Calculus

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Preface

Calculus began with the systematic study of differentiation and integration in Euclidean space. From Newton and Leibniz onward, these two operations were understood to be closely related through the fundamental theorem of calculus. The development of geometry in the nineteenth and twentieth centuries showed that the same ideas extend far beyond ordinary coordinates. On a smooth manifold, derivatives are described by tangent and cotangent spaces, differential forms replace ordinary integrands, and Stokes' theorem provides the natural general form of the fundamental theorem of calculus.

A new phenomenon appears in this setting. A closed differential form is locally the differential of another form, but it need not admit a global primitive. De Rham cohomology measures precisely this failure and thereby connects local differential calculus with the global topology of a manifold. The passage from differentiation to integration, and from local formulas to global invariants, is the central theme of these notes.

For this reason, the notes are titled simply *Calculus*. Their purpose is not to repeat elementary calculus, but to study what its basic constructions become on smooth manifolds. The exposition develops smooth maps, tangent and cotangent spaces, vector bundles, differential forms, de Rham cohomology, integration, Stokes' theorem, and the Mayer–Vietoris sequence, with emphasis on intrinsic definitions and the relation between local and global geometry.

Prerequisites. The reader is assumed to be familiar with linear algebra, multivariable calculus, and elementary point-set topology. Some experience with abstract algebra is helpful but not essential for the first several chapters. The discussion of locally free sheaves, fine sheaves, and the de Rham resolution also uses basic sheaf theory and homological algebra, including cochain complexes, long exact sequences, derived functors, and acyclic resolutions. These topics are recalled when they enter the argument, but are not developed from the beginning.

Chapter 1

Smooth Manifolds and Smooth Functions

A smooth manifold is a topological space which is locally modeled on Euclidean space, together with a compatible notion of smoothness on overlapping local models. Coordinates are indispensable for defining and verifying this local structure, but the resulting notion of smoothness is intrinsic: it does not depend on the particular coordinates used.

This chapter develops three complementary descriptions of a smooth manifold. The first is the classical language of charts and atlases. The second is the local-to-global language of smooth functions and gluing. The third is the sheaf-theoretic language of the structure sheaf $\mathscr{C}_M^\infty$ and its local rings. The latter will provide the algebraic foundation for tangent and cotangent spaces in the next chapter.

Unless otherwise specified, all manifolds are finite-dimensional, Hausdorff, second countable, without boundary, and all smooth functions and maps are infinitely differentiable.

1.1 Topological Manifolds and Charts

Definition 1.1. Let $n \geq 0$. A *topological manifold of dimension* n is a Hausdorff, second-countable topological space M such that every point $p \in M$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Definition 1.2. A *chart* on a topological n-manifold M is a pair (U, φ), where $U \subseteq M$ is open and

$$\varphi : U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$$

is a homeomorphism onto an open subset of $\mathbb{R}^n$.

The open set U is called a *coordinate neighborhood*, and φ is called a *coordinate map.* Writing

$$\varphi = (x^1, \ldots, x^n),$$

the component functions

$$x^i : U \longrightarrow \mathbb{R}$$

are called *local coordinates* on U.

A chart (U, φ) is said to be *centered at* $p \in U$ if $\varphi(p) = 0$.

Remark 1.3. The Hausdorff condition ensures that distinct points can be separated, while second countability excludes excessively large locally Euclidean spaces and implies several global countability properties used later. In particular, second-countable manifolds are paracompact.

The dimension in the preceding definition is intrinsic by the following topological theorem.

Theorem 1.4 (Invariance of dimension). *Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be nonempty open subsets. If U and V are homeomorphic, then $m = n$.*

We take Theorem 1.4 as a standard topological input. It implies that a nonempty connected topological manifold cannot be locally Euclidean of two different dimensions.

Proposition 1.5 (Restriction of charts). *Let (U, φ) be a chart on M, and let $V \subseteq U$ be open. Then*

$$(V, \varphi|_V)$$

is a chart on M.

Proof. Since φ is a homeomorphism and V is open in U, the image $\varphi(V)$ is open in $\varphi(U)$ and hence open in $\mathbb{R}^n$. The restriction

$$\varphi|_V : V \longrightarrow \varphi(V)$$

is again a homeomorphism. □

Corollary 1.6 (Coordinate neighborhoods form a basis). *Let M be a topological manifold, let $W \subseteq M$ be open, and let $p \in W$. Then there exists a chart (U, φ) such that*

$$p \in U \subseteq W.$$

Proof. Choose any chart (V, ψ) about p. Then $U = V \cap W$ is an open neighborhood of p, and Proposition 1.5 shows that

$$(U, \psi|_U)$$

is a chart. □

Example 1.7 (Euclidean spaces). The Euclidean space $\mathbb{R}^n$ is a topological n-manifold with the single chart

$$(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n}).$$

Every open subset $U \subseteq \mathbb{R}^n$ is a topological n-manifold with the chart (U, id_U).

Example 1.8 (Zero-dimensional manifolds). A zero-dimensional manifold is a countable discrete space. Indeed, every point has an open neighborhood homeomorphic to the one-point space $\mathbb{R}^0$, so every singleton is open. Second countability then implies that the set of points is countable.

1.2 Smooth Atlases and Smooth Structures

Let (U, φ) and (V, ψ) be two charts on a topological n-manifold M. Since $U \cap V$ is open in both U and V, the subsets

$$\varphi(U \cap V) \quad \text{and} \quad \psi(U \cap V)$$

are open in $\mathbb{R}^n$.

Definition 1.9. Two charts (U, φ) and (V, ψ) are *smoothly compatible* if $U \cap V = \varnothing$, or if both transition maps

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

and

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$$

are smooth maps between open subsets of $\mathbb{R}^n$.

Since the two transition maps are inverses, smooth compatibility means that either transition map is a diffeomorphism between Euclidean open subsets.

Definition 1.10. A *smooth atlas* on a topological manifold M is a family

$$\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$$

of pairwise smoothly compatible charts such that

$$M = \bigcup_{\alpha \in A} U_\alpha.$$

A chart (V, ψ) is said to be *compatible with* $\mathcal{A}$ if it is smoothly compatible with every chart in $\mathcal{A}$.

Smooth compatibility of charts is reflexive and symmetric, but it is not itself a transitive relation. Nevertheless, compatibility with a fixed atlas does satisfy the following crucial property.

Lemma 1.11 (Compatibility lemma)**.** *Let $\mathcal{A}$ be a smooth atlas on M. If two charts (U, φ) and (V, ψ) are both compatible with $\mathcal{A}$, then they are compatible with each other.*

Proof. Fix $p \in U \cap V$. Since $\mathcal{A}$ covers M, there is a chart $(W, \theta) \in \mathcal{A}$ such that $p \in W$. On the portion corresponding to $U \cap V \cap W$, the transition map factors as

$$\psi \circ \varphi^{-1} = \left(\psi \circ \theta^{-1}\right) \circ \left(\theta \circ \varphi^{-1}\right).$$

Both factors are smooth because (U, φ) and (V, ψ) are compatible with (W, θ). Hence $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of $\varphi(p)$.

As p was arbitrary, the transition map is smooth on all of $\varphi(U \cap V)$. Interchanging the two charts proves that its inverse is also smooth. □

Definition 1.12. A smooth atlas $\mathcal{A}$ is *maximal* if it is not properly contained in another smooth atlas.

Theorem 1.13 (Maximal atlas)**.** *Every smooth atlas $\mathcal{A}$ is contained in a unique maximal smooth atlas.*

Proof. Let $\overline{\mathcal{A}}$ be the collection of all charts compatible with every chart in $\mathcal{A}$. By Lemma 1.11, any two members of $\overline{\mathcal{A}}$ are compatible. Since $\mathcal{A} \subseteq \overline{\mathcal{A}}$, their domains cover M. Therefore $\overline{\mathcal{A}}$ is a smooth atlas.

Suppose that $\mathcal{B}$ is a smooth atlas containing $\mathcal{A}$. Every chart in $\mathcal{B}$ is compatible with every chart in $\mathcal{A}$, so

$$\mathcal{B} \subseteq \overline{\mathcal{A}}.$$

It follows that $\overline{\mathcal{A}}$ is maximal.

If $\mathcal{M}$ is any maximal atlas containing $\mathcal{A}$, then the same argument gives

$$\mathcal{M} \subseteq \overline{\mathcal{A}}.$$

Since $\mathcal{M}$ is maximal and $\overline{\mathcal{A}}$ is an atlas, the two atlases are equal. Thus the maximal atlas is unique. □

Definition 1.14. A *smooth structure* on a topological manifold is a maximal smooth atlas. A *smooth manifold* is a topological manifold equipped with a smooth structure.

In practice, one specifies a smooth manifold by giving a convenient atlas, rather than listing every chart in its maximal atlas.

Proposition 1.15 (Equivalent atlases)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be smooth atlases on the same topological manifold. The following conditions are equivalent:*

(1) $\mathcal{A}$ and $\mathcal{B}$ determine the same maximal atlas;

(2) every chart in $\mathcal{A}$ is compatible with every chart in $\mathcal{B}$;

(3) $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas.

Proof. If the two atlases determine the same maximal atlas, then all their charts belong to that maximal atlas and are therefore pairwise compatible. This proves (1) $\Rightarrow$ (2).

Condition (2) implies that every pair of charts in $\mathcal{A} \cup \mathcal{B}$ is compatible. Since the domains cover M, their union is an atlas. Thus (2) $\Rightarrow$ (3).

Finally, if $\mathcal{A} \cup \mathcal{B}$ is an atlas, its maximal extension contains both $\mathcal{A}$ and $\mathcal{B}$. By uniqueness in Theorem 1.13, the maximal extensions of $\mathcal{A}$ and $\mathcal{B}$ coincide. Hence (3) $\Rightarrow$ (1). □

1.3 Smooth Maps

Definition 1.16. Let M and N be smooth manifolds, and let $f : M \longrightarrow N$ be a map. We say that f is *smooth at* $p \in M$ if there exist charts (U, φ) about p and (V, ψ) about $f(p)$ such that $f(U) \subseteq V$ and

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$$

is smooth.

The map f is *smooth* if it is smooth at every point of M.

No continuity is assumed in this definition. It will follow from smoothness.

Lemma 1.17 (Independence of charts)**.** *Suppose that $f : M \longrightarrow N$ is smooth at p with respect to one pair of charts. Then its coordinate representative is smooth with respect to every sufficiently restricted pair of charts about p and $f(p)$.*

Proof. Choose charts (U, φ) about p and (V, ψ) about $f(p)$ such that $f(U) \subseteq V$ and

$$\psi \circ f \circ \varphi^{-1}$$

is smooth.

Let $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ be any other charts about p and $f(p)$. The original coordinate representative is continuous. Hence, after shrinking the source neighborhood about p, we may assume that it is contained in $U \cap \widetilde{U}$ and is mapped by f into $V \cap \widetilde{V}$.

On this restricted domain,

$$\widetilde{\psi} \circ f \circ \widetilde{\varphi}^{-1} = \left(\widetilde{\psi} \circ \psi^{-1}\right) \circ \left(\psi \circ f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \widetilde{\varphi}^{-1}\right).$$

The first and third factors are smooth transition maps, and the middle factor is smooth by hypothesis. The Euclidean chain rule therefore proves the result. □

Proposition 1.18 (Smooth maps are continuous)**.** *Every smooth map between smooth manifolds is continuous.*

Proof. Let $f : M \longrightarrow N$ be smooth and fix $p \in M$. Choose charts (U, φ) and (V, ψ) as in the definition of smoothness. On U, the map f can be written as

$$f|_U = \psi^{-1} \circ \left(\psi \circ f \circ \varphi^{-1}\right) \circ \varphi.$$

All three maps on the right are continuous. Thus f is continuous in a neighborhood of every point, and hence continuous on M. □

Proposition 1.19 (Locality of smoothness). *Let $f : M \longrightarrow N$ be a map. Then f is smooth if and only if there exists an open cover $\{U_\alpha\}$ of M such that every restriction*

$$f|_{U_\alpha} : U_\alpha \longrightarrow N$$

is smooth.

Proof. If f is smooth, then every restriction to an open subset is smooth.

Conversely, let $p \in M$ and choose α such that $p \in U_\alpha$. Since $f|_{U_\alpha}$ is smooth at p, the required local coordinate representation exists. Therefore f is smooth at every point of M. □

Proposition 1.20 (Composition). *The composite of two smooth maps is smooth.*

Proof. Let

$$M \xrightarrow{f} N \xrightarrow{g} P$$

be smooth maps, and let $p \in M$. Choose charts

$$(U, \varphi), \qquad (V, \psi), \qquad (W, \theta)$$

about p, $f(p)$, and $g(f(p))$, respectively, and shrink them so that

$$f(U) \subseteq V, \qquad g(V) \subseteq W.$$

Then

$$\theta \circ (g \circ f) \circ \varphi^{-1} = \left(\theta \circ g \circ \psi^{-1}\right) \circ \left(\psi \circ f \circ \varphi^{-1}\right).$$

Both factors are smooth maps between Euclidean open subsets, so their composite is smooth. □

Remark 1.21. Smooth manifolds and smooth maps form a category, denoted **Man**. Identity maps are smooth, and Proposition 1.20 gives closure under composition.

Definition 1.22. A bijective smooth map $f : M \longrightarrow N$ is a *diffeomorphism* if its inverse is smooth. Two manifolds are *diffeomorphic* if there exists a diffeomorphism between them.

Proposition 1.23 (Coordinate maps are diffeomorphisms). *If (U, φ) is a chart on a smooth n-manifold M, then*

$$\varphi : U \longrightarrow \varphi(U)$$

is a diffeomorphism, where U has the induced smooth structure and $\varphi(U)$ has its standard Euclidean smooth structure.

Proof. Using the chart (U, φ) on U and the identity chart on $\varphi(U)$, the coordinate representative of φ is

$$\mathrm{id}_{\varphi(U)}.$$

The same is true for φ^{-1}. Hence both maps are smooth. □

Proposition 1.24 (Component criterion). *A map*

$$f = (f^1, \dots, f^r) : M \longrightarrow \mathbb{R}^r$$

is smooth if and only if each component

$$f^i : M \longrightarrow \mathbb{R}$$

is smooth.

Proof. In a chart (U, φ) on M, the coordinate representative of f is

$$f \circ \varphi^{-1} = \left(f^1 \circ \varphi^{-1}, \ldots, f^r \circ \varphi^{-1}\right).$$

A map between Euclidean open subsets is smooth if and only if each of its components is smooth. □

Proposition 1.25 (Smoothness in target coordinates). *Let $f : M \longrightarrow N$ be a continuous map. The following are equivalent:*

(1) f is smooth;

(2) for every chart

$$(V, y^1, \ldots, y^n)$$

on N, each function

$$y^i \circ f : f^{-1}(V) \longrightarrow \mathbb{R}$$

is smooth;

(3) there exists an atlas on N for which the condition in (2) *holds.*

Proof. Suppose first that f is smooth. Each coordinate function $y^i : V \longrightarrow \mathbb{R}$ is smooth by Proposition 1.23 and Proposition 1.24. Therefore

$$y^i \circ f$$

is smooth by Proposition 1.20. This proves (1) ⇒ (2), and (2) ⇒ (3) is immediate.

Assume (3). Let $p \in M$, and choose a target chart

$$(V, y^1, \ldots, y^n)$$

from the specified atlas such that $f(p) \in V$. By continuity, $f^{-1}(V)$ is open. The map

$$y \circ f = \left(y^1 \circ f, \ldots, y^n \circ f\right) : f^{-1}(V) \longrightarrow \mathbb{R}^n$$

is smooth by the component criterion. Since y is a target coordinate map, this is precisely the local coordinate condition for f to be smooth near p. Thus f is smooth. □

1.4 Elementary Constructions and Gluing

1.4.1 Open submanifolds

Proposition 1.26 (Open submanifolds). *Let M be a smooth n-manifold and let $U \subseteq M$ be open. The restrictions to U of the charts of M determine a natural smooth n-manifold structure on U. The inclusion*

$$j : U \longrightarrow M$$

is smooth.

Proof. If (V, φ) is a chart on M, then

$$(U \cap V, \varphi|_{U \cap V})$$

is a chart on U. The restricted charts cover U, and their transition maps are restrictions of the transition maps on M, hence are smooth.

In such coordinates, the inclusion is represented by the identity map on an open subset of $\mathbb{R}^n$, so it is smooth. □

We always equip an open subset of a smooth manifold with this induced smooth structure.

1.4.2 Products

Theorem 1.27 (Product manifold)**.** *Let M and N be smooth manifolds of dimensions m and n. Then $M \times N$ has a natural smooth structure of dimension $m+n$. If (U,φ) is a chart on M and (V,ψ) is a chart on N, the corresponding product chart is*

$$(U \times V, \varphi \times \psi),$$

where

$$(\varphi \times \psi)(p,q) = (\varphi(p), \psi(q)).$$

Proof. The product of two Hausdorff spaces is Hausdorff, and the product of two second-countable spaces is second countable. Moreover,

$$(\varphi \times \psi)(U \times V) = \varphi(U) \times \psi(V),$$

which is open in

$$\mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}.$$

If $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ are another pair of charts, the transition map between product charts is

$$(\widetilde{\varphi} \times \widetilde{\psi}) \circ (\varphi \times \psi)^{-1} = \left(\widetilde{\varphi} \circ \varphi^{-1}, \widetilde{\psi} \circ \psi^{-1}\right).$$

Its components are smooth, so it is smooth by the Euclidean component criterion. The inverse is treated in the same way. □

Proposition 1.28. *The projections*

$$\mathrm{pr}_M : M \times N \longrightarrow M, \qquad \mathrm{pr}_N : M \times N \longrightarrow N$$

are smooth.

Proof. In product coordinates, the projections are represented by the standard linear projections

$$\mathbb{R}^{m+n} \longrightarrow \mathbb{R}^m$$

and

$$\mathbb{R}^{m+n} \longrightarrow \mathbb{R}^n.$$

□

Theorem 1.29 (Universal property of products)**.** *Let P, M, and N be smooth manifolds. A map*

$$f : P \longrightarrow M \times N$$

is smooth if and only if both component maps

$$\mathrm{pr}_M \circ f : P \longrightarrow M$$

and

$$\mathrm{pr}_N \circ f : P \longrightarrow N$$

are smooth.

Proof. If f is smooth, both component maps are smooth by composition.

Conversely, suppose the component maps are smooth. In a product chart on $M \times N$, the coordinate representative of f is the ordered pair of the coordinate representatives of its two components. It is therefore smooth by the component criterion. □

Thus $M \times N$, together with its two projections, is the categorical product of M and N in **Man**.

1.4.3 Gluing

Theorem 1.30 (Gluing of smooth maps)**.** *Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of a smooth manifold M. Suppose that for each α there is a smooth map*

$$f_\alpha : U_\alpha \longrightarrow N$$

such that

$$f_\alpha|_{U_\alpha\cap U_\beta} = f_\beta|_{U_\alpha\cap U_\beta}$$

for all α, β. Then there exists a unique smooth map

$$f : M \longrightarrow N$$

such that

$$f|_{U_\alpha} = f_\alpha$$

for every α.

Proof. The compatibility condition implies that the set-theoretic maps f_α glue uniquely to a map $f : M \longrightarrow N$. Its restriction to each U_α is smooth, so Proposition 1.19 shows that f is smooth. Uniqueness is immediate because the sets U_α cover M. □

Proposition 1.31 (Gluing of local inverses)**.** *Let $f : M \longrightarrow N$ be a bijective smooth map. Suppose every point $p \in M$ has an open neighborhood U_p such that $f(U_p)$ is open in N and*

$$f|_{U_p} : U_p \longrightarrow f(U_p)$$

is a diffeomorphism. Then f is a diffeomorphism.

Proof. The open sets $f(U_p)$ cover N. On each $f(U_p)$, the inverse of f is the smooth map

$$\left(f|_{U_p}\right)^{-1}.$$

Since f is bijective, these local inverses agree on overlaps. By Theorem 1.30, they glue to a smooth global inverse $f^{-1} : N \longrightarrow M$. □

1.4.4 Basic examples

Example 1.32 (The standard sphere)**.** Let

$$S^n = \left\{ (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1 \right\}.$$

Let

$$N = (0, \ldots, 0, 1), \qquad S = (0, \ldots, 0, -1).$$

Stereographic projection from N and S gives maps

$$\sigma_N : S^n \setminus \{N\} \longrightarrow \mathbb{R}^n, \qquad \sigma_S : S^n \setminus \{S\} \longrightarrow \mathbb{R}^n.$$

Writing $x' = (x^1, \ldots, x^n)$, these maps are

$$\sigma_N(x', x^{n+1}) = \frac{x'}{1 - x^{n+1}},$$

and

$$\sigma_S(x', x^{n+1}) = \frac{x'}{1 + x^{n+1}}.$$

On the overlap, their transition map is

$$\sigma_S \circ \sigma_N^{-1}(u) = \frac{u}{\|u\|^2}, \qquad u \in \mathbb{R}^n \setminus \{0\}.$$

This map is smooth and is its own inverse. The two stereographic charts therefore define the standard smooth structure on S^n.

Example 1.33 (Real projective space)**.** The real projective space $\mathbb{RP}^n$ is the set of one-dimensional linear subspaces of $\mathbb{R}^{n+1}$. It can equivalently be written as the quotient

$$\mathbb{RP}^n = S^n/\{x \sim -x\}.$$

With its quotient topology, it is Hausdorff and second countable.

Write a point as

$$[x^0 : \cdots : x^n].$$

For each i, set

$$U_i = \left\{[x^0 : \cdots : x^n] : x^i \neq 0\right\}.$$

Define

$$\varphi_i : U_i \longrightarrow \mathbb{R}^n$$

by dividing all homogeneous coordinates by x^i and omitting the ith coordinate:

$$\varphi_i([x^0 : \cdots : x^n]) = \left(\frac{x^0}{x^i}, \ldots, \widehat{\frac{x^i}{x^i}}, \ldots, \frac{x^n}{x^i}\right).$$

The transition maps are rational functions whose denominators are nonzero on their domains. Hence the charts (U_i, φ_i) define the standard smooth structure on $\mathbb{RP}^n$.

Example 1.34 (Tori)**.** The n-dimensional torus is

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ factors}}.$$

It carries the product smooth structure. We shall also use the notation

$$T^n \simeq \mathbb{R}^n/\mathbb{Z}^n,$$

although the general theory of quotient manifolds will not be required in this chapter.

1.5 The Sheaf of Smooth Functions

For an open subset $U \subseteq M$, let

$$\mathscr{C}_M^\infty(U) = C^\infty(U)$$

be the $\mathbb{R}$-algebra of smooth real-valued functions on U. Restriction of functions defines maps

$$\rho_{UV} : \mathscr{C}_M^\infty(U) \longrightarrow \mathscr{C}_M^\infty(V)$$

whenever $V \subseteq U$ is open.

