integer coefficients. We denote the set of such characters by $R^+(G)$, and the subgroup generated by $R^+(G)$ by $R(G)$. Thus

$$R(G) = \mathbb{Z}\chi_1 \oplus \cdots \oplus \mathbb{Z}\chi_h.$$

An element of $R(G)$ is called a *virtual character*. Since the product of two characters is a character, $R(G)$ is a subring of the ring $F_c(G)$ of all class functions on G. Since $\{\chi_i\}$ is a basis of $F_c(G)$, we have

$$\mathbb{C} \otimes_{\mathbb{Z}} R(G) \simeq F_c(G).$$

Definition 2.33. Let $(M, +)$ be a unitary commutative monoid containing zero. A *Grothendieck group* of M is an abelian group $K(M)$ together with a monoid homomorphism

$$M \to K(M)$$

satisfying the following universal property: for any abelian group A and any monoid homomorphism $f : M \to A$, there exists a unique group homomorphism

$$\widetilde{f} : K(M) \to A$$

such that the diagram

$$\begin{array}{ccc} M & \longrightarrow & K(M) \\ & \searrow^{f} & \big\downarrow \exists!\, \widetilde{f} \\ & & A \end{array}$$

commutes.

Therefore $R(G)$ can also be viewed as the Grothendieck group of the category of finitely generated $\mathbb{C}[G]$-modules.

If $H \leq G$, restriction defines a homomorphism

$$\mathrm{Res}_H^G : R(G) \to R(H).$$

Similarly, induction defines a homomorphism

$$\mathrm{Ind}_H^G : R(H) \to R(G).$$

2.9 Artin Induction Theorem

Let $R(G)$ denote the abelian group of virtual characters of G. If $H \leq G$, we write Ind_H^G and Res_H^G for induction and restriction of virtual characters.

Theorem 2.34 (Artin induction theorem)**.** *Every virtual character of G is a rational linear combination of characters induced from one-dimensional characters of cyclic subgroups. Equivalently,*

$$R(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is generated over $\mathbb{Q}$ by characters of the form

$$\mathrm{Ind}_C^G \lambda,$$

where $C \leq G$ is cyclic and λ is a one-dimensional character of C.

Proof. Let M be the subgroup of $R(G)$ generated by all characters

$$\mathrm{Ind}_C^G \lambda,$$

where C runs through the cyclic subgroups of G, and λ runs through the irreducible characters of C. Since C is cyclic, all irreducible characters of C are one-dimensional.

We first show that M has full rank in $R(G)$. It is enough to prove that the complex span of M is the whole space $F_c(G)$ of class functions on G. Suppose that $f \in F_c(G)$ is orthogonal to every generator $\mathrm{Ind}_C^G \lambda$ of M. Then by Frobenius reciprocity,

$$0 = \langle f, \mathrm{Ind}_C^G \lambda \rangle_G = \langle \mathrm{Res}_C^G f, \lambda \rangle_C$$

for every cyclic subgroup $C \leq G$ and every irreducible character λ of C. Since irreducible characters form an orthonormal basis of the class functions on C, it follows that

$$\mathrm{Res}_C^G f = 0$$

for every cyclic subgroup $C \leq G$. For any $g \in G$, taking $C = \langle g \rangle$ gives $f(g) = 0$. Thus $f = 0$. Hence the orthogonal complement of the complex span of M is zero, and the complex span of M is all of $F_c(G)$.

Therefore M has the same rank as $R(G)$. Since $R(G)$ is a free abelian group of finite rank, $R(G)/M$ is finite. Hence for every $\chi \in R(G)$, there exists a positive integer N such that

$$N\chi \in M.$$

Thus $\chi \in M \otimes_{\mathbb{Z}} \mathbb{Q}$, which is exactly the assertion. □

Corollary 2.35 (Finite-index form)**.** *Let $M \subseteq R(G)$ be the subgroup generated by the characters*

$$\operatorname{Ind}_C^G \lambda,$$

where $C \leq G$ is cyclic and λ is a one-dimensional character of C. Then M has finite index in $R(G)$.

Proof. In the proof of Artin induction, we showed that M has the same rank as $R(G)$. Since $R(G)$ is a free abelian group of finite rank, a subgroup of the same rank has finite index. □

Corollary 2.36 (Detection by cyclic subgroups)**.** *Let $\chi \in R(G)$ be a virtual character. If*

$$\operatorname{Res}_C^G \chi = 0$$

for every cyclic subgroup $C \leq G$, then $\chi = 0$.

Proof. For every $g \in G$, take $C = \langle g \rangle$. Since $\operatorname{Res}_C^G \chi = 0$, evaluating at g gives $\chi(g) = 0$. Thus χ vanishes on every element of G, so $\chi = 0$. □

Corollary 2.37 (Irreducibles occur in cyclic inductions)**.** *Let χ be an irreducible character of G. Then there exists a cyclic subgroup $C \leq G$ and a one-dimensional character λ of C such that χ is a constituent of $\operatorname{Ind}_C^G \lambda$. Equivalently,*

$$\langle \chi, \operatorname{Ind}_C^G \lambda \rangle_G \neq 0$$

for some (C, λ).

Proof. By Artin induction, we may write

$$\chi = \sum_{C,\lambda} a_{C,\lambda} \operatorname{Ind}_C^G \lambda, \qquad a_{C,\lambda} \in \mathbb{Q}.$$

Taking inner product with χ, we get

$$1 = \langle \chi, \chi \rangle_G = \sum_{C,\lambda} a_{C,\lambda} \langle \operatorname{Ind}_C^G \lambda, \chi \rangle_G.$$

Therefore at least one term has

$$\langle \operatorname{Ind}_C^G \lambda, \chi \rangle_G \neq 0.$$

Since this inner product is the multiplicity of χ in $\operatorname{Ind}_C^G \lambda$, the claim follows. □

Corollary 2.38 (A vanishing criterion for class functions)**.** *Let f be a class function on G. If*

$$\langle f, \operatorname{Ind}_C^G \lambda \rangle_G = 0$$

for every cyclic subgroup $C \leq G$ and every one-dimensional character λ of C, then $f = 0$.

Proof. By Frobenius reciprocity,

$$\langle f, \operatorname{Ind}_C^G \lambda \rangle_G = \langle \operatorname{Res}_C^G f, \lambda \rangle_C.$$

Thus $\operatorname{Res}_C^G f$ is orthogonal to every irreducible character of C. Since C is cyclic, its irreducible characters form an orthonormal basis for its class functions. Hence $\operatorname{Res}_C^G f = 0$. Taking $C = \langle g \rangle$ for each $g \in G$, we get $f(g) = 0$ for all g, so $f = 0$. □