Theorem 1.35 (Sheaf of smooth functions)**.** *The assignment*

$$U \longmapsto \mathscr{C}_M^\infty(U)$$

defines a sheaf of commutative $\mathbb{R}$*-algebras on* M*.*

Proof. The presheaf identities for restriction maps are immediate.

Let $U \subseteq M$ be open and let $\{U_\alpha\}$ be an open cover of U. Suppose

$$f_\alpha \in C^\infty(U_\alpha)$$

and

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}.$$

The functions glue uniquely as set maps to a function

$$f : U \longrightarrow \mathbb{R}.$$

Since smoothness is local and $f|_{U_\alpha} = f_\alpha$, the function f is smooth. This proves existence.

If two smooth functions on U have the same restrictions to all U_α, then they agree at every point of U, proving uniqueness. □

Definition 1.36. The sheaf $\mathscr{C}^\infty_M$ is called the *structure sheaf* of the smooth manifold M.

Proposition 1.37 (Structure sheaf of an open submanifold)**.** *Let $j : U \longrightarrow M$ be an open submanifold. Then there is a canonical isomorphism*

$$\mathscr{C}^\infty_U \simeq \mathscr{C}^\infty_M|_U.$$

Proof. For every open set $V \subseteq U$, a function $V \longrightarrow \mathbb{R}$ is smooth with respect to the induced smooth structure on U if and only if it is smooth as a function on the open subset V of M. Thus both sheaves assign the same algebra $C^\infty(V)$ to every open $V \subseteq U$, with the same restriction maps. □

Let $f : M \longrightarrow N$ be smooth. For every open set $V \subseteq N$, composition with f gives an algebra homomorphism

$$f^*_V : C^\infty(V) \longrightarrow C^\infty(f^{-1}(V)), \qquad h \longmapsto h \circ f.$$

Proposition 1.38 (Pullback of smooth functions)**.** *The maps f^*_V commute with restrictions and therefore define a morphism of sheaves of $\mathbb{R}$-algebras*

$$f^\sharp : \mathscr{C}^\infty_N \longrightarrow f_*\mathscr{C}^\infty_M.$$

For smooth maps

$$M \xrightarrow{f} N \xrightarrow{g} P,$$

one has

$$(g \circ f)^* = f^* \circ g^*$$

on local sections, and

$$\mathrm{id}^*_M = \mathrm{id}_{\mathscr{C}^\infty_M}.$$

Proof. If $W \subseteq V \subseteq N$ are open and $h \in C^\infty(V)$, then

$$(h|_W) \circ f = (h \circ f)|_{f^{-1}(W)}.$$

Thus pullback commutes with restriction maps and defines a sheaf morphism.

The functorial identities follow from associativity of composition:

$$(g \circ f)^* h = h \circ g \circ f = f^*(g^* h).$$

The identity statement is immediate. □

Smoothness of a continuous map can be characterized entirely by its action on local smooth functions.

Theorem 1.39 (Smoothness detected by the structure sheaf)**.** *Let $f : M \longrightarrow N$ be continuous. The following are equivalent:*

(1) f is smooth;

(2) for every open set $V \subseteq N$ and every $h \in C^\infty(V)$, the composite

$$h \circ f : f^{-1}(V) \longrightarrow \mathbb{R}$$

is smooth;

(3) for every target coordinate chart

$$(V, y^1, \ldots, y^n),$$

the functions

$$y^i \circ f : f^{-1}(V) \longrightarrow \mathbb{R}$$

are smooth.

Proof. The implication (1) ⇒ (2) follows from the stability of smooth maps under composition. The implication (2) ⇒ (3) follows by taking $h = y^i$.

The implication (3) ⇒ (1) is Proposition 1.25. □

Remark 1.40. Theorem 1.39 explains why the sheaf $\mathscr{C}^\infty_M$ contains the complete local smooth information of the manifold. Nevertheless, ordinary sheaves of commutative rings do not record all operations naturally available on smooth functions. A categorical characterization of smooth manifolds by structure sheaves requires the more refined language of C^∞-rings. We shall not need that theory here.

1.6 Germs and Local Rings

Definition 1.41. Let $p \in M$. Two smooth functions

$$f : U \longrightarrow \mathbb{R}, \qquad g : V \longrightarrow \mathbb{R},$$

defined on neighborhoods of p, have the same *germ at p* if there exists an open neighborhood

$$W \subseteq U \cap V$$

of p such that

$$f|_W = g|_W.$$

The germ of f at p is denoted by $[f]_p$.

The set of germs at p is the stalk

$$\mathscr{C}^\infty_{M,p} = \varinjlim_{p \in U} C^\infty(U).$$

Addition and multiplication of representatives make $\mathscr{C}^\infty_{M,p}$ a commutative $\mathbb{R}$-algebra.

Definition 1.42. Evaluation at p is the algebra homomorphism

$$\mathrm{ev}_p : \mathscr{C}^\infty_{M,p} \longrightarrow \mathbb{R}, \qquad [f]_p \longmapsto f(p).$$

Define

$$\mathfrak{m}_p = \ker(\mathrm{ev}_p) = \{[f]_p : f(p) = 0\}.$$

Theorem 1.43 (Local ring of smooth germs). *The stalk $\mathscr{C}^\infty_{M,p}$ is a local ring whose unique maximal ideal is $\mathfrak{m}_p$.*

Proof. Let $[f]_p \in \mathscr{C}^\infty_{M,p}$. Suppose first that

$$[f]_p \notin \mathfrak{m}_p.$$

Then $f(p) \neq 0$. By continuity, there exists an open neighborhood W of p on which f is nowhere zero. The function

$$\frac{1}{f} : W \longrightarrow \mathbb{R}$$

is smooth, and its germ satisfies

$$[f]_p \left[\frac{1}{f}\right]_p = [1]_p.$$

Thus every element outside $\mathfrak{m}_p$ is a unit.

Conversely, no element of $\mathfrak{m}_p$ is a unit. Indeed, if $[f]_p \in \mathfrak{m}_p$ and

$$[f]_p[g]_p = [1]_p,$$

then evaluating at p would give

$$0 = f(p)g(p) = 1,$$

a contradiction.

Hence the nonunits of $\mathscr{C}^\infty_{M,p}$ form precisely the ideal $\mathfrak{m}_p$. A commutative ring whose nonunits form an ideal is local, and that ideal is its unique maximal ideal. □

Corollary 1.44 (Residue field). *There is a canonical isomorphism*

$$\mathscr{C}^\infty_{M,p}/\mathfrak{m}_p \simeq \mathbb{R}.$$

Proof. The evaluation map is surjective because constant functions realize every real value. Its kernel is $\mathfrak{m}_p$, so the result follows from the first isomorphism theorem. □

A smooth map induces homomorphisms not only on local sections but also on stalks.

Proposition 1.45 (Pullback on stalks). *Let $f : M \longrightarrow N$ be smooth and let $p \in M$. Pullback induces an algebra homomorphism*

$$f_p^\sharp : \mathscr{C}^\infty_{N,f(p)} \longrightarrow \mathscr{C}^\infty_{M,p}, \qquad [h]_{f(p)} \longmapsto [h \circ f]_p.$$

This homomorphism is local:

$$(f_p^\sharp)^{-1}(\mathfrak{m}_p) = \mathfrak{m}_{f(p)}.$$

Proof. If two functions agree on a neighborhood of $f(p)$, then their pullbacks agree on the inverse image of that neighborhood, which is a neighborhood of p. Thus the map on germs is well defined.

For a germ $[h]_{f(p)}$,

$$f_p^\sharp([h]_{f(p)}) \in \mathfrak{m}_p$$

if and only if

$$(h \circ f)(p) = 0.$$

Since

$$(h \circ f)(p) = h(f(p)),$$

this is equivalent to

$$[h]_{f(p)} \in \mathfrak{m}_{f(p)}.$$

□

Corollary 1.46. *The pair*

$$(M, \mathscr{C}^\infty_M)$$

is a locally ringed space. Every smooth map

$$f : M \longrightarrow N$$

determines a morphism of locally ringed spaces

$$(f, f^\sharp) : (M, \mathscr{C}^\infty_M) \longrightarrow (N, \mathscr{C}^\infty_N).$$

Proof. Theorem 1.43 shows that every stalk of $\mathscr{C}^\infty_M$ is local. Proposition 1.45 shows that the stalk homomorphisms induced by a smooth map are local. □

Remark 1.47. We shall use the locally ringed space language only as a convenient organization of local smooth functions. We do not identify smooth manifolds with arbitrary locally ringed spaces, nor do we use this corollary as an alternative definition of a smooth manifold.

The maximal ideal $\mathfrak{m}_p$ consists of functions vanishing at p. Its quotient

$$\mathfrak{m}_p/\mathfrak{m}_p^2$$

retains precisely their first-order parts. In the next chapter we shall prove the natural identification

$$T_p^*M \simeq \mathfrak{m}_p/\mathfrak{m}_p^2,$$

while the tangent space will be described by derivations

$$T_pM = \mathrm{Der}_{\mathbb{R}}\left(\mathscr{C}^\infty_{M,p}, \mathbb{R}\right).$$

Thus the first-order geometry of a smooth manifold will be extracted directly from the local algebra of its structure sheaf.

Chapter 2

Tangent and Cotangent Spaces

The local ring $\mathscr{C}^\infty_{M,p}$ records all smooth functions near a point p, while its maximal ideal $\mathfrak{m}_p$ records those vanishing at p. The first-order geometry of M at p is extracted from this local algebra in two dual ways. Tangent vectors are derivations of smooth germs, and cotangent vectors are first-order germs modulo products of vanishing germs. Coordinates identify these intrinsic constructions with the familiar vector spaces of directional derivatives and differentials.

2.1 Derivations and Tangent Spaces

Definition 2.1. Let M be a smooth manifold and let $p \in M$. A *derivation at* p is an $\mathbb{R}$-linear map

$$D : \mathscr{C}^\infty_{M,p} \longrightarrow \mathbb{R}$$

satisfying the Leibniz identity

$$D(fg) = f(p)Dg + g(p)Df.$$

The vector space of derivations at p is called the *tangent space* of M at p and is denoted by

$$T_pM = \mathrm{Der}_{\mathbb{R}}\left(\mathscr{C}^\infty_{M,p}, \mathbb{R}\right).$$

Here and below, a germ and any representative of that germ are denoted by the same letter when no confusion can arise.

Lemma 2.2. *Let $D \in T_pM$.*

(1) D annihilates constant germs.

(2) D annihilates $\mathfrak{m}_p^2$.

Proof. Since $1 = 1 \cdot 1$, the Leibniz identity gives

$$D1 = 2D1,$$

so $D1 = 0$. By $\mathbb{R}$-linearity, D annihilates every constant germ.

If $f, g \in \mathfrak{m}_p$, then $f(p) = g(p) = 0$, and therefore

$$D(fg) = f(p)Dg + g(p)Df = 0.$$

Every element of $\mathfrak{m}_p^2$ is a finite sum of such products, so D annihilates $\mathfrak{m}_p^2$. □

The following elementary decomposition is the main analytic ingredient in the coordinate description of tangent vectors.

Lemma 2.3 (Hadamard lemma)**.** *Let $U \subseteq \mathbb{R}^n$ be open, let $a \in U$, and let $f \in C^\infty(U)$. After replacing U by a convex neighborhood of a, there exist smooth functions $g_1, \dots, g_n \in C^\infty(U)$ such that*

$$f(x) - f(a) = \sum_{i=1}^{n} (x^i - a^i) g_i(x),$$

and

$$g_i(a) = \frac{\partial f}{\partial x^i}(a).$$

Proof. For x in a sufficiently small convex neighborhood of a, define

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}\big(a + t(x-a)\big)\, dt.$$

Differentiation under the integral sign shows that each g_i is smooth. By the fundamental theorem of calculus,

$$\begin{aligned} f(x) - f(a) &= \int_0^1 \frac{d}{dt} f\big(a + t(x-a)\big)\, dt \\ &= \sum_{i=1}^{n} (x^i - a^i) \int_0^1 \frac{\partial f}{\partial x^i}\big(a + t(x-a)\big)\, dt. \end{aligned}$$

Evaluating at $x = a$ gives the stated value of $g_i(a)$. □

Let (U, x) be a chart about p, with

$$x = (x^1, \dots, x^n).$$

For each i, define

$$\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$$

by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial (f \circ x^{-1})}{\partial r^i}\big(x(p)\big),$$

where $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$. The ordinary product rule shows that these maps are derivations.

Theorem 2.4 (Coordinate basis)**.** *Let $(U, x^1, \dots, x^n)$ be a chart about $p \in M$. Then*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$$

form a basis of T_pM. In particular,

$$\dim T_pM = \dim M.$$

Proof. After subtracting constants from the coordinate functions, we may apply Lemma 2.3 in the coordinate domain to write every germ f at p in the form

$$f = f(p) + \sum_{i=1}^{n} \big(x^i - x^i(p)\big) g_i,$$

where

$$g_i(p) = \left.\frac{\partial f}{\partial x^i}\right|_p.$$

For any $D \in T_pM$, the Leibniz identity and Lemma 2.2 give

$$Df = \sum_{i=1}^{n} D(x^i) g_i(p) = \sum_{i=1}^{n} D(x^i) \left. \frac{\partial f}{\partial x^i} \right|_p .$$

Hence

$$D = \sum_{i=1}^{n} D(x^i) \left. \frac{\partial}{\partial x^i} \right|_p ,$$

so the coordinate derivations span T_pM.

If

$$\sum_{i=1}^{n} a^i \left. \frac{\partial}{\partial x^i} \right|_p = 0,$$

then applying this derivation to x^j gives $a^j = 0$. Thus the coordinate derivations are linearly independent. □

Proposition 2.5 (Tangent space of an open submanifold)**.** *Let $U \subseteq M$ be open and let $p \in U$. Restriction of germs gives a canonical identification*

$$\mathscr{C}^\infty_{U,p} = \mathscr{C}^\infty_{M,p},$$

and hence a canonical identification

$$T_pU = T_pM.$$

Proof. A neighborhood of p in U is also a neighborhood of p in M after possibly shrinking it, and every germ on M restricts to the same germ on U. Thus the two stalks are canonically equal. Their derivation spaces are therefore equal. □

2.2 Cotangent Spaces and First-Order Germs

Definition 2.6. The *cotangent space* of M at p is the dual vector space

$$T_p^*M = \mathrm{Hom}_{\mathbb{R}}(T_pM, \mathbb{R}).$$

For a germ $f \in \mathscr{C}^\infty_{M,p}$, its *differential at* p is the covector

$$d_pf \in T_p^*M$$

defined by

$$d_pf(v) = v(f)$$

for $v \in T_pM$.

The assignment $f \mapsto d_pf$ is $\mathbb{R}$-linear and satisfies

$$d_p(fg) = f(p)d_pg + g(p)d_pf.$$

In particular, it sends $\mathfrak{m}_p$ to T_p^*M and annihilates $\mathfrak{m}_p^2$.

Theorem 2.7 (Cotangent space as first-order germs)**.** *The map*

$$\mathfrak{m}_p \longrightarrow T_p^*M, \qquad f \longmapsto d_pf,$$

induces a natural vector-space isomorphism

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \xrightarrow{\simeq} T_p^*M.$$

Proof. By Lemma 2.2, every tangent vector annihilates $\mathfrak{m}_p^2$. Hence d_pf depends only on the class of f modulo $\mathfrak{m}_p^2$, and the displayed map is well defined.

Choose coordinates $(x^1, \ldots, x^n)$ about p and put

$$u^i = x^i - x^i(p).$$

The proof of Theorem 2.4 shows that every $f \in \mathfrak{m}_p$ satisfies

$$f \equiv \sum_{i=1}^n \left.\frac{\partial f}{\partial x^i}\right|_p u^i \pmod{\mathfrak{m}_p^2}.$$

Thus the classes of $u^1, \ldots, u^n$ span $\mathfrak{m}_p/\mathfrak{m}_p^2$.

Moreover,

$$d_pu^i\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta^i_j.$$

Hence $d_pu^1, \ldots, d_pu^n$ are the dual basis to the coordinate basis of T_pM. The induced map therefore sends a spanning set to a basis and is an isomorphism. □

Definition 2.8. For a coordinate system $(x^1, \ldots, x^n)$ about p, write

$$\left.dx^i\right|_p = d_px^i.$$

Corollary 2.9 (Coordinate basis of the cotangent space)**.** *The covectors*

$$\left.dx^1\right|_p, \ldots, \left.dx^n\right|_p$$

form the basis of T_p^*M dual to

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

In particular,

$$\left.dx^i\right|_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta^i_j.$$

Proposition 2.10 (Change of coordinates)**.** *Let $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ be two coordinate systems on an open neighborhood of p. Then*

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum_{i=1}^n \frac{\partial y^i}{\partial x^j}(p) \left.\frac{\partial}{\partial y^i}\right|_p,$$

and

$$\left.dy^i\right|_p = \sum_{j=1}^n \frac{\partial y^i}{\partial x^j}(p) \left.dx^j\right|_p.$$

Proof. Since the y-coordinate vectors form a basis, write

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum_i a^i_j \left.\frac{\partial}{\partial y^i}\right|_p.$$

Applying both sides to y^k gives

$$\frac{\partial y^k}{\partial x^j}(p) = a^k_j.$$

This proves the first formula.

For the second formula, evaluate both sides on the x-coordinate basis. One has

$$\left.dy^i\right|_p\left(\left.\frac{\partial}{\partial x^k}\right|_p\right) = \frac{\partial y^i}{\partial x^k}(p),$$

which is also the value of the right-hand side on the same basis vector. □

2.3 Curves and Tangent Vectors

Definition 2.11. A *smooth curve through* p is a smooth map

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$$

with $\gamma(0) = p$.

Two such curves γ and η are *tangent at* 0 if, for one and hence every chart x about p,

$$(x \circ \gamma)'(0) = (x \circ \eta)'(0).$$

The independence of the chart follows from the ordinary chain rule applied to a transition map.

Proposition 2.12 (Curve realization)**.** *Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ be a smooth curve through p. The formula*

$$D_\gamma f = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$$

defines a derivation at p. The resulting map from tangent classes of curves through p to T_pM is a bijection.

Proof. The ordinary product rule gives

$$D_\gamma(fg) = f(p)D_\gamma g + g(p)D_\gamma f,$$

so D_γ is a derivation.

In coordinates $(x^1, \ldots, x^n)$ about p,

$$D_\gamma = \sum_{i=1}^{n} (x^i \circ \gamma)'(0) \left.\frac{\partial}{\partial x^i}\right|_p .$$

Hence two curves determine the same derivation exactly when they are tangent at 0.

Conversely, let

$$v = \sum_{i=1}^{n} v^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

For sufficiently small $\varepsilon > 0$, the curve

$$\gamma(t) = x^{-1}\big(x(p) + t(v^1, \ldots, v^n)\big)$$

is defined for $|t| < \varepsilon$ and satisfies $D_\gamma = v$. $\square$

Definition 2.13. The tangent vector represented by a curve γ at time t_0 is called its *velocity* and is denoted by

$$\gamma'(t_0) \in T_{\gamma(t_0)}M.$$

Equivalently,

$$\gamma'(t_0)f = \left.\frac{d}{dt}\right|_{t=t_0} f(\gamma(t)).$$

Proposition 2.14 (Velocity in local coordinates)**.** *Let $(x^1, \ldots, x^n)$ be coordinates near $\gamma(t_0)$. Then*

$$\gamma'(t_0) = \sum_{i=1}^{n} \frac{d(x^i \circ \gamma)}{dt}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)} .$$

Proof. The coefficient of the ith coordinate vector is obtained by applying $\gamma'(t_0)$ to x^i:

$$\gamma'(t_0)x^i = \frac{d(x^i \circ \gamma)}{dt}(t_0).$$

The conclusion follows from the coordinate basis theorem. $\square$

2.4 The Differential of a Smooth Map

Definition 2.15. Let $f : M \longrightarrow N$ be smooth and let $p \in M$. The *differential of f at p* is the linear map

$$df_p : T_pM \longrightarrow T_{f(p)}N$$

defined by

$$(df_p v)(g) = v(g \circ f)$$

for $v \in T_pM$ and $g \in \mathscr{C}^\infty_{N,f(p)}$.

Equivalently, $df_p v$ is the composite

$$\mathscr{C}^\infty_{N,f(p)} \xrightarrow{f_p^\sharp} \mathscr{C}^\infty_{M,p} \xrightarrow{v} \mathbb{R}.$$

Proposition 2.16. *The differential df_p is a well-defined linear map. If v is represented by a curve γ through p, then $df_p v$ is represented by the curve $f \circ \gamma$.*

Proof. Since $f_p^\sharp$ is an algebra homomorphism compatible with evaluation, the composite of v with $f_p^\sharp$ satisfies the Leibniz identity at $f(p)$. Thus $df_p v \in T_{f(p)}N$. Linearity in v is immediate.

For a germ g at $f(p)$,

$$\begin{aligned}(df_p v)(g) &= v(g \circ f) \\ &= \left.\frac{d}{dt}\right|_{t=0} g(f(\gamma(t))),\end{aligned}$$

which is precisely the derivation represented by $f \circ \gamma$. □

Theorem 2.17 (Chain rule)**.** *For smooth maps*

$$M \xrightarrow{f} N \xrightarrow{g} P$$

and $p \in M$,

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

Proof. For $v \in T_pM$ and a germ h at $g(f(p))$,

$$\begin{aligned}\big(d(g \circ f)_p v\big)(h) &= v(h \circ g \circ f) \\ &= \big(dg_{f(p)}(df_p v)\big)(h).\end{aligned}$$

The two tangent vectors therefore agree on every germ. □

Corollary 2.18. *If $f : M \longrightarrow N$ is a diffeomorphism, then df_p is an isomorphism and*

$$d(f^{-1})_{f(p)} = (df_p)^{-1}.$$

Proof. Apply the chain rule to

$$f^{-1} \circ f = \mathrm{id}_M$$

and

$$f \circ f^{-1} = \mathrm{id}_N\,.$$

□

Proposition 2.19 (Coordinate formula)**.** *Let*

$$(x^1, \dots, x^m)$$

be coordinates near $p \in M$*, and let*

$$(y^1, \dots, y^n)$$

be coordinates near $f(p) \in N$*. Then*

$$df_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \sum_{a=1}^{n} \frac{\partial (y^a \circ f)}{\partial x^j}(p)\, \frac{\partial}{\partial y^a}\bigg|_{f(p)}.$$

Thus the matrix of df_p *in the coordinate bases is the Jacobian matrix*

$$\left[\frac{\partial (y^a \circ f)}{\partial x^j}(p) \right].$$

Proof. Write

$$df_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = \sum_a A^a_j \, \frac{\partial}{\partial y^a}\bigg|_{f(p)}.$$

Applying both sides to the coordinate function y^b gives

$$A^b_j = \frac{\partial}{\partial x^j}\bigg|_p (y^b \circ f) = \frac{\partial (y^b \circ f)}{\partial x^j}(p).$$

□

Definition 2.20. The dual of df_p is the *cotangent pullback*

$$df_p^* : T^*_{f(p)} N \longrightarrow T^*_p M,$$

defined by

$$(df_p^* \alpha)(v) = \alpha(df_p v).$$

Proposition 2.21 (Compatibility with first-order germs)**.** *Under the natural identifications*

$$T^*_{f(p)} N \simeq \mathfrak{m}_{f(p)} / \mathfrak{m}^2_{f(p)}$$

and

$$T^*_p M \simeq \mathfrak{m}_p / \mathfrak{m}_p^2,$$

the map df_p^* *is induced by the local homomorphism*

$$f_p^\sharp : \mathscr{C}^\infty_{N, f(p)} \longrightarrow \mathscr{C}^\infty_{M,p}.$$

Equivalently,

$$df_p^*(d_{f(p)} g) = d_p(g \circ f).$$

Proof. For $v \in T_p M$,

$$\begin{aligned} \big(df_p^*(d_{f(p)} g)\big)(v) &= d_{f(p)} g(df_p v) \\ &= (df_p v)(g) \\ &= v(g \circ f) \\ &= d_p(g \circ f)(v). \end{aligned}$$

□

2.5 The Tangent and Cotangent Bundles

Definition 2.22. The *tangent bundle as a set* is the disjoint union

$$TM = \coprod_{p \in M} T_pM,$$

with projection

$$\pi_M : TM \longrightarrow M, \qquad v \in T_pM \longmapsto p.$$

Let $(U, x^1, \ldots, x^n)$ be a coordinate chart. Every $v \in T_pM$ with $p \in U$ has a unique expression

$$v = \sum_{i=1}^{n} v^i \left. \frac{\partial}{\partial x^i} \right|_p .$$

Define

$$\widetilde{x} : \pi_M^{-1}(U) \longrightarrow x(U) \times \mathbb{R}^n$$

by

$$\widetilde{x}(v) = \big(x(p), v^1, \ldots, v^n\big).$$

Theorem 2.23 (Smooth structure on the tangent bundle)**.** *The maps $\widetilde{x}$ associated with all coordinate charts of M form an atlas on TM. They define a natural smooth structure of dimension $2n$ for which*

$$\pi_M : TM \longrightarrow M$$

is smooth.

Proof. Each $\widetilde{x}$ is a bijection from $\pi_M^{-1}(U)$ onto $x(U) \times \mathbb{R}^n$. We use these bijections to transport a topology and smooth structure to TM.

Let (U, x) and (V, y) be overlapping charts. By Proposition 2.10, the corresponding transition map is

$$(a, v) \longmapsto \big((y \circ x^{-1})(a), D(y \circ x^{-1})_a v\big) .$$

The derivative of a smooth Euclidean map depends smoothly on the base point, so this transition map is smooth. Its inverse has the same form with x and y interchanged. Thus the induced charts are smoothly compatible and define a smooth structure on TM.

In induced coordinates, the projection is

$$(a, v) \longmapsto a,$$

so it is smooth. ☐

Proposition 2.24. *The projection*

$$\pi_M : TM \longrightarrow M$$

is a smooth submersion.

Proof. In the induced coordinates of Theorem 2.23, π_M is the standard projection

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Its differential is surjective at every point. ☐

Definition 2.25. The *cotangent bundle as a set* is

$$T^*M = \coprod_{p\in M} T_p^*M,$$

with projection

$$\pi_M^* : T^*M \longrightarrow M.$$

If $(U, x^1, \ldots, x^n)$ is a chart, every $\alpha \in T_p^*M$ has a unique expression

$$\alpha = \sum_{i=1}^{n} \alpha_i \, dx^i\big|_p .$$

This defines a bijection

$$\widetilde{x}^* : (\pi_M^*)^{-1}(U) \longrightarrow x(U) \times (\mathbb{R}^n)^*.$$

Theorem 2.26 (Smooth structure on the cotangent bundle)**.** *The maps $\widetilde{x}^*$ define a natural smooth structure of dimension $2n$ on T^*M. In overlapping coordinates, the fiber coordinates transform by the dual inverse Jacobian.*

Proof. Let x and y be overlapping coordinate systems. If

$$\alpha = \sum_i \alpha_i \, dx^i = \sum_a \beta_a \, dy^a,$$

then Proposition 2.10 gives

$$\alpha_i = \sum_a \beta_a \frac{\partial y^a}{\partial x^i}.$$

Thus, in matrix notation,

$$\beta = \left(D(y \circ x^{-1})^{-1}\right)^* \alpha.$$

The resulting transition maps are smooth because matrix inversion is smooth on $GL(n, \mathbb{R})$. They therefore define the claimed smooth structure. □

Proposition 2.27 (Smoothness of the tangent map)**.** *For every smooth map $f : M \longrightarrow N$, the map*

$$Tf : TM \longrightarrow TN, \qquad v \in T_pM \longmapsto df_p v,$$

is smooth.

Proof. In tangent bundle coordinates induced by charts x on M and y on N, the map Tf is represented by

$$(a, v) \longmapsto \left((y \circ f \circ x^{-1})(a), D(y \circ f \circ x^{-1})_a v\right).$$

Both the coordinate representative and its Jacobian depend smoothly on a, so this map is smooth. □

2.6 Functoriality and Products

Theorem 2.28 (Tangent functor)**.** *The assignments*

$$M \longmapsto TM, \qquad f \longmapsto Tf$$

define a covariant functor

$$T : \mathbf{Man} \longrightarrow \mathbf{Man}.$$

More precisely,

$$T(\mathrm{id}_M) = \mathrm{id}_{TM}$$

and

$$T(g \circ f) = Tg \circ Tf.$$

Moreover, the projections form a natural transformation

$$\pi : T \Longrightarrow \mathrm{Id}_{\mathbf{Man}}.$$

Proof. The two functorial identities are the identity rule and chain rule for differentials. For naturality, one has

$$\pi_N \circ Tf = f \circ \pi_M$$

by the definition of Tf. □

Theorem 2.29 (Tangent bundle of a product)**.** *There is a natural diffeomorphism over* $M \times N$

$$T(M \times N) \xrightarrow{\simeq} TM \times TN$$

given by

$$v \longmapsto \big(d(\mathrm{pr}_M)v, d(\mathrm{pr}_N)v\big).$$

At (p, q) *this restricts to the natural linear isomorphism*

$$T_{(p,q)}(M \times N) \simeq T_pM \oplus T_qN.$$

Proof. In product coordinates, the displayed map is the standard identification

$$\mathbb{R}^{m+n} \longrightarrow \mathbb{R}^m \oplus \mathbb{R}^n.$$

It is therefore a smooth fiberwise linear bijection with smooth inverse. Naturality follows from the universal property of products and the chain rule. □

Corollary 2.30 (Cotangent space of a product)**.** *There is a natural linear isomorphism*

$$T^*_{(p,q)}(M \times N) \simeq T^*_pM \oplus T^*_qN.$$

Under this identification,

$$(\alpha, \beta) \longmapsto \mathrm{pr}_M^* \alpha + \mathrm{pr}_N^* \beta.$$

Proof. Dualize the isomorphism in Theorem 2.29. The displayed formula is the dual of the pair of tangent projections. □

Chapter 3

Rank, Submanifolds, and Fiber Products

The differential linearizes a smooth map at each point. Rank conditions on this linearization determine local normal forms, while submanifolds are the subsets that become linear coordinate subspaces in suitable charts. The main purpose of this chapter is to organize regular level sets, transverse preimages, intersections, and fiber products as instances of a single local principle.

3.1 Rank and Local Normal Forms

Definition 3.1. Let $f : M \longrightarrow N$ be smooth and let $p \in M$. The *rank of f at p* is

$$\operatorname{rank}_p(f) = \operatorname{rank}(df_p).$$

The map f is

(1) an *immersion at p* if df_p is injective;

(2) a *submersion at p* if df_p is surjective;

(3) a *local diffeomorphism at p* if there exists an open neighborhood U of p such that $f(U)$ is open and $f|_U : U \longrightarrow f(U)$ is a diffeomorphism.

The corresponding unqualified terms mean that the condition holds at every point.

Definition 3.2. A smooth map $f : M \longrightarrow N$ is an *embedding* if it is an immersion and a homeomorphism from M onto $f(M)$ with the subspace topology.

Proposition 3.3 (Maximal rank is an open condition)**.** *Let $f : M^m \longrightarrow N^n$ be smooth.*

(1) The set of points at which f is an immersion is open.

(2) The set of points at which f is a submersion is open.

Consequently, an immersion or submersion at one point has constant maximal rank on a neighborhood of that point.

Proof. Choose coordinates near p and $f(p)$. The differential is represented by an $n \times m$ Jacobian matrix whose entries vary continuously.

If df_p is injective, then $m \leq n$ and some $m \times m$ minor is nonzero at p. The same minor remains nonzero on a neighborhood of p, so the rank remains m there. This proves the first assertion.

If df_p is surjective, then $n \leq m$ and some $n \times n$ minor is nonzero at p. Again it remains nonzero nearby, proving the second assertion. $\square$

We use the following analytic theorems in their manifold form. Their proofs reduce in coordinates to the corresponding theorems for Euclidean open sets.

Theorem 3.4 (Inverse function theorem). *Let $f : M^n \longrightarrow N^n$ be smooth. If*

$$df_p : T_pM \longrightarrow T_{f(p)}N$$

is an isomorphism, then f is a local diffeomorphism at p.

Theorem 3.5 (Constant rank theorem). *Let $f : M^m \longrightarrow N^n$ be smooth and suppose that f has constant rank r on a neighborhood of p. Then there exist coordinates*

$$(x^1, \ldots, x^m)$$

centered at p and coordinates

$$(y^1, \ldots, y^n)$$

centered at $f(p)$ such that

$$(y \circ f \circ x^{-1})(u^1, \ldots, u^m) = (u^1, \ldots, u^r, 0, \ldots, 0).$$

Corollary 3.6 (Local form of an immersion). *Let $f : M^m \longrightarrow N^n$ be an immersion at p. There are coordinates near p and $f(p)$ in which*

$$f(u^1, \ldots, u^m) = (u^1, \ldots, u^m, 0, \ldots, 0).$$

In particular, every immersion is locally an embedding.

Proof. By Proposition 3.3, f has constant rank m near p. Apply the constant rank theorem. In the resulting coordinates, f is the standard inclusion, hence an embedding on a sufficiently small neighborhood. □

Corollary 3.7 (Local form of a submersion). *Let $f : M^m \longrightarrow N^n$ be a submersion at p. There are coordinates near p and $f(p)$ in which*

$$f(u^1, \ldots, u^m) = (u^1, \ldots, u^n).$$

Proof. By Proposition 3.3, f has constant rank n near p. Apply the constant rank theorem. □

Corollary 3.8 (Submersions are open). *Every smooth submersion is an open map.*

Proof. Let $W \subseteq M$ be open and let $q = f(p) \in f(W)$ with $p \in W$. Shrinking the local coordinates in Corollary 3.7, we may assume that the coordinate neighborhood of p is contained in W. In these coordinates, f is a projection, and projections between Euclidean spaces are open. Hence q has an open neighborhood contained in $f(W)$. Therefore $f(W)$ is open. □

3.2 Embedded Submanifolds

Definition 3.9. Let M be an m-dimensional smooth manifold. A subset $S \subseteq M$ is an *embedded submanifold of dimension k* if for every $p \in S$ there is a chart

$$(U, x^1, \ldots, x^m)$$

of M about p such that

$$x(U \cap S) = x(U) \cap \left(\mathbb{R}^k \times \{0\}\right).$$

Such a chart is called an *adapted chart* for S in M.

The first k coordinate functions in an adapted chart restrict to local coordinates on S.

Theorem 3.10 (Induced smooth structure)**.** *Let $S \subseteq M$ be an embedded submanifold of dimension k. The maps*

$$(x^1, \dots, x^k)|_{U \cap S} : U \cap S \longrightarrow \mathbb{R}^k$$

obtained from adapted charts define a smooth structure on S. With this structure, the topology of S is the subspace topology inherited from M.

Proof. In an adapted chart, the restriction

$$(x^1, \dots, x^k)|_{U \cap S}$$

is a homeomorphism from $U \cap S$ onto

$$\Big\{(u^1, \dots, u^k) \in \mathbb{R}^k : (u^1, \dots, u^k, 0, \dots, 0) \in x(U)\Big\}.$$

This set is open in $\mathbb{R}^k$, being the inverse image of the open set $x(U)$ under the standard inclusion

$$\mathbb{R}^k \longrightarrow \mathbb{R}^m, \qquad u \longmapsto (u, 0).$$

Let x and y be two adapted charts. The transition map between the restricted charts is obtained by restricting $y \circ x^{-1}$ to the coordinate k-plane and then taking its first k components. It is therefore smooth. The restricted charts form a smooth atlas on S.

Each $U \cap S$ is open in the subspace topology, and the restricted coordinate map is a homeomorphism for that topology. Hence the topology determined by the atlas is exactly the subspace topology. □

Proposition 3.11 (The inclusion of a submanifold)**.** *Let $S \subseteq M$ be an embedded submanifold and let*

$$i : S \longrightarrow M$$

be the inclusion. Then i is an embedding. For every $p \in S$, the differential

$$di_p : T_pS \longrightarrow T_pM$$

is injective and identifies T_pS with the linear subspace spanned in an adapted chart by

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^k}\right|_p.$$

Proof. In an adapted chart, the inclusion is represented by the standard inclusion

$$\mathbb{R}^k \longrightarrow \mathbb{R}^m, \qquad (u^1, \dots, u^k) \longmapsto (u^1, \dots, u^k, 0, \dots, 0).$$

It is therefore a smooth immersion. By Theorem 3.10, the topology on S is the subspace topology, so the inclusion is an embedding. The statement about tangent spaces follows by differentiating the standard inclusion. □

We henceforth identify T_pS with its image in T_pM under di_p.

Theorem 3.12 (Smooth maps into a submanifold)**.** *Let $S \subseteq M$ be an embedded submanifold, and let $f : P \longrightarrow S$ be a map. Then f is smooth if and only if the composite*

$$P \xrightarrow{f} S \xrightarrow{i} M$$

is smooth.

Proof. If $f : P \longrightarrow S$ is smooth, then $i \circ f$ is smooth because the inclusion is smooth.

Conversely, suppose that $i \circ f : P \longrightarrow M$ is smooth. Since S has the subspace topology, f is continuous. Let

$$(U, x^1, \dots, x^m)$$

be an adapted chart of M for S. On $f^{-1}(U \cap S)$, the coordinate representative of f as a map into S is

$$(x^1 \circ f, \dots, x^k \circ f).$$

These functions are smooth because they are components of the smooth map $i \circ f$ in the ambient chart. Hence f is smooth. $\square$

Theorem 3.13 (Images of embeddings)**.** *Let $f : S \longrightarrow M$ be an embedding. Then $f(S)$ is an embedded submanifold of M, and*

$$f : S \longrightarrow f(S)$$

is a diffeomorphism.

Proof. Fix $p \in S$. By Corollary 3.6, there are an open neighborhood U of p and a chart y on a neighborhood V of $f(p)$ such that, in suitable coordinates on U, the restriction $f|_U$ is the standard inclusion. After shrinking U and V, its image is

$$f(U) = V \cap y^{-1}\left(\mathbb{R}^{\dim S} \times \{0\}\right).$$

Because $f : S \longrightarrow f(S)$ is a homeomorphism and U is open in S, the subset $f(U)$ is open in $f(S)$. Thus there exists an open subset $W \subseteq M$ such that

$$f(U) = W \cap f(S).$$

Replacing V by $V \cap W$, we obtain an adapted chart of M for $f(S)$ near $f(p)$. Therefore $f(S)$ is an embedded submanifold.

The map $f : S \longrightarrow f(S)$ is a bijective smooth map. Its inverse is smooth locally in the coordinates supplied by the immersion theorem, and the local inverses glue. Hence it is a diffeomorphism. $\square$

Example 3.14 (Graphs)**.** Let $f : M \longrightarrow N$ be smooth. The map

$$\Gamma_f : M \longrightarrow M \times N, \qquad p \longmapsto (p, f(p))$$

is an embedding. Indeed,

$$\mathrm{pr}_M \circ \Gamma_f = \mathrm{id}_M,$$

so $d\Gamma_f$ is injective, and pr_M restricts to a continuous inverse on the image. Consequently, the graph

$$\mathrm{Graph}(f) = \{(p, f(p)) : p \in M\}$$

is an embedded submanifold of $M \times N$ diffeomorphic to M.

3.3 Regular Values and Level Sets

Definition 3.15. Let $f : M \longrightarrow N$ be smooth.

(1) A point $p \in M$ is a *regular point* of f if df_p is surjective; otherwise it is a *critical point.*

(2) A point $q \in N$ is a *regular value* of f if every point of $f^{-1}(q)$ is regular. A point which is not a regular value is a *critical value.*

A point outside the image of f is a regular value by convention.

Theorem 3.16 (Regular level set theorem)**.** *Let $f : M^m \longrightarrow N^n$ be smooth, and let $q \in N$ be a regular value. If $f^{-1}(q)$ is nonempty, then it is an embedded submanifold of M of dimension*

$$m - n.$$

Proof. Let $p \in f^{-1}(q)$. Since df_p is surjective, the submersion theorem provides coordinates near p and q in which

$$f(u^1, \dots, u^m) = (u^1, \dots, u^n).$$

Choose the target coordinates centered at q. In these coordinates, the level set $f^{-1}(q)$ is defined by

$$u^1 = \cdots = u^n = 0.$$

After reordering the source coordinates, this is the standard coordinate subspace of dimension $m - n$. Hence the source chart is adapted to the level set. Since p was arbitrary, the level set is an embedded submanifold. □

Proposition 3.17 (Tangent space of a regular level set)**.** *Under the hypotheses of Theorem 3.16, for every $p \in f^{-1}(q)$,*

$$T_p\big(f^{-1}(q)\big) = \ker(df_p) \subseteq T_pM.$$

Proof. Let

$$i : f^{-1}(q) \longrightarrow M$$

be the inclusion. Since $f \circ i$ is constant, the chain rule gives

$$df_p \circ di_p = 0.$$

After identifying $T_p(f^{-1}(q))$ with its image under di_p, this gives

$$T_p\big(f^{-1}(q)\big) \subseteq \ker(df_p).$$

Both spaces have dimension $m - n$: the first by the regular level set theorem, and the second by the rank-nullity theorem because df_p is surjective. Therefore they are equal. □

Theorem 3.18 (Constant-rank level set theorem)**.** *Let $f : M^m \longrightarrow N^n$ be smooth, and let $q \in N$. Suppose that f has constant rank r on a neighborhood of $f^{-1}(q)$. If $f^{-1}(q)$ is nonempty, then it is an embedded submanifold of codimension r.*

Proof. Fix $p \in f^{-1}(q)$ and apply the constant rank theorem with source and target coordinates centered at p and q. In these coordinates,

$$f(u^1, \dots, u^m) = (u^1, \dots, u^r, 0, \dots, 0).$$

The fiber over q, represented by the origin in the target chart, is therefore locally defined by

$$u^1 = \cdots = u^r = 0.$$

This is a coordinate subspace of codimension r. □

Example 3.19 (The sphere)**.** Let

$$F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}, \qquad F(x) = \|x\|^2.$$

For $p, v \in \mathbb{R}^{n+1}$,

$$dF_p(v) = 2\langle p, v\rangle.$$

Hence 1 is a regular value and

$$S^n = F^{-1}(1)$$

is an embedded n-manifold. Moreover,

$$T_pS^n = \ker(dF_p) = \{v \in \mathbb{R}^{n+1} : \langle p, v\rangle = 0\} = p^\perp.$$

3.4 Transversality

Definition 3.20. Let $f : M \longrightarrow N$ be smooth and let $S \subseteq N$ be an embedded submanifold. We say that f is *transverse to S at $p \in M$*, written

$$f \pitchfork_p S,$$

if either $f(p) \notin S$, or

$$df_p(T_pM) + T_{f(p)}S = T_{f(p)}N.$$

The map f is *transverse to S*, written $f \pitchfork S$, if it is transverse at every point of M.

Proposition 3.21. *Every submersion $f : M \longrightarrow N$ is transverse to every embedded submanifold of N.*

Proof. If $f(p) \in S$, then

$$df_p(T_pM) = T_{f(p)}N,$$

so the transversality condition is automatic. □

Theorem 3.22 (Transverse preimage theorem)**.** *Let $f : M^m \longrightarrow N^n$ be smooth, and let $S \subseteq N$ be an embedded submanifold of dimension k. If $f \pitchfork S$, then*

$$f^{-1}(S)$$

is an embedded submanifold of M of dimension

$$m + k - n.$$

For every $p \in f^{-1}(S)$,

$$T_p\big(f^{-1}(S)\big) = (df_p)^{-1}(T_{f(p)}S).$$

Proof. Fix $p \in f^{-1}(S)$ and choose an adapted chart

$$(V, y^1, \dots, y^n)$$

for S near $f(p)$ such that

$$S \cap V = \{y^{k+1} = \cdots = y^n = 0\}.$$

Define

$$G : f^{-1}(V) \longrightarrow \mathbb{R}^{n-k}$$

by

$$G = (y^{k+1} \circ f, \dots, y^n \circ f).$$

Then

$$f^{-1}(S) \cap f^{-1}(V) = G^{-1}(0).$$

The kernel of the differential of

$$(y^{k+1}, \dots, y^n) : V \longrightarrow \mathbb{R}^{n-k}$$

at a point of S is exactly the tangent space of S. Therefore the transversality condition

$$df_p(T_pM) + T_{f(p)}S = T_{f(p)}N$$

is equivalent to the surjectivity of dG_p. By Proposition 3.3, G is a submersion near p. The regular level set theorem now shows that $G^{-1}(0)$ is locally an embedded submanifold of codimension $n - k$.

Moreover,

$$\begin{aligned} T_p\big(f^{-1}(S)\big) &= \ker(dG_p) \\ &= \{v \in T_pM : df_p(v) \in T_{f(p)}S\} \\ &= (df_p)^{-1}(T_{f(p)}S). \end{aligned}$$

The dimension formula follows from the codimension. □

Definition 3.23. Two embedded submanifolds $S, T \subseteq M$ are *transverse*, written

$$S \pitchfork T,$$

if for every $p \in S \cap T$,

$$T_pS + T_pT = T_pM.$$

Corollary 3.24 (Transverse intersections). *Let $S, T \subseteq M^m$ be transverse embedded submanifolds of dimensions s and t. Then $S \cap T$ is an embedded submanifold of M of dimension*

$$s + t - m,$$

and

$$T_p(S \cap T) = T_pS \cap T_pT$$

for every $p \in S \cap T$.

Proof. Let $i_S : S \longrightarrow M$ be the inclusion. The condition $S \pitchfork T$ is precisely the condition that $i_S \pitchfork T$. Hence the transverse preimage theorem shows that

$$i_S^{-1}(T) = S \cap T$$

is an embedded submanifold of S of dimension

$$s + t - m.$$

The composite inclusion $S \cap T \longrightarrow S \longrightarrow M$ is an embedding, so its image is an embedded submanifold of M. The tangent-space formula in Theorem 3.22 gives

$$T_p(S \cap T) = (di_S)_p^{-1}(T_pT) = T_pS \cap T_pT.$$

□

3.5 Fiber Products

Definition 3.25. Let

$$f : M \longrightarrow P, \qquad g : N \longrightarrow P$$

be smooth. The maps f and g are *transverse*, written

$$f \pitchfork g,$$

if for every pair (x, y) satisfying $f(x) = g(y) = z$,

$$df_x(T_xM) + dg_y(T_yN) = T_zP.$$

Their set-theoretic *fiber product* is

$$M \times_P N = \{(x, y) \in M \times N : f(x) = g(y)\}.$$

Proposition 3.26 (The diagonal). *For every p-dimensional smooth manifold P, the diagonal*

$$\Delta_P = \{(z, z) : z \in P\} \subseteq P \times P$$

is an embedded submanifold of dimension p. At (z, z),

$$T_{(z,z)}\Delta_P = \{(w, w) : w \in T_zP\}.$$

Proof. Choose a chart $y : V \longrightarrow \mathbb{R}^p$ near z. On $V \times V$, compose the product chart with the diffeomorphism

$$\mathbb{R}^p \times \mathbb{R}^p \longrightarrow \mathbb{R}^p \times \mathbb{R}^p, \qquad (u, v) \longmapsto (u, v - u).$$

In the resulting coordinates, the diagonal is given by the vanishing of the last p coordinates. Hence it is an embedded submanifold. Differentiating the diagonal embedding

$$P \longrightarrow P \times P, \qquad z \longmapsto (z, z)$$

gives the tangent-space formula. □

Proposition 3.27. *Let*

$$F = (f, g) : M \times N \longrightarrow P \times P.$$

Then

$$f \pitchfork g$$

if and only if

$$F \pitchfork \Delta_P.$$

Proof. At a point (x, y) with $f(x) = g(y) = z$, the differential of F has image

$$df_x(T_xM) \oplus dg_y(T_yN) \subseteq T_zP \oplus T_zP.$$

Consider the surjective linear map

$$T_zP \oplus T_zP \longrightarrow T_zP, \qquad (a, b) \longmapsto a - b.$$

Its kernel is $T_{(z,z)}\Delta_P$. Therefore

$$\operatorname{im}(dF_{(x,y)}) + T_{(z,z)}\Delta_P = T_zP \oplus T_zP$$

if and only if

$$df_x(T_xM) + dg_y(T_yN) = T_zP.$$

□

Theorem 3.28 (Fiber product theorem)**.** *Let*

$$f : M^m \longrightarrow P^p, \qquad g : N^n \longrightarrow P^p$$

be transverse smooth maps. Then $M \times_P N$ is an embedded submanifold of $M \times N$ of dimension

$$m + n - p.$$

At $(x, y) \in M \times_P N$, with $z = f(x) = g(y)$,

$$T_{(x,y)}(M \times_P N) = \{(u, v) \in T_xM \oplus T_yN : df_x(u) = dg_y(v)\}.$$

Equivalently,

$$T_{(x,y)}(M \times_P N) \simeq T_xM \times_{T_zP} T_yN.$$

Proof. By definition,

$$M \times_P N = (f, g)^{-1}(\Delta_P).$$

Propositions 3.26 and 3.27, together with the transverse preimage theorem, show that this is an embedded submanifold. Its dimension is

$$(m + n) + p - 2p = m + n - p.$$

The tangent-space formula follows from Theorem 3.22:

$$\begin{aligned} T_{(x,y)}(M \times_P N) &= (d(f, g)_{(x,y)})^{-1}\big(T_{(z,z)}\Delta_P\big) \\ &= \{(u, v) : df_x(u) = dg_y(v)\}. \end{aligned}$$

□

Let

$$\mathrm{pr}_M : M \times_P N \longrightarrow M, \qquad \mathrm{pr}_N : M \times_P N \longrightarrow N$$

be the restrictions of the product projections.

Theorem 3.29 (Universal property of a transverse fiber product)**.** *Let Q be a smooth manifold, and suppose that smooth maps*

$$a : Q \longrightarrow M, \qquad b : Q \longrightarrow N$$

satisfy

$$f \circ a = g \circ b.$$

Then there exists a unique smooth map

$$h : Q \longrightarrow M \times_P N$$

such that

$$\mathrm{pr}_M \circ h = a, \qquad \mathrm{pr}_N \circ h = b.$$

Proof. The universal property of the product gives a unique smooth map

$$(a, b) : Q \longrightarrow M \times N.$$

The equality $f \circ a = g \circ b$ implies that its image lies in $M \times_P N$. Since $M \times_P N$ is an embedded submanifold, Theorem 3.12 shows that the induced map

$$h : Q \longrightarrow M \times_P N$$

is smooth. Uniqueness follows from the two projection identities. ☐

Example 3.30 (Regular level sets as fiber products)**.** Let $f : M \longrightarrow P$ be smooth and let $z \in P$. The inclusion

$$\{z\} \longrightarrow P$$

is transverse to f exactly when z is a regular value. In that case,

$$M \times_P \{z\} \simeq f^{-1}(z),$$

and the fiber product theorem recovers the regular level set theorem.

Example 3.31 (Intersections as fiber products)**.** Let $S, T \subseteq M$ be transverse embedded submanifolds. Then

$$S \times_M T$$

is naturally diffeomorphic to $S \cap T$. The tangent-space formula for fiber products becomes

$$T_p(S \cap T) = T_pS \cap T_pT.$$

Thus regular level sets, transverse preimages, and transverse intersections are all governed by the same fiber-product construction. The next chapter passes from pointwise tangent spaces to smoothly varying families of vector spaces.

Chapter 4

Vector Bundles and Locally Free Sheaves

A vector bundle is a family of finite-dimensional vector spaces varying smoothly over a manifold. Its local trivializations are glued by $GL_r(\mathbb{R})$-valued transition functions, while its smooth sections form a locally free module over the structure sheaf. These two descriptions are equivalent. This chapter develops the geometric and sheaf-theoretic languages in parallel and explains how natural tensor operations and pullback bundles arise from the same local linear structure.

Throughout this chapter, bundle morphisms between bundles over a fixed base are understood to cover the identity unless another base map is displayed.

4.1 Smooth Vector Bundles and Transition Functions

Definition 4.1. Let M be a smooth manifold. A *smooth real vector bundle of rank* r over M consists of a smooth manifold E, a smooth surjective map

$$\pi : E \longrightarrow M,$$

and a vector-space structure on every fiber

$$E_p = \pi^{-1}(p),$$

such that every $p \in M$ has an open neighborhood U and a diffeomorphism

$$\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^r$$

satisfying

$$\mathrm{pr}_U \circ \Phi = \pi$$

and such that the restriction

$$\Phi_p : E_p \longrightarrow \{p\} \times \mathbb{R}^r \simeq \mathbb{R}^r$$

is linear for every $p \in U$.

The map Φ is called a *local trivialization.* A rank-one vector bundle is called a *line bundle.*

Example 4.2 (Trivial bundles)**.** For every manifold M and every finite-dimensional real vector space V, the projection

$$M \times V \longrightarrow M$$

is a vector bundle. When $V = \mathbb{R}^r$, it is denoted by

$$\underline{\mathbb{R}}^r_M$$

or simply by $M \times \mathbb{R}^r$.

Definition 4.3. Let $\pi_E : E \longrightarrow M$ and $\pi_F : F \longrightarrow N$ be smooth vector bundles. A *bundle morphism over* $f : M \longrightarrow N$ is a smooth map

$$\Phi : E \longrightarrow F$$

such that

$$\pi_F \circ \Phi = f \circ \pi_E$$

and such that each induced map

$$\Phi_p : E_p \longrightarrow F_{f(p)}$$

is linear.

A bundle morphism over id_M is called a *bundle morphism over* M. A bundle morphism is a *bundle isomorphism* if it has an inverse which is also a bundle morphism.

Proposition 4.4 (Local matrix criterion)**.** *Let $\Phi : E \longrightarrow F$ be a fiber-preserving map over $f : M \longrightarrow N$. Choose local trivializations of E over $U \subseteq M$ and of F over $V \subseteq N$, with $f(U) \subseteq V$. Then Φ is a smooth bundle morphism on $E|_U$ if and only if in these trivializations it has the form*

$$(p, v) \longmapsto \big(f(p), A(p)v\big),$$

where

$$A : U \longrightarrow \mathrm{Hom}(\mathbb{R}^r, \mathbb{R}^s)$$

is smooth.

Proof. Fiberwise linearity forces the second component to be linear in v, so it is represented by a matrix $A(p)$. If Φ is smooth, the columns of $A(p)$ are obtained by evaluating the second component of $\Phi(p, -)$ on the standard basis vectors, hence vary smoothly with p. Conversely, if the entries of A are smooth, then the displayed map is smooth and fiberwise linear. □

Let $E \longrightarrow M$ be a rank-r vector bundle and let

$$\Phi_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^r$$

be local trivializations. On an overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$, the composite

$$\Phi_\alpha \circ \Phi_\beta^{-1}$$

is a bundle automorphism of the trivial bundle over $U_{\alpha\beta}$. Hence it has the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = \big(p, g_{\alpha\beta}(p)v\big).$$

Theorem 4.5 (Transition functions)**.** *The maps*

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL_r(\mathbb{R})$$

are smooth and satisfy

$$g_{\alpha\alpha} = I, \qquad g_{\alpha\beta}^{-1} = g_{\beta\alpha}, \qquad g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$$

on all relevant intersections.

Proof. Smoothness follows from Proposition 4.4. The first identity expresses that a trivialization composed with its inverse is the identity. The second follows by taking inverses. On a triple overlap, associativity gives

$$\Phi_\alpha \circ \Phi_\gamma^{-1} = (\Phi_\alpha \circ \Phi_\beta^{-1}) \circ (\Phi_\beta \circ \Phi_\gamma^{-1}),$$

which is precisely the cocycle identity. □

The preceding construction can be reversed.

Theorem 4.6 (Reconstruction from transition functions)**.** *Let $\{U_\alpha\}$ be an open cover of M, and suppose that smooth maps*

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL_r(\mathbb{R})$$

satisfy

$$g_{\alpha\alpha} = I, \qquad g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}.$$

Then there exists a rank-r smooth vector bundle $E \longrightarrow M$ whose transition functions are the given maps. It is unique up to a bundle isomorphism compatible with the prescribed local trivializations.

Proof. Form the disjoint union

$$\coprod_\alpha U_\alpha \times \mathbb{R}^r$$

and impose the relation

$$(p, v)_\beta \sim (p, g_{\alpha\beta}(p)v)_\alpha.$$

The cocycle identities imply that this is an equivalence relation. Let E be the quotient set. Projection to the first coordinate descends to a map

$$\pi : E \longrightarrow M.$$

For each α, the natural map

$$U_\alpha \times \mathbb{R}^r \longrightarrow \pi^{-1}(U_\alpha)$$

is bijective, and the transition maps between these identifications are

$$(p, v) \longmapsto (p, g_{\alpha\beta}(p)v).$$

They are diffeomorphisms satisfying the cocycle condition. The standard gluing construction for manifolds therefore equips E with a unique smooth structure for which these maps are charts. Fiberwise addition and scalar multiplication are the ordinary operations in each local trivialization and agree on overlaps because the transition functions are linear.

If E' is another bundle with the same local data, the identity maps in the chosen trivializations agree on overlaps. They therefore glue to a bundle isomorphism $E \longrightarrow E'$. □

Example 4.7 (The tangent and cotangent bundles)**.** The tangent bundle $TM \longrightarrow M$ constructed in Chapter 2 is a rank-n vector bundle when $\dim M = n$. In coordinates $(x^1, \dots, x^n)$, the coordinate vector fields

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$$

form a local frame. The dual vector bundle is the cotangent bundle $T^*M \longrightarrow M$, with dual local frame

$$dx^1, \dots, dx^n.$$

Example 4.8 (The tautological line bundle)**.** Let $\mathbb{RP}^n$ be the space of lines in $\mathbb{R}^{n+1}$. Define

$$L = \left\{(\ell, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in \ell\right\}.$$

Projection to ℓ makes L a line bundle. Over the standard chart

$$U_i = \{[x^0 : \dots : x^n] : x^i \neq 0\},$$

every line has a unique representative whose ith coordinate is 1. This representative is a nowhere-vanishing local section and hence gives a local trivialization. The bundle is called tautological because its fiber over a line ℓ is the line ℓ itself.

4.2 Sections, Frames, and the Section Sheaf

Definition 4.9. Let $E \longrightarrow M$ be a vector bundle and let $U \subseteq M$ be open. A *smooth section of E over U* is a smooth map

$$s : U \longrightarrow E$$

satisfying

$$\pi \circ s = \mathrm{id}_U .$$

The vector space of smooth sections over U is denoted by

$$\Gamma(U, E).$$

When $U = M$, we also write $\Gamma(E)$.

Pointwise addition and multiplication by smooth functions make $\Gamma(U, E)$ a module over $C^\infty(U)$.

Definition 4.10. A *local frame* of a rank-r vector bundle E over U is an ordered r-tuple of smooth sections

$$(e_1, \dots, e_r)$$

such that

$$(e_1(p), \dots, e_r(p))$$

is a basis of E_p for every $p \in U$.

Theorem 4.11 (Frame criterion)**.** *A rank-r vector bundle E is trivial over an open set U if and only if it admits a local frame over U.*

Proof. A trivialization pulls the standard basis of $\mathbb{R}^r$ back to a local frame. Conversely, if $(e_1, \dots, e_r)$ is a local frame, define

$$\Psi : U \times \mathbb{R}^r \longrightarrow E|_U, \qquad \Psi(p, a^1, \dots, a^r) = \sum_{i=1}^{r} a^i e_i(p).$$

This is a fiberwise linear bijection. In any pre-existing local trivialization, it is represented by an invertible smooth matrix. Its inverse is therefore smooth, so Ψ is a bundle isomorphism. □

Proposition 4.12 (Local expression of a section)**.** *Let $(e_1, \dots, e_r)$ be a local frame of E over U. Every section $s \in \Gamma(U, E)$ has a unique expression*

$$s = \sum_{i=1}^{r} s^i e_i$$

with $s^i \in C^\infty(U)$. Conversely, every such expression defines a smooth section.

Proof. Pointwise linear algebra gives unique coefficient functions s^i. In the trivialization defined by the frame, the section is represented by the map

$$p \longmapsto (s^1(p), \dots, s^r(p)).$$

Thus the section is smooth exactly when all coefficients are smooth. □

Theorem 4.13 (The sheaf of smooth sections)**.** *The assignment*

$$U \longmapsto \Gamma(U, E)$$

with the usual restriction maps defines a sheaf of $\mathscr{C}^\infty_M$-modules, denoted by $\mathscr{E}$ or $\mathscr{S}(E)$.

Proof. Suppose $U = \bigcup U_\alpha$ and sections

$$s_\alpha \in \Gamma(U_\alpha, E)$$

agree on overlaps. They glue uniquely as set maps to a map $s : U \to E$. Smoothness is local, so s is smooth. Since each s_α is a section, $\pi \circ s = \mathrm{id}_U$. Uniqueness is immediate.

The $\mathscr{C}^\infty_M$-module structure is defined pointwise and commutes with restriction. □

Proposition 4.14. *If E has rank r, then its section sheaf $\mathscr{E}$ is locally free of rank r as a $\mathscr{C}^\infty_M$-module.*

Proof. On an open set U admitting a frame $(e_1, \dots, e_r)$, Proposition 4.12 gives an isomorphism

$$(\mathscr{C}^\infty_M|_U)^{\oplus r} \longrightarrow \mathscr{E}|_U, \qquad (f^1, \dots, f^r) \longmapsto \sum_i f^i e_i.$$

□

A bundle morphism over M acts on sections pointwise.

Proposition 4.15 (Bundle morphisms and section sheaves)**.** *Let $\Phi : E \longrightarrow F$ be a bundle morphism over M. Then*

$$\Phi_* : \mathscr{E} \longrightarrow \mathscr{F}, \qquad s \longmapsto \Phi \circ s,$$

is a morphism of $\mathscr{C}^\infty_M$-modules. Conversely, every $\mathscr{C}^\infty_M$-linear morphism

$$\eta : \mathscr{E} \longrightarrow \mathscr{F}$$

arises from a unique bundle morphism $\Phi : E \longrightarrow F$ over M.

Proof. The first assertion follows from fiberwise linearity of Φ.

Conversely, choose frames $(e_1, \dots, e_r)$ of E and $(u_1, \dots, u_s)$ of F over an open set U. Write

$$\eta(e_j) = \sum_{i=1}^{s} a^i_j u_i, \qquad a^i_j \in C^\infty(U).$$

The matrix $A = (a^i_j)$ defines a smooth bundle morphism on $E|_U$ by the local matrix criterion. On overlaps, the local morphisms induce the same map on every local section because they were all constructed from η. Evaluating on local frames shows that they agree pointwise, so they glue to a global bundle morphism Φ. Uniqueness follows because local frame values span every fiber. □

4.3 Vector Bundles and Locally Free Sheaves

Definition 4.16. A sheaf $\mathscr{F}$ of $\mathscr{C}^\infty_M$-modules is *locally free of rank r* if every point of M has an open neighborhood U such that

$$\mathscr{F}|_U \simeq (\mathscr{C}^\infty_M|_U)^{\oplus r}.$$

It is *locally free of finite rank* if the rank is finite and locally constant on M.

Let $\mathrm{Vect}(M)$ denote the category of finite-rank smooth vector bundles over M and bundle morphisms over M. Let

$$\mathrm{LocFree}_{\mathrm{fin}}(\mathscr{C}^\infty_M)$$

denote the category of finite-rank locally free $\mathscr{C}^\infty_M$-modules.

Theorem 4.17 (Vector bundles and locally free sheaves)**.** *The section-sheaf construction defines an equivalence of categories*

$$\mathscr{S} : \mathrm{Vect}(M) \xrightarrow{\simeq} \mathrm{LocFree}_{\mathrm{fin}}(\mathscr{C}^\infty_M).$$

Proof. Proposition 4.14 shows that $\mathscr{S}$ lands in the stated category, and Proposition 4.15 shows that it is fully faithful.

It remains to prove essential surjectivity. Let $\mathscr{F}$ be locally free of rank r, after restricting to a connected component if necessary. Choose an open cover $\{U_\alpha\}$ and isomorphisms

$$\theta_\alpha : \mathscr{F}|_{U_\alpha} \longrightarrow (\mathscr{C}^\infty_M|_{U_\alpha})^{\oplus r}.$$

On an overlap, the composite

$$\theta_\alpha \circ \theta_\beta^{-1}$$

is an automorphism of a free module sheaf. Relative to the standard basis it is represented by an invertible matrix of smooth functions, hence by a smooth map

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow GL_r(\mathbb{R}).$$

Associativity of composition gives the cocycle identities. By Theorem 4.6, these transition functions construct a vector bundle $E \longrightarrow M$.

The local identifications between $\mathscr{F}|_{U_\alpha}$ and the section sheaf of $E|_{U_\alpha}$ agree on overlaps because both are glued by the same matrices $g_{\alpha\beta}$. They therefore glue to an isomorphism

$$\mathscr{F} \simeq \mathscr{S}(E).$$

□

The fiber of a vector bundle can be recovered directly from the stalk of its section sheaf.

Proposition 4.18 (Fiber from the stalk)**.** *Let $E \longrightarrow M$ be a vector bundle with section sheaf $\mathscr{E}$. For every $p \in M$, evaluation induces a natural isomorphism*

$$\mathscr{E}_p/\mathfrak{m}_p\mathscr{E}_p \xrightarrow{\simeq} E_p.$$

Proof. Evaluation of a section germ at p gives a surjective linear map

$$\mathrm{ev}_p : \mathscr{E}_p \longrightarrow E_p.$$

Surjectivity follows by choosing a local frame and taking a constant linear combination of its members. Clearly

$$\mathfrak{m}_p\mathscr{E}_p \subseteq \ker(\mathrm{ev}_p).$$

Choose a local frame $(e_1, \ldots, e_r)$ and write

$$s = \sum_i f^i e_i.$$

If $s(p) = 0$, then every $f^i(p) = 0$, so $f^i \in \mathfrak{m}_p$. Hence $s \in \mathfrak{m}_p\mathscr{E}_p$. The kernel is therefore exactly $\mathfrak{m}_p\mathscr{E}_p$, and the first isomorphism theorem gives the result. □

Remark 4.19. For an arbitrary locally free sheaf $\mathscr{F}$, the vector space

$$\mathscr{F}_p/\mathfrak{m}_p\mathscr{F}_p$$

should therefore be regarded as its fiber at p. Under Theorem 4.17, this construction recovers the ordinary geometric fiber.

4.4 Natural Operations on Vector Bundles

Finite-dimensional linear algebra may be applied fiberwise to vector bundles.

Theorem 4.20 (Direct sum, dual, tensor product, and exterior powers)**.** *Let E and F be finite-rank vector bundles over M. The fiberwise vector spaces*

$$E_p \oplus F_p, \qquad E_p^*, \qquad E_p \otimes F_p, \qquad \mathrm{Hom}(E_p, F_p), \qquad \Lambda^k E_p$$

assemble naturally into smooth vector bundles

$$E \oplus F, \qquad E^*, \qquad E \otimes F, \qquad \mathrm{Hom}(E, F), \qquad \Lambda^k E.$$

There is a natural bundle isomorphism

$$\mathrm{Hom}(E, F) \simeq E^* \otimes F.$$

If E has rank r, the line bundle

$$\det(E) = \Lambda^r E$$

is called its determinant bundle*.*

Proof. Choose transition functions $g_{\alpha\beta}$ for E and $h_{\alpha\beta}$ for F. The corresponding transition functions are obtained by applying the relevant linear-algebraic construction:

$$g_{\alpha\beta} \oplus h_{\alpha\beta}, \qquad (g_{\alpha\beta}^{-1})^*, \qquad g_{\alpha\beta} \otimes h_{\alpha\beta},$$

$$A \longmapsto h_{\alpha\beta} A g_{\alpha\beta}^{-1}, \qquad \Lambda^k g_{\alpha\beta}.$$

These maps are smooth and satisfy the cocycle identities. Reconstruction therefore produces the desired bundles. The isomorphism

$$E_p^* \otimes F_p \simeq \mathrm{Hom}(E_p, F_p)$$

is canonical in each fiber and has constant matrix in local frames, hence is smooth. □

The sheaf equivalence respects these operations.

Proposition 4.21 (Sheaf form of the natural operations)**.** *Let $\mathscr{E}$ and $\mathscr{F}$ be the section sheaves of E and F. There are natural isomorphisms*

$$\mathscr{S}(E \oplus F) \simeq \mathscr{E} \oplus \mathscr{F},$$

$$\mathscr{S}(E^*) \simeq \mathscr{H}om_{\mathscr{C}_M^\infty}(\mathscr{E}, \mathscr{C}_M^\infty),$$

$$\mathscr{S}(E \otimes F) \simeq \mathscr{E} \otimes_{\mathscr{C}_M^\infty} \mathscr{F},$$

and

$$\mathscr{S}(\Lambda^k E) \simeq \Lambda^k_{\mathscr{C}_M^\infty} \mathscr{E}.$$

Proof. All assertions are local. On an open set where E and F are trivial, they reduce to the corresponding canonical isomorphisms for finite free modules. These local isomorphisms are natural and therefore agree on overlaps. □

4.5 Pullback Bundles

Let $f : N \longrightarrow M$ be smooth and let $\pi : E \longrightarrow M$ be a vector bundle.

Definition 4.22. The *pullback set* is the fiber product

$$f^*E = N \times_M E = \{(q, e) \in N \times E : f(q) = \pi(e)\}.$$

It is equipped with the projection

$$\pi_N : f^*E \longrightarrow N, \qquad (q, e) \longmapsto q.$$

Theorem 4.23 (Pullback bundle)**.** *The set f^*E has a unique natural smooth vector-bundle structure for which*

$$\widetilde{f} : f^*E \longrightarrow E, \qquad (q, e) \longmapsto e,$$

is a bundle morphism over f. If E has transition functions $g_{\alpha\beta}$, then f^*E has transition functions

$$g_{\alpha\beta} \circ f$$

on the cover $\{f^{-1}(U_\alpha)\}$.

Proof. A local trivialization

$$E|_{U_\alpha} \simeq U_\alpha \times \mathbb{R}^r$$

pulls back to a bijection

$$(f^*E)|_{f^{-1}(U_\alpha)} \longrightarrow f^{-1}(U_\alpha) \times \mathbb{R}^r.$$

The changes of trivialization are

$$(q, v) \longmapsto \big(q, g_{\alpha\beta}(f(q))v\big).$$

They are smooth and satisfy the cocycle identities, so the reconstruction theorem gives the claimed vector-bundle structure. In these trivializations, $\widetilde{f}$ is

$$(q, v) \longmapsto (f(q), v),$$

and is therefore smooth and fiberwise linear. Uniqueness follows because the pulled-back trivializations are required to be smooth. ◻

Theorem 4.24 (Universal property of the pullback bundle)**.** *Let $G \longrightarrow N$ be a vector bundle and let*

$$\Phi : G \longrightarrow E$$

be a bundle morphism over $f : N \longrightarrow M$. There exists a unique bundle morphism over N

$$\widehat{\Phi} : G \longrightarrow f^*E$$

such that

$$\widetilde{f} \circ \widehat{\Phi} = \Phi.$$

Proof. The only possible map is

$$\widehat{\Phi}(v) = \big(\pi_G(v), \Phi(v)\big).$$

The defining relation of the fiber product holds because Φ covers f. In local trivializations the map is represented by the same smooth matrix as Φ, so it is a smooth bundle morphism. Uniqueness is immediate from the two projections of the fiber product. ◻

Proposition 4.25 (Functoriality of pullback)**.** *There are canonical bundle isomorphisms*

$$\mathrm{id}_M^* E \simeq E$$

and, for $P \xrightarrow{g} N \xrightarrow{f} M$*,*

$$(f \circ g)^* E \simeq g^*(f^* E).$$

Pullback also commutes naturally with direct sums, duals, tensor products, Hom bundles, exterior powers, and determinant bundles.

Proof. Every asserted map is the evident fiberwise canonical isomorphism. In local pulled-back frames its matrix is constant, so it is smooth. The coherence identities follow from the corresponding identities for fiber products and finite-dimensional vector spaces. □

Proposition 4.26 (Sheaf-theoretic pullback)**.** *Let* $\mathscr{E}$ *be the section sheaf of* E*. The section sheaf of* f^*E *is naturally isomorphic to*

$$f^{-1}\mathscr{E} \otimes_{f^{-1}\mathscr{C}^\infty_M} \mathscr{C}^\infty_N.$$

Proof. The assertion is local on N. If E is trivial of rank r over U, then both sides restrict over $f^{-1}(U)$ to

$$(\mathscr{C}^\infty_N|_{f^{-1}(U)})^{\oplus r}.$$

The local identifications transform on overlaps by the matrices $g_{\alpha\beta} \circ f$, so they glue to the stated global isomorphism. □

The constructions of this chapter provide the tangent, cotangent, and exterior power bundles used below. We next develop partitions of unity, which supply the global extension and acyclicity arguments needed for de Rham theory.

Chapter 5

Partitions of Unity and Fine Sheaves

The local theory of smooth manifolds is governed by charts and locally free modules. The distinctive global feature of the smooth category is the existence of partitions of unity. They make it possible to average local constructions, extend local sections, construct bundle metrics, and prove that every sheaf of modules over $\mathscr{C}^\infty_M$ is cohomologically acyclic. The purpose of this chapter is to develop these consequences in a form suited to later use in de Rham theory.

5.1 Supports and Locally Finite Families

Definition 5.1. Let $s \in \Gamma(U, E)$ be a section of a vector bundle. Its *support* is

$$\operatorname{supp}(s) = \overline{\{p \in U : s(p) \neq 0\}}^{U}.$$

For a function $f \in C^\infty(U)$, this agrees with the usual support.

Definition 5.2. A family of subsets $\{A_i\}_{i \in I}$ of a topological space is *locally finite* if every point has a neighborhood meeting only finitely many A_i. A family of sections is locally finite if the family of their supports is locally finite.

Lemma 5.3 (Locally finite sums)**.** *Let $\{s_i\}_{i \in I}$ be a locally finite family of smooth sections of a vector bundle $E \longrightarrow M$. Then*

$$s = \sum_{i \in I} s_i$$

is a well-defined smooth section. The analogous statement holds for smooth functions and for sections of any sheaf of $\mathscr{C}^\infty_M$-modules.

Proof. Near any point, only finitely many s_i are nonzero. Thus the sum is locally a finite sum of smooth sections and is therefore smooth. The same argument applies to module sheaves. □

We shall use the following standard consequence of paracompactness and local compactness.

Theorem 5.4 (Locally finite precompact refinement)**.** *Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of a smooth manifold M. There exist a locally finite index set of open subsets $\{W_i\}_{i \in I}$, an open cover $\{V_i\}_{i \in I}$, and a map $a : I \longrightarrow A$ such that*

$$\overline{V_i} \subseteq W_i, \qquad \overline{W_i} \text{ is compact}, \qquad \overline{W_i} \subseteq U_{a(i)}.$$

The sets W_i may be chosen to lie in coordinate neighborhoods.

We take this theorem as a standard topological input. Its proof uses second countability, local compactness, and paracompactness of manifolds.

5.2 Bump Functions

The analytic input behind partitions of unity is the existence of smooth cutoff functions on Euclidean space.

Lemma 5.5 (Euclidean bump functions)**.** *Let $K \subseteq U \subseteq \mathbb{R}^n$, where K is compact and U is open. There exists $\rho \in C_c^\infty(U)$ such that*

$$0 \le \rho \le 1$$

and

$$\rho = 1$$

on a neighborhood of K.

The usual proof begins with the smooth function

$$\eta(t) = \begin{cases} \exp(-1/t), & t > 0, \\ 0, & t \le 0, \end{cases}$$

and combines translated and rescaled radial cutoffs. We use the lemma as a standard fact from smooth analysis.

Theorem 5.6 (Bump functions on manifolds)**.** *Let $K \subseteq U \subseteq M$, where K is compact and U is open. There exists $\rho \in C_c^\infty(U)$ satisfying*

$$0 \le \rho \le 1$$

and

$$\rho = 1$$

on a neighborhood of K.

Proof. For every $p \in K$, choose a coordinate neighborhood W_p with compact closure contained in U. By the Euclidean bump-function lemma, there exists a smooth function ψ_p supported in W_p and positive on a neighborhood of p. Compactness gives finitely many points $p_1, \dots, p_m$ such that the sets on which ψ_{p_j} is positive cover K. Put

$$\psi = \sum_{j=1}^{m} \psi_{p_j}.$$

Then $\psi > 0$ on K, and its support is a compact subset of U. By compactness, there is $c > 0$ such that $\psi \ge c$ on K. Choose a smooth function

$$\chi : \mathbb{R} \longrightarrow [0, 1]$$

with $\chi = 0$ on a neighborhood of 0 and $\chi = 1$ on $[c, \infty)$. Then

$$\rho = \chi \circ \psi$$

has compact support in U and equals 1 on a neighborhood of K. $\square$

5.3 Partitions of Unity

Definition 5.7. A family $\{\rho_i\}_{i\in I}$ of smooth functions on M is a *smooth partition of unity* if

(1) $0 \le \rho_i \le 1$ for all i;

(2) the family $\{\mathrm{supp}(\rho_i)\}$ is locally finite;

(3)
$$\sum_{i\in I} \rho_i = 1.$$

It is *subordinate* to an open cover $\{U_\alpha\}_{\alpha\in A}$ if for every i there is an index $a(i) \in A$ such that

$$\mathrm{supp}(\rho_i) \subseteq U_{a(i)}.$$

Theorem 5.8 (Existence of partitions of unity)**.** *Every open cover of a smooth manifold admits a smooth locally finite partition of unity subordinate to it.*

Proof. Let $\{U_\alpha\}$ be an open cover. Choose the families V_i and W_i from Theorem 5.4. By Theorem 5.6, there is a smooth function ψ_i such that

$$0 \le \psi_i \le 1, \qquad \psi_i = 1 \text{ on a neighborhood of } \overline{V_i}, \qquad \mathrm{supp}(\psi_i) \subseteq W_i.$$

Because the W_i are locally finite, the sum

$$\psi = \sum_i \psi_i$$

is smooth. Since the V_i cover M, one has $\psi > 0$ everywhere. Define

$$\rho_i = \frac{\psi_i}{\psi}.$$

Then $\{\rho_i\}$ is locally finite,

$$\sum_i \rho_i = 1,$$

and

$$\mathrm{supp}(\rho_i) \subseteq \mathrm{supp}(\psi_i) \subseteq W_i \subseteq U_{a(i)}.$$

Thus it is subordinate to the original cover. □

Corollary 5.9 (Cutoff near a closed set)**.** *Let $A \subseteq U \subseteq M$, where A is closed and U is open. There exists $\rho \in C^\infty(M)$ such that*

$$0 \le \rho \le 1, \qquad \rho = 1 \textit{ on a neighborhood of } A, \qquad \mathrm{supp}(\rho) \subseteq U.$$

Proof. Apply Theorem 5.8 to the cover

$$\{U, M \setminus A\}.$$

Let ρ_U and $\rho_{M\setminus A}$ denote the sums of all members of the partition whose supports lie in the indicated sets. Then

$$\rho_U + \rho_{M\setminus A} = 1.$$

The support of $\rho_{M\setminus A}$ is disjoint from A, so this function vanishes on a neighborhood of A. Therefore $\rho_U = 1$ near A, and its support is contained in U. □

Corollary 5.10 (Smooth Urysohn lemma). *Let A and B be disjoint closed subsets of M. There exists a smooth function*

$$f : M \longrightarrow [0,1]$$

such that

$$f|_A = 0, \qquad f|_B = 1.$$

Proof. Apply Corollary 5.9 to

$$B \subseteq M \setminus A.$$

□

5.4 Globalizing Sections and Bundle Constructions

Lemma 5.11 (Extension by zero). *Let $E \longrightarrow M$ be a vector bundle, let $U \subseteq M$ be open, and let $s \in \Gamma(U, E)$. Suppose $\rho \in C^\infty(M)$ satisfies*

$$\operatorname{supp}(\rho) \subseteq U.$$

Then the section

$$\widetilde{s}(p) = \begin{cases} \rho(p)s(p), & p \in U, \\ 0, & p \notin U \end{cases}$$

is a smooth global section of E.

Proof. The formula is smooth on U. If $p \notin \operatorname{supp}(\rho)$, then ρ vanishes on a neighborhood of p, and the section is identically zero there. These two local descriptions agree on their overlap, so they glue to a smooth global section. □

Proposition 5.12 (Extension from a neighborhood of a closed set). *Let $A \subseteq M$ be closed, let U be a neighborhood of A, and let $s \in \Gamma(U, E)$. There exists a global section*

$$\widetilde{s} \in \Gamma(M, E)$$

which agrees with s on a neighborhood of A.

Proof. Choose a cutoff function ρ with

$$\rho = 1 \text{ near } A, \qquad \operatorname{supp}(\rho) \subseteq U.$$

Apply Lemma 5.11 to ρs. □

Corollary 5.13 (Prescribed fiber value). *For every $p \in M$ and every $v \in E_p$, there exists a global smooth section $s \in \Gamma(M, E)$ satisfying*

$$s(p) = v.$$

Proof. Choose a local frame near p and a local section taking the value v at p. Extend it by Proposition 5.12, with $A = \{p\}$. □

Theorem 5.14 (Existence of bundle metrics). *Every finite-rank real smooth vector bundle $E \longrightarrow M$ admits a smooth positive-definite bundle metric.*

Proof. Choose a trivializing cover $\{U_\alpha\}$ and on each $E|_{U_\alpha}$ pull back the standard Euclidean inner product to obtain a local metric g_α. Let $\{\rho_\alpha\}$ be a subordinate partition of unity. Define

$$g_p(v, w) = \sum_\alpha \rho_\alpha(p)(g_\alpha)_p(v, w).$$

The sum is locally finite and hence smooth. It is symmetric and bilinear. For $v \neq 0$, at least one $\rho_\alpha(p)$ is positive, and every local metric is positive definite, so

$$g_p(v, v) > 0.$$

Thus g is a bundle metric. □

5.5 Fine Sheaves

We now express the partition-of-unity mechanism directly in sheaf-theoretic language.

Definition 5.15. Let $\mathscr{F}$ be a sheaf of abelian groups on M and let $h : \mathscr{F} \longrightarrow \mathscr{F}$ be an endomorphism. Its support is

$$\operatorname{supp}(h) = \overline{\{p \in M : h_p \neq 0\}}.$$

The sheaf $\mathscr{F}$ is *fine* if for every locally finite open cover $\{U_i\}_{i\in I}$ there exist sheaf endomorphisms

$$h_i : \mathscr{F} \longrightarrow \mathscr{F}$$

such that

$$\operatorname{supp}(h_i) \subseteq U_i,$$

the family $\{h_i\}$ is locally finite, and

$$\sum_i h_i = \mathrm{id}_{\mathscr{F}}.$$

Theorem 5.16 ($\mathscr{C}^\infty_M$-modules are fine)**.** *Every sheaf of $\mathscr{C}^\infty_M$-modules is fine.*

Proof. Let $\mathscr{F}$ be a $\mathscr{C}^\infty_M$-module sheaf and let $\{U_i\}$ be a locally finite open cover. Choose a subordinate partition of unity and, by summing all members assigned to the same U_i, index it as $\{\rho_i\}$ with

$$\operatorname{supp}(\rho_i) \subseteq U_i.$$

Define

$$h_i(s) = \rho_i s$$

for every local section s of $\mathscr{F}$. This is a sheaf endomorphism, and

$$\operatorname{supp}(h_i) \subseteq \operatorname{supp}(\rho_i) \subseteq U_i.$$

Local finiteness and the identity

$$\sum_i h_i(s) = \left(\sum_i \rho_i\right) s = s$$

prove that $\mathscr{F}$ is fine. $\square$

Corollary 5.17. *The section sheaf of every smooth vector bundle is fine. In particular, $\mathscr{C}^\infty_M$ itself and every finite-rank locally free $\mathscr{C}^\infty_M$-module are fine.*

5.6 Acyclicity of Fine Sheaves

The main cohomological consequence of partitions of unity is the following.

Theorem 5.18 (Acyclicity of fine sheaves)**.** *Let X be a paracompact Hausdorff space and let $\mathscr{F}$ be a fine sheaf of abelian groups on X. Then*

$$H^q(X, \mathscr{F}) = 0$$

for every $q > 0$.

Proof. We indicate the standard Čech-theoretic contraction. Let $\mathfrak{U} = \{U_i\}$ be a locally finite open cover and choose fine endomorphisms h_i subordinate to it. For a Čech q-cochain c, define an operator of degree -1 by

$$(sc)_{i_0 \ldots i_{q-1}} = \sum_j h_j\big(c_{j i_0 \ldots i_{q-1}}\big).$$

Each term is initially defined on

$$U_j \cap U_{i_0} \cap \cdots \cap U_{i_{q-1}}.$$

Because h_j is supported in U_j, the resulting section extends by zero to $U_{i_0} \cap \cdots \cap U_{i_{q-1}}$. Local finiteness makes the sum well defined. A direct alternating-sign computation gives

$$\delta s + s\delta = \mathrm{id}$$

in positive degrees. Hence

$$\check{H}^q(\mathfrak{U}, \mathscr{F}) = 0$$

for $q > 0$.

Every open cover of a paracompact Hausdorff space admits a locally finite refinement, and Čech cohomology over such refinements computes sheaf cohomology. Therefore $H^q(X, \mathscr{F}) = 0$ for $q > 0$. □

Corollary 5.19 (Acyclicity of smooth module sheaves)**.** *For every sheaf $\mathscr{F}$ of $\mathscr{C}^\infty_M$-modules,*

$$H^q(M, \mathscr{F}) = 0$$

for all $q > 0$.

Proof. Apply Theorems 5.16 and 5.18. □

Corollary 5.20 (Exactness of global sections)**.** *The global-section functor is exact on sheaves of $\mathscr{C}^\infty_M$-modules. Thus every short exact sequence*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow \Gamma(M, \mathscr{F}') \longrightarrow \Gamma(M, \mathscr{F}) \longrightarrow \Gamma(M, \mathscr{F}'') \longrightarrow 0.$$

Proof. The long exact sequence in sheaf cohomology contains

$$\Gamma(M, \mathscr{F}) \longrightarrow \Gamma(M, \mathscr{F}'') \longrightarrow H^1(M, \mathscr{F}').$$

The last group vanishes by Corollary 5.19, so the map on global sections is surjective. Left exactness is general. □

Remark 5.21*.* This exactness sharply distinguishes the smooth category from algebraic and complex-analytic geometry. The existence of partitions of unity makes every $\mathscr{C}^\infty_M$-module sheaf acyclic, but the resulting global lifts and extensions are generally noncanonical and therefore need not be compatible with additional structures.

Chapter 6

Differential Forms and the de Rham Complex

The cotangent bundle records first-order linear measurements on tangent vectors. Its exterior powers assemble alternating covariant tensors, and their smooth sections are differential forms. The exterior product makes all forms into a graded-commutative algebra, while the exterior derivative equips this algebra with a differential. The resulting de Rham complex is contravariantly functorial in the manifold and is locally exact in positive degrees.

6.1 Differential Forms as Sections

Let M be a smooth manifold of dimension n. By Chapter 4, every exterior power

$$\Lambda^k T^*M$$

is a smooth vector bundle. Its fiber at p is

$$\Lambda^k T_p^*M,$$

the space of alternating k-linear forms on T_pM.

Definition 6.1. Let $U \subseteq M$ be open. A *differential k-form on U* is a smooth section of the vector bundle

$$\Lambda^k T^*U \longrightarrow U.$$

The space of differential k-forms on U is denoted by

$$\Omega^k(U) = \Gamma(U, \Lambda^k T^*U).$$

The associated sheaf of differential k-forms is denoted by

$$\Omega^k_M.$$

Thus

$$\Omega^k_M(U) = \Omega^k(U).$$

A differential k-form ω assigns to each $p \in U$ an alternating multilinear map

$$\omega_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ factors}} \longrightarrow \mathbb{R}$$

which varies smoothly with p.

By convention,

$$\Lambda^0 T^*M = M \times \mathbb{R},$$

so

$$\Omega^0(U) = C^\infty(U).$$

Since T_pM has dimension n,

$$\Omega^k(U) = 0$$

for $k > n$.

Proposition 6.2. *The sheaf Ω^k_M is a locally free $\mathscr{C}^\infty_M$-module of rank*

$$\binom{n}{k}.$$

Equivalently, Ω^k_M is the section sheaf of the vector bundle $\Lambda^k T^*M$.

Proof. The cotangent bundle has rank n. The exterior power $\Lambda^k T^*M$ therefore has rank $\binom{n}{k}$. The conclusion follows from the equivalence between finite-rank smooth vector bundles and finite-rank locally free $\mathscr{C}^\infty_M$-modules established in Theorem 4.17. □

Let $(U, x^1, \dots, x^n)$ be a coordinate chart. The coordinate covectors

$$dx^1, \dots, dx^n$$

form a local frame of $T^*M|_U$. For a strictly increasing multi-index

$$I = (i_1 < \cdots < i_k),$$

write

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Theorem 6.3 (Local expression of differential forms)**.** *Let $(U, x^1, \dots, x^n)$ be a coordinate chart. The forms*

$$dx^I, \qquad I = (i_1 < \cdots < i_k),$$

form a local frame of $\Lambda^k T^*M|_U$. Consequently, every $\omega \in \Omega^k(U)$ has a unique expression

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where

$$\omega_{i_1 \cdots i_k} \in C^\infty(U).$$

Proof. At each $p \in U$, the covectors

$$dx^1\big|_p, \dots, dx^n\big|_p$$

form a basis of T_p^*M. The standard basis theorem for exterior powers shows that the wedges

$$dx^{i_1}\big|_p \wedge \cdots \wedge dx^{i_k}\big|_p, \qquad i_1 < \cdots < i_k,$$

form a basis of $\Lambda^k T_p^*M$. These sections are smooth, so they form a local frame. The unique coefficient functions in this frame are smooth by the local criterion for smooth sections. □

Example 6.4. On a coordinate chart $(U, x^1, \dots, x^n)$, a 1-form has the form

$$\alpha = \sum_{i=1}^{n} a_i \, dx^i,$$

while a top-degree form has the form

$$\omega = a \, dx^1 \wedge \cdots \wedge dx^n.$$

Definition 6.5. A *smooth vector field on U* is a smooth section of TU. We write

$$\mathfrak{X}(U) = \Gamma(U, TU).$$

If $\omega \in \Omega^k(U)$ and

$$X_1, \dots, X_k \in \mathfrak{X}(U),$$

then pointwise evaluation defines a function

$$\omega(X_1, \dots, X_k) : U \longrightarrow \mathbb{R},$$

where

$$\omega(X_1, \dots, X_k)(p) = \omega_p(X_1(p), \dots, X_k(p)).$$

Proposition 6.6. *For smooth vector fields $X_1, \dots, X_k$, the function*

$$\omega(X_1, \dots, X_k)$$

is smooth. Moreover, the assignment

$$(X_1, \dots, X_k) \longmapsto \omega(X_1, \dots, X_k)$$

is alternating and $C^\infty(U)$-multilinear.

Proof. On a coordinate chart, write

$$\omega = \sum_I \omega_I \, dx^I$$

and

$$X_j = \sum_{a=1}^{n} X_j^a \frac{\partial}{\partial x^a}.$$

Evaluation is a finite sum of products of the smooth coefficient functions ω_I and X_j^a, and is therefore smooth. Alternation and $C^\infty(U)$-multilinearity hold pointwise. ☐

The preceding construction characterizes differential forms intrinsically.

Theorem 6.7 (Multilinear characterization)**.** *For every open set $U \subseteq M$, evaluation defines a natural isomorphism of $C^\infty(U)$-modules*

$$\Omega^k(U) \xrightarrow{\simeq} \mathrm{Alt}_{C^\infty(U)}\big(\mathfrak{X}(U)^k, C^\infty(U)\big).$$

Proof. Proposition 6.6 gives the stated map. To prove injectivity, fix $p \in U$ and tangent vectors

$$v_1, \dots, v_k \in T_pM.$$

By Corollary 5.13, each v_i extends to a smooth vector field X_i on U. If a form evaluates to zero on all smooth vector fields, then

$$\omega_p(v_1, \dots, v_k) = \omega(X_1, \dots, X_k)(p) = 0.$$

Thus $\omega = 0$.

Conversely, let

$$A : \mathfrak{X}(U)^k \longrightarrow C^\infty(U)$$

be alternating and $C^\infty(U)$-multilinear. For $v_1, \dots, v_k \in T_pM$, choose smooth vector fields X_i with $X_i(p) = v_i$ and set

$$\omega_p(v_1, \dots, v_k) = A(X_1, \dots, X_k)(p).$$

We first show that this is independent of the chosen extensions. It suffices to show that if $X(p) = 0$, then

$$A(X, X_2, \dots, X_k)(p) = 0.$$

Choose a local frame $E_1, \dots, E_n$ near p and write

$$X = \sum_i f^i E_i.$$

Since $X(p) = 0$, every $f^i(p) = 0$. By C^∞-linearity,

$$A(X, X_2, \dots, X_k)(p) = \sum_i f^i(p) A(E_i, X_2, \dots, X_k)(p) = 0.$$

Thus ω_p is well defined and alternating multilinear.

To verify smoothness, choose a local frame $E_1, \dots, E_n$ of TU. The coefficients of ω in the dual exterior frame are the smooth functions

$$A(E_{i_1}, \dots, E_{i_k}).$$

Hence ω is a smooth form. By construction its evaluation map is A. □

6.2 Exterior Products

The exterior product on covectors is defined fiberwise and therefore extends to differential forms.

Definition 6.8. Let

$$\alpha \in \Omega^k(U), \qquad \beta \in \Omega^\ell(U).$$

Their *exterior product* is the $(k+\ell)$-form

$$\alpha \wedge \beta \in \Omega^{k+\ell}(U)$$

defined by

$$(\alpha \wedge \beta)_p = \alpha_p \wedge \beta_p.$$

In terms of tangent vectors, one may write

$$(\alpha \wedge \beta)_p(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma) \alpha_p(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta_p(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

Equivalently, the sum may be taken over (k, ℓ)-shuffles without the factor $1/(k!\ell!)$.

Proposition 6.9. *The exterior product of two smooth differential forms is smooth. It defines morphisms of sheaves*

$$\Omega^k_M \times \Omega^\ell_M \longrightarrow \Omega^{k+\ell}_M$$

and

$$\Omega^k_M \otimes_{\mathscr{C}^\infty_M} \Omega^\ell_M \longrightarrow \Omega^{k+\ell}_M.$$

Proof. In local coordinates, write

$$\alpha = \sum_I a_I \, dx^I, \qquad \beta = \sum_J b_J \, dx^J.$$

Then

$$\alpha \wedge \beta = \sum_{I,J} a_I b_J \, dx^I \wedge dx^J.$$

The coefficients are smooth, so the product is a smooth form. The construction commutes with restriction and is $\mathscr{C}^\infty_M$-bilinear. □

Theorem 6.10 (Algebraic properties of the exterior product)**.** *Let*

$$\alpha \in \Omega^k(U), \qquad \beta \in \Omega^\ell(U), \qquad \gamma \in \Omega^m(U).$$

Then:

(1) the exterior product is bilinear;

(2)

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma);$$

(3)

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha;$$

(4) if $f \in C^\infty(U) = \Omega^0(U)$, then

$$f \wedge \alpha = f\alpha.$$

Proof. Every identity holds pointwise and is the corresponding identity in the exterior algebra $\Lambda^\bullet T^*_p M$. □

Corollary 6.11. *The direct sum*

$$\Omega^\bullet(U) = \bigoplus_{k \geq 0} \Omega^k(U)$$

is a graded-commutative $\mathbb{R}$-algebra. Sheafwise,

$$\Omega^\bullet_M = \bigoplus_{k \geq 0} \Omega^k_M$$

is a sheaf of graded-commutative $\mathscr{C}^\infty_M$-algebras.

Example 6.12 (Coordinate volume form)**.** On a coordinate chart $(U, x^1, \dots, x^n)$, the form

$$dx^1 \wedge \cdots \wedge dx^n$$

is nowhere zero. Every top-degree form on U is a unique smooth multiple of this form.

6.3 Pullback of Differential Forms

Let

$$f : M \longrightarrow N$$

be smooth. The differential

$$d_p f : T_p M \longrightarrow T_{f(p)} N$$

induces, by alternating multilinear pullback, a map

$$\Lambda^k (d_p f)^* : \Lambda^k T^*_{f(p)} N \longrightarrow \Lambda^k T^*_p M.$$

Definition 6.13. For $\omega \in \Omega^k(V)$, where $V \subseteq N$ is open, the *pullback of ω by f* is the k-form on $f^{-1}(V)$ defined by

$$(f^*\omega)_p(v_1, \dots, v_k) = \omega_{f(p)}\big(d_p f(v_1), \dots, d_p f(v_k)\big).$$

For $k = 0$, this is the ordinary pullback of smooth functions:

$$f^* h = h \circ f.$$

Theorem 6.14 (Smoothness and local formula for pullback)**.** *The pullback of a smooth differential form is smooth. If $(x^1, \dots, x^m)$ are coordinates on M and $(y^1, \dots, y^n)$ are coordinates on N, and if*

$$\omega = \sum_I a_I \, dy^I,$$

then

$$f^*\omega = \sum_I (a_I \circ f) \, d(y^{i_1} \circ f) \wedge \cdots \wedge d(y^{i_k} \circ f).$$

In particular,

$$f^*(dy^i) = d(y^i \circ f).$$

Proof. For a 1-form dy^i, one has, for $v \in T_p M$,

$$(f^* dy^i)_p(v) = dy^i\big|_{f(p)} (d_p f(v)) = (d_p f(v))(y^i) = v(y^i \circ f) = d(y^i \circ f)\big|_p (v).$$

Thus

$$f^*(dy^i) = d(y^i \circ f).$$

Pullback is defined fiberwise by a linear map, so it preserves exterior products. Applying it to the local expression of ω gives the displayed formula. Every term in that formula is smooth, proving smoothness. $\square$

Proposition 6.15 (Functoriality of pullback)**.** *Let*

$$M \xrightarrow{f} N \xrightarrow{g} P$$

be smooth maps. Then

$$\mathrm{id}_M^* = \mathrm{id}_{\Omega^k(M)}$$

and

$$(g \circ f)^* = f^* \circ g^*.$$

These identities also hold on local sections and therefore define morphisms of sheaves

$$f^* : \Omega^k_N \longrightarrow f_* \Omega^k_M.$$

Proof. The identity statement follows from

$$d_p \operatorname{id}_M = \operatorname{id}_{T_pM}.$$

For composition, the chain rule gives

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f.$$

Substituting this into the pointwise definition of pullback proves

$$(g \circ f)^* = f^* g^*.$$

Compatibility with restrictions is immediate from the local definition. □

Proposition 6.16 (Compatibility with exterior products). *For differential forms α and β on N,*

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta.$$

Thus pullback is a homomorphism of graded algebras

$$f^* : \Omega^\bullet(N) \longrightarrow \Omega^\bullet(M).$$

Proof. The assertion holds pointwise because pullback by a linear map preserves the exterior product of alternating covariant tensors. □

Proposition 6.17 (Jacobian formula). *Let $f : U \longrightarrow V$ be a smooth map between open subsets of $\mathbb{R}^n$, with source coordinates $x^1, \dots, x^n$ and target coordinates $y^1, \dots, y^n$. Then*

$$f^*(dy^1 \wedge \cdots \wedge dy^n) = \det\left(\frac{\partial(y^i \circ f)}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n.$$

Proof. By Theorem 6.14,

$$f^*(dy^i) = \sum_{j=1}^{n} \frac{\partial(y^i \circ f)}{\partial x^j} dx^j.$$

Taking the wedge product of these n expressions and using the determinant transformation rule for a wedge of 1-forms gives the formula. □

Remark 6.18. Proposition 6.17 is the differential-form version of the Jacobian determinant in the change-of-variables theorem. It will be used in the construction of integration on oriented manifolds.

6.4 The Exterior Derivative

The differential of a smooth function is a 1-form. The exterior derivative extends this operation to forms of every degree.

6.4.1 Exterior differentiation on Euclidean open sets

Let $U \subseteq \mathbb{R}^n$ be open with standard coordinates $x^1, \dots, x^n$. If

$$f \in C^\infty(U),$$

define

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j.$$

This agrees with the differential defined pointwise in Chapter 2.

Definition 6.19. If

$$\omega = \sum_I a_I \, dx^I \in \Omega^k(U),$$

define

$$d\omega = \sum_I da_I \wedge dx^I \in \Omega^{k+1}(U).$$

Proposition 6.20 (Euclidean exterior derivative)**.** *On an open subset $U \subseteq \mathbb{R}^n$, the operator*

$$d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$$

has the following properties:

(1) it is $\mathbb{R}$-linear;

(2) for $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^\ell(U)$,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta;$$

(3)

$$d^2 = 0;$$

(4) for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$,

$$df(X) = X(f).$$

Proof. Linearity follows directly from the coordinate formula.

It suffices to prove the graded Leibniz rule for forms

$$\alpha = f \, dx^I, \qquad \beta = g \, dx^J,$$

where $|I| = k$. Since $d(dx^I) = 0$ by definition,

$$d(\alpha \wedge \beta) = d(fg) \wedge dx^I \wedge dx^J.$$

Using the ordinary product rule,

$$d(fg) = g \, df + f \, dg.$$

Moving the 1-form dg past the k-form dx^I introduces the sign $(-1)^k$, and therefore

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

For a basic term $f \, dx^I$,

$$d^2(f \, dx^I) = d(df \wedge dx^I) = d^2 f \wedge dx^I.$$

Now

$$d^2 f = \sum_{i,j} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i.$$

The coefficients are symmetric in i and j, while the wedge products are antisymmetric, so the sum vanishes. Hence $d^2 = 0$ on every form.

Finally, if

$$X = \sum_j X^j \frac{\partial}{\partial x^j},$$

then

$$df(X) = \sum_j \frac{\partial f}{\partial x^j} X^j = X(f).$$

$\square$

The coordinate definition is natural under smooth maps between Euclidean open sets.

Proposition 6.21 (Euclidean naturality)**.** *Let*

$$f : U \longrightarrow V$$

be a smooth map between Euclidean open sets. Then

$$f^*(d\omega) = d(f^*\omega)$$

for every differential form ω on V.

Proof. For a smooth function h on V, Theorem 6.14 gives

$$f^*(dh) = d(h \circ f) = d(f^*h).$$

Every form on V is a finite sum of expressions

$$h\, dy^{i_1} \wedge \cdots \wedge dy^{i_k}.$$

Both f^*d and df^* satisfy the same graded product rule with respect to such expressions, and

$$d(d(y^i \circ f)) = 0.$$

Thus the identity on functions implies the identity on all forms. □

6.4.2 Existence on manifolds

Theorem 6.22 (Existence of the exterior derivative)**.** *Let M be a smooth manifold. There exists an $\mathbb{R}$-linear operator*

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

whose restriction to every coordinate chart is the Euclidean exterior derivative. It satisfies

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta,$$

$$d^2 = 0,$$

and

$$df(X) = X(f)$$

for every smooth function f and smooth vector field X.

Proof. Let $\omega \in \Omega^k(M)$ and let (U, x) be a coordinate chart. Define $d_U(\omega|_U)$ by the Euclidean coordinate formula. If (V, y) is another chart, then on $U \cap V$ the coordinate change

$$y \circ x^{-1}$$

is a diffeomorphism between Euclidean open sets. By Proposition 6.21, the two local definitions agree on the overlap. Hence the local forms

$$d_U(\omega|_U)$$

glue uniquely to a global form $d\omega$.

All stated identities are local. They hold in every coordinate chart by Proposition 6.20, and therefore hold globally. □

The exterior derivative is itself local in the sheaf-theoretic sense.

Proposition 6.23 (Locality)**.** *If $V \subseteq U \subseteq M$ are open and $\omega \in \Omega^k(U)$, then*

$$(d\omega)|_V = d(\omega|_V).$$

Consequently, exterior differentiation defines a morphism of sheaves of $\mathbb{R}$-vector spaces

$$d : \Omega^k_M \longrightarrow \Omega^{k+1}_M.$$

Proof. Both sides are computed by the same coordinate formula on every coordinate chart contained in V. □

6.4.3 Uniqueness

Lemma 6.24 (Locality of an abstract exterior derivative). *Suppose*

$$D : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

is an $\mathbb{R}$-linear operator of degree 1 satisfying the graded Leibniz rule and $D^2 = 0$, and suppose

$$Df = df$$

for all $f \in C^\infty(M)$. If a form ω vanishes on a neighborhood of p, then

$$(D\omega)_p = 0.$$

Proof. Choose an open neighborhood U of p on which ω vanishes. By the bump-function theorem, there exists $\rho \in C^\infty(M)$ such that

$$\rho = 1$$

on a neighborhood of p and

$$\operatorname{supp}(\rho) \subseteq U.$$

Then

$$\rho\omega = 0.$$

Applying D and using the graded Leibniz rule gives

$$0 = D(\rho\omega) = d\rho \wedge \omega + \rho D\omega.$$

At p, the first term vanishes because $\omega_p = 0$, while $\rho(p) = 1$. Thus

$$(D\omega)_p = 0.$$

□

Theorem 6.25 (Uniqueness of the exterior derivative). *The exterior derivative is the unique $\mathbb{R}$-linear operator*

$$D : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

of degree 1 such that:

(1) for homogeneous α,

$$D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge D\beta;$$

(2)

$$D^2 = 0;$$

(3) for every smooth function f,

$$Df = df.$$

Proof. By Lemma 6.24, the value of $D\omega$ at a point depends only on the germ of ω there. We may therefore work on a coordinate chart $(U, x^1, \ldots, x^n)$.

Since

$$Dx^i = dx^i,$$

one has

$$D(dx^i) = D^2 x^i = 0.$$

For a basic form

$$\omega = f\,dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

the graded Leibniz rule gives

$$D\omega = df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

This is precisely the coordinate formula for $d\omega$. Since such forms span $\Omega^k(U)$, one has $D = d$ locally and hence globally. □

Theorem 6.26 (Naturality of the exterior derivative)**.** *For every smooth map*

$$f : M \longrightarrow N,$$

pullback commutes with exterior differentiation:

$$f^* \circ d = d \circ f^*.$$

Proof. The identity is local on both manifolds. In compatible coordinate charts it is Proposition 6.21. □

Corollary 6.27 (The de Rham algebra as a functor)**.** *The assignment*

$$M \longmapsto (\Omega^\bullet(M), d)$$

and

$$f \longmapsto f^*$$

defines a contravariant functor

$$\Omega^\bullet : \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{CDGA}_{\mathbb{R}},$$

where $\mathbf{CDGA}_{\mathbb{R}}$ *denotes the category of graded-commutative differential* $\mathbb{R}$*-algebras.*

Proof. Proposition 6.15 gives contravariant functoriality, Proposition 6.16 gives compatibility with multiplication, and Theorem 6.26 gives compatibility with the differential. □

6.5 Closed Forms, Exact Forms, and the de Rham Complex

Definition 6.28. A differential form $\omega \in \Omega^k(M)$ is *closed* if

$$d\omega = 0.$$

It is *exact* if there exists $\eta \in \Omega^{k-1}(M)$ such that

$$\omega = d\eta.$$

Write

$$Z^k(M) = \ker\bigl(d : \Omega^k(M) \to \Omega^{k+1}(M)\bigr)$$

and

$$B^k(M) = \mathrm{im}\bigl(d : \Omega^{k-1}(M) \to \Omega^k(M)\bigr).$$

Since $d^2 = 0$, every exact form is closed:

$$B^k(M) \subseteq Z^k(M).$$

Definition 6.29. The *de Rham complex* of M is the cochain complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0.$$

Its cohomology will be introduced and studied systematically in the next chapter. Sheafwise, exterior differentiation gives the complex

$$0 \longrightarrow \mathscr{C}^\infty_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots .$$

The relation between this complex and the constant sheaf will also be established there.

Example 6.30 (The de Rham complex in dimension three)**.** On an open subset of $\mathbb{R}^3$, the de Rham complex is

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \longrightarrow 0.$$

After choosing the Euclidean metric and standard orientation, the three nonzero exterior derivatives correspond to the classical operators gradient, curl, and divergence. The de Rham complex itself, however, is defined without choosing a metric.

6.6 Parameter Integration and the Homotopy Operator

The local exactness of the de Rham complex follows from an explicit chain homotopy. We use the closed interval only as a parameter space. A form or map on $M \times [0,1]$ is called smooth if it is locally the restriction of a smooth form or map defined on an open subset of $M \times \mathbb{R}$. This convention does not require the general theory of manifolds with boundary.

For $t \in [0,1]$, let

$$i_t : M \longrightarrow M \times [0,1], \qquad i_t(p) = (p,t).$$

Using the natural splitting

$$T_{(p,t)}(M \times \mathbb{R}) \simeq T_pM \oplus \mathbb{R},$$

write

$$\partial_t = (0,1).$$

Let η be a smooth k-form on $M \times [0,1]$. Define a family of $(k-1)$-forms α_t on M by

$$(\alpha_t)_p(v_1, \ldots, v_{k-1}) = \eta_{(p,t)}\big(\partial_t, (v_1,0), \ldots, (v_{k-1},0)\big).$$

Also put

$$\beta_t = i_t^*\eta.$$

Pointwise, this is the unique decomposition

$$\eta = dt \wedge \alpha_t + \beta_t,$$

where α_t and β_t contain no dt factor.

Lemma 6.31 (Smooth parameter integration)**.** *Let $\{\alpha_t\}_{0\le t\le 1}$ be a smooth family of differential k-forms on M. Integrating the coefficient functions in local coordinates defines a smooth form*

$$\int_0^1 \alpha_t \, dt \in \Omega^k(M).$$

Moreover,

$$d\left(\int_0^1 \alpha_t \, dt\right) = \int_0^1 d\alpha_t \, dt.$$

Proof. On a coordinate chart, write

$$\alpha_t = \sum_I a_I(x,t)\, dx^I.$$

Define

$$\int_0^1 \alpha_t\, dt = \sum_I \left(\int_0^1 a_I(x,t)\, dt \right) dx^I.$$

Differentiation under the integral sign shows that the coefficient functions are smooth and that spatial partial derivatives commute with parameter integration. The stated identity with d follows from the coordinate formula for the exterior derivative. The construction agrees on overlaps because it is characterized pointwise by integration of the family of covectors. □

Definition 6.32. The *cylinder homotopy operator* is the degree -1 linear map

$$K : \Omega^k(M \times [0,1]) \longrightarrow \Omega^{k-1}(M)$$

defined by

$$K\eta = \int_0^1 \alpha_t\, dt,$$

where α_t is obtained from the dt-component of η as above. For $k = 0$, set $K = 0$.

Theorem 6.33 (Cylinder homotopy formula)**.** *For every differential form η on $M \times [0,1]$,*

$$i_1^*\eta - i_0^*\eta = dK\eta + Kd\eta.$$

Equivalently,

$$i_1^* - i_0^* = dK + Kd$$

as maps of graded vector spaces.

Proof. Write

$$\eta = dt \wedge \alpha_t + \beta_t.$$

A coordinate calculation gives

$$d\eta = dt \wedge \left(\frac{\partial \beta_t}{\partial t} - d_M \alpha_t \right) + d_M \beta_t,$$

where d_M denotes exterior differentiation in the M variables. Hence

$$Kd\eta = \int_0^1 \left(\frac{\partial \beta_t}{\partial t} - d_M \alpha_t \right) dt.$$

By the fundamental theorem of calculus and Lemma 6.31,

$$Kd\eta = \beta_1 - \beta_0 - d_M \int_0^1 \alpha_t\, dt.$$

Since

$$\beta_t = i_t^*\eta$$

and

$$dK\eta = d_M \int_0^1 \alpha_t\, dt,$$

we obtain

$$dK\eta + Kd\eta = i_1^*\eta - i_0^*\eta.$$

□

The formula extends immediately to a smooth homotopy.

Corollary 6.34 (Homotopy formula)**.** *Let*

$$F : M \times [0,1] \longrightarrow N$$

be smooth, and write

$$f_t = F \circ i_t.$$

Define

$$K_F = K \circ F^* : \Omega^k(N) \longrightarrow \Omega^{k-1}(M).$$

Then

$$f_1^* - f_0^* = dK_F + K_F d.$$

Proof. Apply Theorem 6.33 to $F^*\omega$ and use naturality of the exterior derivative:

$$i_t^* F^* = (F \circ i_t)^* = f_t^*.$$

□

Remark 6.35. Corollary 6.34 says that the cochain maps f_0^* and f_1^* are chain homotopic. Its consequence for de Rham cohomology will be developed together with other computational tools in a later chapter.

6.7 The Poincaré Lemma

Definition 6.36. An open set $U \subseteq \mathbb{R}^n$ is *star-shaped with respect to* $a \in U$ if

$$(1-t)a + tx \in U$$

for every $x \in U$ and $0 \le t \le 1$.

Every open ball is star-shaped with respect to its center.

Theorem 6.37 (Poincaré lemma)**.** *Let $U \subseteq \mathbb{R}^n$ be star-shaped. If*

$$\omega \in \Omega^k(U), \qquad k > 0,$$

and

$$d\omega = 0,$$

then there exists $\eta \in \Omega^{k-1}(U)$ such that

$$\omega = d\eta.$$

Thus every closed differential form of positive degree on a star-shaped open set is exact.

Proof. After translating coordinates, assume that U is star-shaped with respect to 0. Define

$$F : U \times [0,1] \longrightarrow U, \qquad F(x,t) = tx.$$

Then

$$f_1 = \mathrm{id}_U$$

and f_0 is the constant map with value 0. By Corollary 6.34,

$$\mathrm{id}_U^* - f_0^* = dK_F + K_F d.$$

Since $k > 0$ and f_0 is constant, its differential is zero and therefore

$$f_0^*\omega = 0.$$

Because $d\omega = 0$, the homotopy formula gives

$$\omega = d(K_F\omega).$$

Thus $\eta = K_F\omega$ is a primitive of ω. □

For reference, the primitive constructed in the proof has an explicit radial formula. If U is star-shaped with respect to 0 and $\omega \in \Omega^k(U)$, then

$$(K_F\omega)_x(v_1, \ldots, v_{k-1}) = \int_0^1 t^{k-1}\omega_{tx}(x, v_1, \ldots, v_{k-1})\, dt.$$

No vector-field or flow theory is involved in this formula.

Corollary 6.38 (Local Poincaré lemma)**.** *Let M be a smooth manifold and let $p \in M$. There exists an open neighborhood U of p such that every closed form in $\Omega^k(U)$ of positive degree is exact.*

Proof. Choose a coordinate neighborhood of p whose image is an open ball in $\mathbb{R}^n$. Transport Theorem 6.37 through the coordinate diffeomorphism. □

The degree-zero part has a different form.

Proposition 6.39. *Let $U \subseteq M$ be open and let $f \in C^\infty(U)$. Then*

$$df = 0$$

if and only if f is locally constant. In particular, if U is connected, then f is constant.

Proof. If f is locally constant, then its differential vanishes. Conversely, suppose $df = 0$. In a coordinate chart, every partial derivative of f vanishes. The ordinary mean-value theorem shows that f is constant on each sufficiently small coordinate ball. Hence it is locally constant. A locally constant function on a connected space is constant. □

Corollary 6.40 (Local exactness of the de Rham complex)**.** *Let $\mathbb{R}_M$ denote the sheaf of locally constant real-valued functions on M. The sequence of sheaves*

$$0 \longrightarrow \mathbb{R}_M \longrightarrow \mathscr{C}^\infty_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots$$

is exact.

Proof. Exactness at $\mathscr{C}^\infty_M$ is Proposition 6.39. Exactness in every positive degree is the local Poincaré lemma. Exactness of a sequence of sheaves may be checked on stalks, so these local statements prove the result. □

Remark 6.41*.* Each sheaf Ω^k_M is a locally free $\mathscr{C}^\infty_M$-module and hence is fine and acyclic by Chapter 5. Therefore Corollary 6.40 is not merely a local exact sequence: it is a fine resolution of the constant sheaf $\mathbb{R}_M$. The cohomological consequences of this observation are the subject of the next chapter.

Chapter 7

de Rham Cohomology and the de Rham Resolution

The de Rham complex associates a cochain complex of real vector spaces to every smooth manifold. Its cohomology measures the failure of a closed form to possess a global primitive. The Poincaré lemma shows that this failure is not local: every positive-degree closed form is locally exact. It is therefore a genuinely global invariant.

This chapter first develops de Rham cohomology directly from differential forms. We establish functoriality, the graded algebra structure, homotopy invariance, and several elementary examples. Only after the invariant itself has been understood do we identify it with the sheaf cohomology of the constant sheaf by means of the fine de Rham resolution.

7.1 Definition and Elementary Properties

Recall from Chapter 6 that

$$Z^k(M) = \ker\big(d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)\big)$$

is the vector space of closed k-forms and

$$B^k(M) = \operatorname{im}\big(d : \Omega^{k-1}(M) \longrightarrow \Omega^k(M)\big)$$

is the vector space of exact k-forms. Since $d^2 = 0$, one has

$$B^k(M) \subseteq Z^k(M).$$

Definition 7.1. The *kth de Rham cohomology group* of M is the quotient vector space

$$H^k_{\mathrm{dR}}(M) = \frac{Z^k(M)}{B^k(M)}.$$

If $\omega \in Z^k(M)$, its class is denoted by

$$[\omega] \in H^k_{\mathrm{dR}}(M).$$

Thus two closed forms ω and ω' determine the same class if and only if

$$\omega' - \omega = d\eta$$

for some $\eta \in \Omega^{k-1}(M)$.

Equivalently,
$$H^k_{\mathrm{dR}}(M) = H^k\big(\Omega^\bullet(M), d\big),$$
the kth cohomology of the global de Rham complex.

Proposition 7.2 (Degree zero)**.** *There is a natural identification*
$$H^0_{\mathrm{dR}}(M) \simeq \Gamma(M, \mathbb{R}_M),$$
where $\mathbb{R}_M$ is the sheaf of locally constant real-valued functions. Consequently,
$$H^0_{\mathrm{dR}}(M) \simeq \mathrm{Map}(\pi_0(M), \mathbb{R}).$$
In particular, if M is nonempty and connected, then
$$H^0_{\mathrm{dR}}(M) \simeq \mathbb{R}.$$

Proof. There are no forms of degree -1, so
$$B^0(M) = 0.$$
By Proposition 6.39,
$$Z^0(M) = \{f \in C^\infty(M) : df = 0\}$$
is precisely the vector space of locally constant functions. Since the connected components of a manifold are open, a locally constant function is determined by an arbitrary real value on each component. □

Proposition 7.3 (Vanishing above the dimension)**.** *If M has dimension n, then*
$$H^k_{\mathrm{dR}}(M) = 0$$
for every $k > n$.

Proof. For $k > n$, the vector bundle $\Lambda^k T^*M$ is zero. Hence
$$\Omega^k(M) = 0,$$
and therefore its cohomology in degree k is zero. □

Example 7.4 (A point)**.** For the one-point manifold,
$$H^k_{\mathrm{dR}}(\mathrm{pt}) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

7.2 Functoriality

Let $f : M \longrightarrow N$ be smooth. Naturality of the exterior derivative implies that pullback preserves closed and exact forms.

Proposition 7.5 (Pullback in cohomology)**.** *For every smooth map $f : M \longrightarrow N$, pullback induces a linear map*
$$f^* : H^k_{\mathrm{dR}}(N) \longrightarrow H^k_{\mathrm{dR}}(M),$$
defined by
$$f^*[\omega] = [f^*\omega].$$

Proof. If ω is closed, then

$$d(f^*\omega) = f^*(d\omega) = 0.$$

If ω is exact, say $\omega = d\eta$, then

$$f^*\omega = f^*(d\eta) = d(f^*\eta),$$

so $f^*\omega$ is exact. Hence pullback descends to the quotient by exact forms. □

Theorem 7.6 (Contravariant functoriality)**.** *For every $k \geq 0$, the assignment*

$$M \longmapsto H^k_{\mathrm{dR}}(M), \qquad f \longmapsto f^*,$$

defines a contravariant functor

$$H^k_{\mathrm{dR}} : \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}.$$

Thus

$$\mathrm{id}_M^* = \mathrm{id}_{H^k_{\mathrm{dR}}(M)}$$

and, for smooth maps

$$M \xrightarrow{f} N \xrightarrow{g} P,$$

one has

$$(g \circ f)^* = f^* \circ g^*.$$

Proof. The corresponding identities hold on differential forms by Proposition 6.15. Passing to cohomology preserves them. □

Corollary 7.7 (Diffeomorphism invariance)**.** *If $f : M \longrightarrow N$ is a diffeomorphism, then*

$$f^* : H^k_{\mathrm{dR}}(N) \xrightarrow{\simeq} H^k_{\mathrm{dR}}(M)$$

is an isomorphism for every k.

Proof. The inverse is $(f^{-1})^*$ by functoriality. □

7.3 The de Rham Cohomology Algebra

The exterior product of forms descends to cohomology.

Theorem 7.8 (Cohomology product)**.** *For $k, \ell \geq 0$, the rule*

$$H^k_{\mathrm{dR}}(M) \times H^\ell_{\mathrm{dR}}(M) \longrightarrow H^{k+\ell}_{\mathrm{dR}}(M), \qquad ([\alpha], [\beta]) \longmapsto [\alpha \wedge \beta],$$

is well defined and bilinear. With this product,

$$H^\bullet_{\mathrm{dR}}(M) = \bigoplus_{k \geq 0} H^k_{\mathrm{dR}}(M)$$

is a unital graded-commutative $\mathbb{R}$-algebra. In particular,

$$[\alpha][\beta] = (-1)^{k\ell}[\beta][\alpha]$$

for homogeneous classes of degrees k and ℓ.

Proof. If α and β are closed, then the graded Leibniz rule gives

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta = 0,$$

so their product is closed.

Suppose first that α is replaced by $\alpha + d\eta$. Since β is closed,

$$(\alpha + d\eta) \wedge \beta - \alpha \wedge \beta = d\eta \wedge \beta = d(\eta \wedge \beta).$$

Thus the cohomology class does not change. Similarly, replacing β by $\beta + d\theta$ changes the product by

$$\alpha \wedge d\theta = (-1)^k d(\alpha \wedge \theta),$$

because $d\alpha = 0$. Hence the product is well defined.

Associativity, bilinearity, the unit $[1]$, and graded commutativity descend from the corresponding properties of the wedge product. ☐

Proposition 7.9. *For every smooth map $f : M \longrightarrow N$, the induced map*

$$f^* : H^\bullet_{\mathrm{dR}}(N) \longrightarrow H^\bullet_{\mathrm{dR}}(M)$$

is a homomorphism of graded-commutative $\mathbb{R}$-algebras.

Proof. Pullback preserves the constant function 1 and satisfies

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta.$$

Passing to cohomology proves the result. ☐

7.4 Homotopy Invariance and First Examples

Definition 7.10. Two smooth maps $f_0, f_1 : M \longrightarrow N$ are *smoothly homotopic* if there exists a smooth map

$$F : M \times [0,1] \longrightarrow N$$

such that

$$F(p,0) = f_0(p), \qquad F(p,1) = f_1(p).$$

The homotopy operator constructed in Chapter 6 gives a chain homotopy

$$f_1^* - f_0^* = dK_F + K_F d.$$

Theorem 7.11 (Homotopy invariance)**.** *Smoothly homotopic maps induce the same homomorphism in de Rham cohomology:*

$$f_0^* = f_1^* : H^k_{\mathrm{dR}}(N) \longrightarrow H^k_{\mathrm{dR}}(M).$$

Proof. Let ω be closed. By Corollary 6.34,

$$f_1^*\omega - f_0^*\omega = dK_F\omega + K_F d\omega = dK_F\omega.$$

Thus the two pullbacks differ by an exact form and determine the same cohomology class. ☐

Definition 7.12. Smooth manifolds M and N are *smoothly homotopy equivalent* if there exist smooth maps

$$f : M \longrightarrow N, \qquad g : N \longrightarrow M$$

such that

$$g \circ f \simeq \mathrm{id}_M, \qquad f \circ g \simeq \mathrm{id}_N$$

through smooth homotopies.

Corollary 7.13 (Homotopy equivalence)**.** *A smooth homotopy equivalence induces an isomorphism of graded de Rham cohomology algebras.*

Proof. By functoriality and homotopy invariance,

$$f^* \circ g^* = (g \circ f)^* = \mathrm{id}$$

on $H^\bullet_{\mathrm{dR}}(M)$, and

$$g^* \circ f^* = (f \circ g)^* = \mathrm{id}$$

on $H^\bullet_{\mathrm{dR}}(N)$. □

Corollary 7.14 (Contractible manifolds)**.** *If a nonempty smooth manifold M is smoothly contractible, then*

$$H^k_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

In particular, the same conclusion holds for every star-shaped open subset of $\mathbb{R}^n$ and for $\mathbb{R}^n$ itself.

Proof. A contractible manifold is smoothly homotopy equivalent to a point. Apply Corollary 7.13 and the computation for a point. □

Corollary 7.15 (Deformation retracts)**.** *Let $S \subseteq M$ be an embedded submanifold. If S is a smooth deformation retract of M, then the inclusion*

$$i : S \longrightarrow M$$

induces an isomorphism

$$i^* : H^\bullet_{\mathrm{dR}}(M) \xrightarrow{\simeq} H^\bullet_{\mathrm{dR}}(S).$$

Proof. A deformation retraction gives a smooth homotopy inverse to the inclusion. □

Example 7.16 (A nontrivial class on the circle)**.** Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

and let $j : S^1 \hookrightarrow \mathbb{R}^2 \setminus \{0\}$ be the inclusion. The 1-form

$$\alpha = j^* \left(\frac{-y\,dx + x\,dy}{x^2 + y^2} \right)$$

is closed because every 2-form on the one-dimensional manifold S^1 vanishes. It is not exact.

Indeed, for

$$c : [0, 2\pi] \longrightarrow S^1, \qquad c(t) = (\cos t, \sin t),$$

one has

$$c^*\alpha = dt.$$

If $\alpha = df$ for a smooth function f on S^1, then

$$d(f \circ c) = dt.$$

The fundamental theorem of calculus would give

$$f(c(2\pi)) - f(c(0)) = 2\pi,$$

contradicting $c(2\pi) = c(0)$. Hence

$$[\alpha] \neq 0$$

in $H^1_{\mathrm{dR}}(S^1)$.

The complete computation of the cohomology of spheres will be obtained later from the Mayer–Vietoris sequence.

7.5 Acyclic Resolutions

Let X be a topological space. For a sheaf $\mathscr{F}$ of abelian groups on X, write

$$H^q(X, \mathscr{F}) = R^q\Gamma(X, \mathscr{F}).$$

Definition 7.17. A sheaf $\mathscr{A}$ on X is Γ-*acyclic* if

$$H^q(X, \mathscr{A}) = 0$$

for every $q > 0$.

Definition 7.18. A *resolution* of a sheaf $\mathscr{F}$ is an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{A}^0 \xrightarrow{d^0} \mathscr{A}^1 \xrightarrow{d^1} \mathscr{A}^2 \longrightarrow \cdots .$$

It is an *acyclic resolution* if every $\mathscr{A}^q$ is Γ-acyclic.

Applying global sections to a resolution gives a cochain complex. It need not remain exact; its cohomology records precisely the derived functors of global sections.

Theorem 7.19 (Acyclic-resolution theorem)**.** *Let*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{A}^0 \xrightarrow{d^0} \mathscr{A}^1 \xrightarrow{d^1} \cdots$$

be an acyclic resolution on X. Then there are natural isomorphisms

$$H^q(X, \mathscr{F}) \simeq H^q\big(\Gamma(X, \mathscr{A}^\bullet)\big)$$

for every $q \geq 0$.

Proof. Set

$$\mathscr{Z}^0 = \mathscr{F}$$

and, for $q \geq 0$, define

$$\mathscr{Z}^{q+1} = \operatorname{im}\big(\mathscr{A}^q \longrightarrow \mathscr{A}^{q+1}\big).$$

Exactness gives short exact sequences

$$0 \longrightarrow \mathscr{Z}^q \longrightarrow \mathscr{A}^q \longrightarrow \mathscr{Z}^{q+1} \longrightarrow 0$$

for every $q \geq 0$.

In degree zero, left exactness of global sections identifies

$$H^0\big(\Gamma(X, \mathscr{A}^\bullet)\big)$$

with $\Gamma(X, \mathscr{F})$.

Let $q > 0$. The degree-q cohomology of the global-section complex is the cokernel of

$$\Gamma(X, \mathscr{A}^{q-1}) \longrightarrow \Gamma(X, \mathscr{Z}^q).$$

The long exact sequence associated with

$$0 \longrightarrow \mathscr{Z}^{q-1} \longrightarrow \mathscr{A}^{q-1} \longrightarrow \mathscr{Z}^q \longrightarrow 0$$

and the acyclicity of $\mathscr{A}^{q-1}$ identify this cokernel with

$$H^1(X, \mathscr{Z}^{q-1}).$$

The same long exact sequences give dimension-shifting isomorphisms

$$H^i(X, \mathscr{Z}^{j+1}) \simeq H^{i+1}(X, \mathscr{Z}^j)$$

for $i > 0$. Iterating yields

$$H^1(X, \mathscr{Z}^{q-1}) \simeq H^q(X, \mathscr{Z}^0) = H^q(X, \mathscr{F}).$$

All maps arise from natural connecting morphisms, so the resulting isomorphism is natural. □

Remark 7.20. The acyclic-resolution theorem is often called the abstract de Rham theorem. It is a general statement of homological algebra and is not specific to smooth manifolds.

7.6 The Fine de Rham Resolution

Let $\mathbb{R}_M$ denote the sheaf of locally constant real-valued functions on M. The local Poincaré lemma gave an exact sequence

$$0 \longrightarrow \mathbb{R}_M \longrightarrow \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots .$$

Proposition 7.21. *For every $q \geq 0$, the sheaf Ω^q_M is a fine and Γ-acyclic sheaf of $\mathscr{C}^\infty_M$-modules.*

Proof. The sheaf Ω^q_M is the section sheaf of the vector bundle $\Lambda^q T^*M$, hence is a locally free $\mathscr{C}^\infty_M$-module sheaf. By Chapter 5, every sheaf of $\mathscr{C}^\infty_M$-modules is fine, and every fine sheaf on the paracompact Hausdorff space M is acyclic. □

Theorem 7.22 (The de Rham resolution)**.** *The exact sequence*

$$0 \longrightarrow \mathbb{R}_M \longrightarrow \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots$$

is a fine acyclic resolution of $\mathbb{R}_M$.

Proof. Exactness is Corollary 6.40, and every term is fine and acyclic by Proposition 7.21. □

Remark 7.23. Two independent mechanisms enter the theorem. The Poincaré lemma supplies local exactness, while partitions of unity supply global acyclicity. Their combination turns the complex of differential forms into a global cohomological model for the constant sheaf.

7.7 The Sheaf-Theoretic de Rham Theorem

Theorem 7.24 (Sheaf-theoretic de Rham theorem)**.** *For every smooth manifold M and every $q \geq 0$, there is a natural isomorphism*

$$\mathrm{DR}^q_M : H^q(M, \mathbb{R}_M) \xrightarrow{\simeq} H^q_{\mathrm{dR}}(M).$$

Proof. Apply Theorem 7.19 to the fine resolution of Theorem 7.22. Its global-section complex is

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots ,$$

whose qth cohomology is $H^q_{\mathrm{dR}}(M)$ by definition. □

Theorem 7.25 (Naturality)**.** *Let $f : M \longrightarrow N$ be smooth. Under the standard pullback map on constant-sheaf cohomology, the diagram*

$$\begin{array}{ccc} H^q(N, \mathbb{R}_N) & \xrightarrow{f^*} & H^q(M, \mathbb{R}_M) \\ \downarrow{\scriptstyle \mathrm{DR}^q_N} & & \downarrow{\scriptstyle \mathrm{DR}^q_M} \\ H^q_{\mathrm{dR}}(N) & \xrightarrow{f^*} & H^q_{\mathrm{dR}}(M) \end{array}$$

commutes.

Proof. The canonical identification

$$f^{-1}\mathbb{R}_N \simeq \mathbb{R}_M$$

and pullback of forms give a morphism of augmented complexes on M,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}\mathbb{R}_N & \longrightarrow & f^{-1}\Omega^0_N & \xrightarrow{d} & f^{-1}\Omega^1_N & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{R}_M & \longrightarrow & \Omega^0_M & \xrightarrow{d} & \Omega^1_M & \longrightarrow & \cdots . \end{array}$$

The vertical maps on forms are induced by ordinary pullback, and they commute with d. Functoriality of the acyclic-resolution comparison therefore gives the displayed commutative diagram. □

The de Rham theorem is compatible with multiplication.

Theorem 7.26 (Multiplicative de Rham theorem)**.** *The isomorphisms* DR^q_M *assemble into a natural isomorphism of graded-commutative* $\mathbb{R}$*-algebras*

$$\mathrm{DR}_M : H^\bullet(M, \mathbb{R}_M) \xrightarrow{\simeq} H^\bullet_{\mathrm{dR}}(M).$$

The cup product on the left corresponds to the product

$$[\alpha][\beta] = [\alpha \wedge \beta]$$

on the right.

Proof. The inclusion

$$\mathbb{R}_M \longrightarrow \Omega^\bullet_M$$

is a quasi-isomorphism of sheaves of graded-commutative differential $\mathbb{R}$-algebras. Hypercohomology therefore carries its multiplication to the multiplication on the cohomology of the global de Rham algebra. Since each Ω^q_M is acyclic,

$$\mathbb{H}^\bullet(M, \Omega^\bullet_M) \simeq H^\bullet(\Omega^\bullet(M), d),$$

and the latter product is represented by wedge products of closed forms. □

Remark 7.27 (Derived formulation)*.* Regarded as a complex concentrated in degree zero, the constant sheaf is quasi-isomorphic to the de Rham complex:

$$\mathbb{R}_M \simeq \Omega^\bullet_M.$$

Consequently,

$$R\Gamma(M, \mathbb{R}_M) \simeq \Gamma(M, \Omega^\bullet_M)$$

in the derived category of real vector spaces. The right-hand side requires no further resolution because all sheaves of forms are Γ-acyclic.

7.8 Comparison with Singular Cohomology

The sheaf-theoretic theorem identifies differential forms with the cohomology of the constant sheaf. The final topological comparison is a standard theorem about locally contractible spaces.

Theorem 7.28 (Constant-sheaf comparison)**.** *Let X be paracompact and locally contractible. Then there is a natural isomorphism of graded* $\mathbb{R}$*-algebras*

$$H^\bullet(X, \mathbb{R}_X) \xrightarrow{\simeq} H^\bullet_{\mathrm{sing}}(X; \mathbb{R}).$$

We take this theorem as a standard input from algebraic topology and sheaf cohomology.

Corollary 7.29 (Classical de Rham theorem)**.** *For every smooth manifold M, there is a natural isomorphism of graded-commutative* $\mathbb{R}$*-algebras*

$$H^\bullet_{\mathrm{dR}}(M) \xrightarrow{\simeq} H^\bullet_{\mathrm{sing}}(M; \mathbb{R}).$$

Proof. A smooth manifold is paracompact and locally contractible. Combine Theorems 7.26 and 7.28. □

Remark 7.30*.* The proof separates three mechanisms. The Poincaré lemma proves that the de Rham complex resolves the constant sheaf. Partitions of unity make this resolution acyclic. The constant-sheaf comparison theorem then identifies sheaf cohomology with singular cohomology.

Chapter 8

Orientation, Integration, and Stokes' Theorem

Differential forms acquire their global geometric meaning through integration. A top-degree form can be integrated only after one has specified how local coordinate systems are to be compared with respect to sign. This is the role of an orientation. Once local integrals have been shown to be invariant under orientation-preserving coordinate changes, partitions of unity assemble them into a global integral. The compatibility of this integral with exterior differentiation is Stokes' theorem.

The chapter proceeds from linear orientations to orientations of manifolds, then constructs integration of compactly supported top forms. We subsequently extend the basic theory to manifolds with boundary, define the induced boundary orientation, and prove Stokes' theorem by reduction to a calculation on Euclidean half-space. The final section explains how integration produces linear functionals on de Rham cohomology.

8.1 Orientations of Vector Spaces

Let V be an n-dimensional real vector space. If

$$\mathcal{B} = (v_1, \dots, v_n)$$

and

$$\mathcal{B}' = (w_1, \dots, w_n)$$

are ordered bases, there is a unique matrix $A \in GL_n(\mathbb{R})$ such that

$$(w_1, \dots, w_n) = (v_1, \dots, v_n)A.$$

Definition 8.1. Two ordered bases of V have the *same orientation* if the determinant of their transition matrix is positive. An *orientation* of V is an equivalence class of ordered bases under this relation. A basis belonging to the chosen class is called *positively oriented.*

Proposition 8.2. *Every nonzero finite-dimensional real vector space has exactly two orientations.*

Proof. The relation above is an equivalence relation because determinants multiply. Every ordered basis belongs to one of two classes according to whether its transition matrix from a fixed basis has positive or negative determinant. Both classes are nonempty: multiplying one basis vector by -1 reverses the sign of the determinant. No third class is possible. □

For the zero vector space, the empty ordered basis determines its unique orientation.

Definition 8.3. Let V and W be oriented vector spaces of the same dimension. A linear isomorphism

$$T : V \longrightarrow W$$

is *orientation preserving* if it carries every positively oriented basis of V to a positively oriented basis of W. It is *orientation reversing* otherwise.

Proposition 8.4. *Let V be an oriented n-dimensional vector space and let*

$$0 \neq \omega \in \Lambda^n V^*.$$

The sign of

$$\omega(v_1, \ldots, v_n)$$

is independent of the chosen positively oriented basis. In particular, the set of nonzero elements of $\Lambda^n V^*$ is divided into positive and negative top covectors.

Proof. If $(w_1, \ldots, w_n) = (v_1, \ldots, v_n)A$, then alternating multilinearity gives

$$\omega(w_1, \ldots, w_n) = \det(A)\, \omega(v_1, \ldots, v_n).$$

For two positively oriented bases, $\det(A) > 0$, so the two values have the same sign. $\square$

Definition 8.5. A nonzero top covector $\omega \in \Lambda^n V^*$ is *positive* if

$$\omega(v_1, \ldots, v_n) > 0$$

for every positively oriented basis $(v_1, \ldots, v_n)$.

For $n > 0$, an orientation of V is equivalently a choice of one of the two connected components of

$$\Lambda^n V^* \setminus \{0\}.$$

Proposition 8.6 (Direct-sum orientation)**.** *Let V and W be oriented vector spaces of dimensions m and n. There is a unique orientation on $V \oplus W$ for which*

$$(v_1, \ldots, v_m, w_1, \ldots, w_n)$$

is positive whenever $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_n)$ are positive. Interchanging the two factors changes the orientation by the sign

$$(-1)^{mn}.$$

Proof. Changing either basis by a matrix of positive determinant changes the block diagonal transition matrix by a matrix of positive determinant. Hence the rule is well defined. Moving the m vectors of V past the n vectors of W is a permutation of sign $(-1)^{mn}$. $\square$

8.2 Orientations of Smooth Manifolds

Let M be a smooth manifold of dimension n.

Definition 8.7. Two charts

$$(U, x^1, \ldots, x^n) \qquad \text{and} \qquad (V, y^1, \ldots, y^n)$$

are *orientation compatible* if, on every nonempty connected component of $U \cap V$,

$$\det\left(\frac{\partial y^i}{\partial x^j}\right) > 0.$$

An *oriented atlas* is a smooth atlas whose charts are pairwise orientation compatible. An *orientation* of M is a maximal oriented atlas. A manifold admitting an orientation is *orientable.*

A chart belonging to the chosen maximal oriented atlas is called an *oriented chart* or a *positive chart.*

Proposition 8.8. *An orientation of M determines an orientation of every tangent space T_pM. In an oriented chart $(U, x^1, \ldots, x^n)$, the coordinate basis*

$$\left(\frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right)$$

is positively oriented. This orientation is independent of the oriented chart.

Proof. On the overlap of two oriented charts, the transition matrix between the two coordinate bases is the Jacobian matrix of one coordinate system with respect to the other. Its determinant is positive by definition. □

Definition 8.9. A smooth map

$$f : M \longrightarrow N$$

between oriented manifolds of the same dimension is *orientation preserving at p* if

$$d_pf : T_pM \longrightarrow T_{f(p)}N$$

is orientation preserving. A diffeomorphism is orientation preserving if it is so at every point, and orientation reversing if its differential reverses orientation at every point.

If M is connected, the sign of the determinant of the differential of a diffeomorphism is locally constant and hence constant. Thus every diffeomorphism between connected oriented manifolds of the same dimension is either orientation preserving everywhere or orientation reversing everywhere.

The orientation can also be described by the determinant line bundle

$$\det(T^*M) = \Lambda^n T^*M.$$

Theorem 8.10 (Equivalent descriptions of orientability)**.** *For an n-dimensional smooth manifold M, the following conditions are equivalent.*

(1) M admits an oriented atlas.

(2) The tangent spaces T_pM admit orientations which are represented locally by coordinate bases.

(3) There exists a nowhere-vanishing smooth n-form on M.

(4) The determinant line bundle $\Lambda^n T^*M$ is trivial.

For a fixed orientation, there exists a nowhere-vanishing top form which is positive at every point.

Proof. The equivalence of (3) and (4) is the usual characterization of a trivial real line bundle by a nowhere-vanishing global section.

Assume (1). Choose an open cover by oriented charts

$$\{(U_i, x_i^1, \ldots, x_i^n)\}_{i\in I}$$

and a smooth partition of unity $\{\rho_i\}$ subordinate to this cover. On U_i, put

$$\omega_i = dx_i^1 \wedge \cdots \wedge dx_i^n.$$

Each ω_i is positive in the orientation defined by the atlas. Since $\operatorname{supp}(\rho_i) \subseteq U_i$, the form $\rho_i\omega_i$ extends by zero to a global smooth top form. The locally finite sum

$$\omega = \sum_i \rho_i\omega_i$$

is smooth. At each point, every nonzero summand is a nonnegative multiple of a positive top covector, and at least one coefficient ρ_i is positive. Therefore ω is positive and nowhere zero. This proves (1) $\Rightarrow$ (3).

Assume (3) and let ω be nowhere zero. Declare a chart $(U, x^1, \ldots, x^n)$ to be positive when

$$\omega = f\, dx^1 \wedge \cdots \wedge dx^n$$

with $f > 0$ on U. Such charts exist near every point, after reversing one coordinate if necessary. On an overlap of two positive charts, the coordinate transformation formula for top forms shows that its Jacobian determinant is positive. Hence the positive charts form an oriented atlas, proving (3) $\Rightarrow$ (1).

The equivalence with (2) follows from Proposition 8.8: an oriented atlas gives locally represented tangent-space orientations, and conversely such a locally represented choice selects precisely those charts whose coordinate bases are positive. $\square$

Definition 8.11. Let M be oriented. A top form $\omega \in \Omega^n(M)$ is *positive at p* if ω_p is positive with respect to the orientation of T_pM. It is *nonnegative* if, in every oriented chart,

$$\omega = f\, dx^1 \wedge \cdots \wedge dx^n$$

with $f \geq 0$.

Corollary 8.12. *A connected orientable manifold has exactly two orientations.*

Proof. Fix one orientation and a positive nowhere-vanishing top form ω. Any other nowhere-vanishing top form has the form $f\omega$ for a nowhere-zero smooth function f. On a connected manifold, f has constant sign. The two possible signs give the original and the opposite orientation. $\square$

The manifold equipped with the opposite orientation is denoted by $-M$.

Proposition 8.13 (Product orientation)**.** *Let M and N be oriented manifolds of dimensions m and n. Under the canonical identification*

$$T_{(p,q)}(M \times N) \simeq T_pM \oplus T_qN,$$

the direct-sum orientation defines the product orientation *on $M \times N$. If ω_M and ω_N are positive top forms, then*

$$\mathrm{pr}_M^*\, \omega_M \wedge \mathrm{pr}_N^*\, \omega_N$$

is positive on $M \times N$. Interchanging the factors changes orientation by $(-1)^{mn}$.

Proof. The assertion follows pointwise from Proposition 8.6. Smoothness is immediate in product charts. $\square$

8.3 Compactly Supported Differential Forms

Definition 8.14. For a smooth manifold M, define

$$\Omega^k_c(M) = \{\omega \in \Omega^k(M) : \operatorname{supp}(\omega) \text{ is compact}\}.$$

These are the *compactly supported smooth k-forms* on M.

Proposition 8.15. *Let $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$. Then*

$$\operatorname{supp}(d\alpha) \subseteq \operatorname{supp}(\alpha)$$

and

$$\operatorname{supp}(\alpha \wedge \beta) \subseteq \operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta).$$

Consequently,

$$d\big(\Omega^k_c(M)\big) \subseteq \Omega^{k+1}_c(M).$$

If one of α and β is compactly supported, then so is $\alpha \wedge \beta$.

Proof. If α vanishes on a neighborhood of a point, then its exterior derivative vanishes on the same neighborhood. If either α or β vanishes on a neighborhood, then their wedge product vanishes there. Taking closures gives the support inclusions. The compact-support statements follow immediately. □

Proposition 8.16 (Proper pullback)**.** *Let $f : M \longrightarrow N$ be a proper smooth map. Then*

$$f^*\big(\Omega^k_c(N)\big) \subseteq \Omega^k_c(M).$$

Proof. For $\omega \in \Omega^k_c(N)$,

$$\operatorname{supp}(f^*\omega) \subseteq f^{-1}(\operatorname{supp}(\omega)).$$

The set on the right is compact because f is proper. □

8.4 Local Integration in Oriented Coordinates

We use the ordinary integral of compactly supported smooth functions on Euclidean space. The principal analytic input is the change-of-variables theorem.

Theorem 8.17 (Euclidean change of variables)**.** *Let $U, V \subseteq \mathbb{R}^n$ be open and let*

$$F : U \longrightarrow V$$

be a diffeomorphism. For every $g \in C^\infty_c(V)$,

$$\int_U g(F(x))\,|\det DF_x|\,dx = \int_V g(y)\,dy.$$

We take this theorem as a standard result from multivariable analysis.

Let $(U, x^1, \dots, x^n)$ be an oriented chart on an oriented n-manifold M, and let $\omega \in \Omega^n_c(U)$. There is a unique $f \in C^\infty_c(x(U))$ such that

$$\omega = (f \circ x)\,dx^1 \wedge \cdots \wedge dx^n.$$

Definition 8.18. The *local integral* of ω in the oriented chart x is

$$\int_U \omega = \int_{x(U)} f(u)\,du^1 \cdots du^n.$$

Proposition 8.19 (Coordinate independence of local integration)**.** *The local integral of a compactly supported top form is independent of the chosen oriented coordinate system.*

Proof. Let $x : U \to x(U)$ and $y : U \to y(U)$ be two oriented coordinate systems, after restricting to a common coordinate neighborhood containing the support. Write

$$\omega = (f \circ x)\, dx^1 \wedge \cdots \wedge dx^n = (g \circ y)\, dy^1 \wedge \cdots \wedge dy^n.$$

Let

$$F = y \circ x^{-1} : x(U) \longrightarrow y(U).$$

By the transformation formula for top forms,

$$f(u) = g(F(u)) \det DF_u.$$

Because both charts are oriented,

$$\det DF_u > 0.$$

The Euclidean change-of-variables theorem therefore gives

$$\int_{x(U)} f(u)\, du = \int_{x(U)} g(F(u)) \det DF_u\, du = \int_{y(U)} g(v)\, dv.$$

□

Corollary 8.20 (Local change of variables)**.** *Let $F : U \longrightarrow V$ be an orientation-preserving diffeomorphism between oriented n-manifolds, and let $\omega \in \Omega_c^n(V)$. If U and V are coordinate neighborhoods, then*

$$\int_U F^*\omega = \int_V \omega.$$

If F is orientation reversing, then

$$\int_U F^*\omega = -\int_V \omega.$$

Proof. In oriented coordinates, the first statement is the Euclidean change-of-variables theorem without an absolute value because the Jacobian determinant is positive. If F reverses orientation, the determinant is negative, producing the additional minus sign. □

8.5 Global Integration on Oriented Manifolds

Let M be an oriented n-manifold and let $\omega \in \Omega_c^n(M)$. Choose an oriented coordinate cover $\{U_i\}_{i\in I}$ and a smooth partition of unity $\{\rho_i\}_{i\in I}$ subordinate to it. Since the family is locally finite and $\operatorname{supp}(\omega)$ is compact, only finitely many of the forms $\rho_i\omega$ are nonzero.

Definition 8.21. Define

$$\int_M \omega = \sum_i \int_{U_i} \rho_i\omega.$$

Theorem 8.22 (Well-definedness of global integration)**.** *The value $\int_M \omega$ is independent of the oriented coordinate cover and the subordinate partition of unity. Hence integration defines a linear map*

$$\int_M : \Omega_c^n(M) \longrightarrow \mathbb{R}.$$

Proof. Let $\{\rho_i\}$ and $\{\sigma_j\}$ be partitions of unity subordinate to two oriented coordinate covers $\{U_i\}$ and $\{V_j\}$. Local finiteness and compactness allow all sums below to be treated as finite sums. Since

$$\sum_j \sigma_j = 1,$$

one has

$$\sum_i \int_{U_i} \rho_i\omega = \sum_{i,j} \int_{U_i \cap V_j} \rho_i\sigma_j\omega.$$

The local integral of each form on $U_i \cap V_j$ is independent of whether it is computed in the coordinates from U_i or from V_j, by Proposition 8.19. Therefore

$$\sum_{i,j} \int_{U_i \cap V_j} \rho_i\sigma_j\omega = \sum_j \int_{V_j} \sigma_j\omega.$$

This proves independence. Linearity follows from linearity of Euclidean integration. □

Proposition 8.23 (Positivity). *Let $\omega \in \Omega^n_c(M)$ be nonnegative. Then*

$$\int_M \omega \geq 0.$$

If ω is positive at some point, then

$$\int_M \omega > 0.$$

Proof. Choose an oriented coordinate partition as above. Every local coefficient of $\rho_i\omega$ is nonnegative, so each local integral is nonnegative. If ω is positive at p, then it is positive on a neighborhood of p. Some partition function is positive at p, and the corresponding local integrand is positive on a nonempty open subset, so its integral is strictly positive. □

Proposition 8.24 (Disjoint unions). *Let*

$$M = \coprod_{\alpha \in A} M_\alpha$$

be an oriented disjoint union. For $\omega \in \Omega^n_c(M)$,

$$\int_M \omega = \sum_{\alpha \in A} \int_{M_\alpha} \omega|_{M_\alpha},$$

where only finitely many terms are nonzero.

Proof. A compact subset of a disjoint union meets only finitely many components. The assertion then follows by taking coordinate partitions separately on those components. □

Theorem 8.25 (Change of variables on manifolds). *Let*

$$F : M \longrightarrow N$$

be a diffeomorphism of oriented n-manifolds, and let $\omega \in \Omega^n_c(N)$. If F preserves orientation, then

$$\int_M F^*\omega = \int_N \omega.$$

If F reverses orientation, then

$$\int_M F^*\omega = -\int_N \omega.$$

Proof. Choose an oriented coordinate cover $\{V_i\}$ of N and a subordinate partition of unity $\{\rho_i\}$. The sets $F^{-1}(V_i)$ form a coordinate cover of M, and $\{\rho_i \circ F\}$ is a subordinate partition of unity. Using local change of variables,

$$\int_M F^*\omega = \sum_i \int_{F^{-1}(V_i)} F^*(\rho_i\omega) = \sum_i \int_{V_i} \rho_i\omega = \int_N \omega$$

when F preserves orientation. The same calculation gives a minus sign in every local term when F reverses orientation. $\square$

Definition 8.26 (Integration over an oriented submanifold)**.** Let $S \subseteq M$ be a compact oriented embedded k-dimensional submanifold, with inclusion $i : S \hookrightarrow M$. For $\omega \in \Omega^k(M)$, define

$$\int_S \omega = \int_S i^*\omega.$$

More generally, if $f : S \to M$ is any smooth map from a compact oriented k-manifold, define

$$\int_S f^*\omega.$$

8.6 Manifolds with Boundary

The standing convention in the preceding chapters excluded boundaries. We now enlarge the category only as far as required for Stokes' theorem.

Put

$$\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n \geq 0\}.$$

Its boundary hyperplane is

$$\partial\mathbb{H}^n = \{x^n = 0\}.$$

Definition 8.27. A *topological n-manifold with boundary* is a Hausdorff, second-countable space locally homeomorphic to relatively open subsets of $\mathbb{H}^n$.

A function on a relatively open subset of $\mathbb{H}^n$ is *smooth* if it is locally the restriction of a smooth function on an open subset of $\mathbb{R}^n$. Transition maps and smooth maps between manifolds with boundary are defined using this notion of smoothness. A maximal compatible smooth atlas defines a *smooth manifold with boundary.*

A point represented in a boundary chart by a point with $x^n > 0$ is called an *interior point*; a point represented by a point with $x^n = 0$ is called a *boundary point.* We use the following standard topological theorem.

Theorem 8.28 (Invariance of the boundary)**.** *A homeomorphism between relatively open subsets of $\mathbb{H}^n$ carries boundary points to boundary points and interior points to interior points.*

This is a consequence of invariance of domain, or equivalently of the local topological distinction between a Euclidean ball and a half-ball. It implies that the subsets

$$\operatorname{Int} M$$

and

$$\partial M$$

are intrinsically defined.

Proposition 8.29. *Let M be an n-manifold with boundary.*

(1) $\operatorname{Int} M$ *is an open smooth n-manifold without boundary.*

(2) ∂M *has a natural structure of a smooth $(n-1)$-manifold without boundary.*

(3) The inclusion

$$i : \partial M \hookrightarrow M$$

is a smooth embedding.

Proof. The first statement follows by restricting boundary charts to the subset $x^n > 0$.

If $(U, x^1, \dots, x^n)$ is a boundary chart, then

$$U \cap \partial M \longrightarrow x(U) \cap \{x^n = 0\},$$

with coordinates $(x^1, \dots, x^{n-1})$, is a chart on ∂M. Transition maps restrict smoothly because their last coordinate vanishes on the boundary hyperplane. These restricted charts give a smooth $(n-1)$-atlas. In the same charts, the inclusion is represented by

$$(x^1, \dots, x^{n-1}) \longmapsto (x^1, \dots, x^{n-1}, 0),$$

so it is an embedding. □

The tangent-space and differential-form constructions of the preceding chapters extend to manifolds with boundary.

Proposition 8.30. *Let p be a point of an n-manifold with boundary. Then T_pM is an n-dimensional vector space. In a boundary chart,*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$$

form a basis. If $p \in \partial M$, then

$$T_p\partial M = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^{n-1}}\right|_p \right\} \subseteq T_pM.$$

Thus $T_p\partial M$ has codimension one in T_pM.

Proof. A smooth germ at a boundary point is locally the restriction of a smooth germ on an open subset of $\mathbb{R}^n$. Coordinate derivations are therefore defined exactly as in Chapter 2. The Hadamard-lemma argument used there applies to a smooth extension and shows that every derivation is a unique linear combination of the coordinate derivations. The description of $T_p\partial M$ follows from the coordinate expression of the inclusion in Proposition 8.29. □

Smooth vector bundles, differential forms, exterior differentiation, and pullback are defined in boundary charts by the same formulas as before. All local proofs remain valid because the relevant objects admit smooth extensions to Euclidean neighborhoods.

Proposition 8.31 (Partitions of unity with boundary)**.** *Every open cover of a smooth manifold with boundary admits a smooth locally finite subordinate partition of unity.*

Proof. Manifolds with boundary are locally compact, Hausdorff, second countable, and therefore paracompact. Euclidean bump functions restrict to relatively open subsets of $\mathbb{H}^n$. The construction of Theorem 5.8 then applies without change. □

An orientation of a manifold with boundary is defined by an oriented atlas of boundary charts, using the sign of the full n-dimensional Jacobian. The construction of integration extends as follows.

Proposition 8.32 (Integration on manifolds with boundary)**.** *Let M be an oriented n-manifold with boundary. There is a well-defined linear map*

$$\int_M : \Omega^n_c(M) \longrightarrow \mathbb{R}$$

obtained from oriented boundary charts and partitions of unity.

Proof. In a boundary chart, integrate the coefficient of a top form over the relatively open subset of $\mathbb{H}^n$. Coordinate changes admit local smooth extensions to Euclidean neighborhoods. Since the boundary hyperplane has measure zero, the Euclidean change-of-variables theorem proves coordinate independence exactly as in Proposition 8.19. The global construction and its independence follow from Proposition 8.31 and the same common-refinement argument as in Theorem 8.22. □

8.7 The Boundary Orientation

Let M be an oriented n-manifold with boundary and let $p \in \partial M$.

Definition 8.33. A vector $v \in T_pM$ is *outward pointing* if, in a boundary chart $(U, x^1, \ldots, x^n)$,

$$dx^n_p(v) < 0.$$

It is *inward pointing* if $dx^n_p(v) > 0$.

Lemma 8.34. *The notions of inward- and outward-pointing vectors are independent of the boundary chart.*

Proof. Let $y = F(x)$ be a transition map between two boundary charts, and let p correspond to $x^n = 0$. Since the transition map preserves the boundary,

$$F^n(x^1, \ldots, x^{n-1}, 0) = 0.$$

Therefore

$$\frac{\partial F^n}{\partial x^j}(p) = 0$$

for $j < n$. The derivative DF_p is invertible, so

$$\frac{\partial F^n}{\partial x^n}(p) \neq 0.$$

Because $F^n(x', t) \geq 0$ for $t \geq 0$ and $F^n(x', 0) = 0$, its one-sided normal derivative is nonnegative; hence it is strictly positive. Consequently,

$$dy^n_p(v) = \frac{\partial F^n}{\partial x^n}(p)\, dx^n_p(v),$$

so the two normal components have the same sign. □

Definition 8.35 (Outward-normal-first convention)**.** An ordered basis

$$(v_1, \ldots, v_{n-1})$$

of $T_p\partial M$ is positive if

$$(\nu, v_1, \ldots, v_{n-1})$$

is a positive basis of T_pM for one, equivalently every, outward-pointing vector $\nu \in T_pM$. This is the *boundary orientation* on ∂M.

Proposition 8.36. *The outward-normal-first convention is independent of the chosen outward vector and defines a smooth orientation of ∂M.*

Proof. If ν and ν' are outward pointing, then in the one-dimensional quotient

$$T_pM/T_p\partial M$$

they lie on the same positive ray. Hence

$$\nu' = a\nu + w$$

for some $a > 0$ and $w \in T_p\partial M$. Replacing ν by ν' in the ordered basis changes its determinant by the positive factor a, so the orientation is unchanged.

In a boundary chart, the convention is represented by a constant sign relative to the coordinate basis of the boundary. Thus it varies smoothly and defines an orientation of ∂M. □

The local sign will be important in Stokes' theorem.

Proposition 8.37 (Coordinate expression for the boundary orientation)**.** *Let $(U, x^1, \ldots, x^n)$ be an oriented boundary chart, with $x^n \geq 0$. Then the boundary orientation on $U \cap \partial M$ is represented by the positive $(n-1)$-form*

$$(-1)^n\, dx^1 \wedge \cdots \wedge dx^{n-1}.$$

Proof. The vector

$$-\frac{\partial}{\partial x^n}$$

is outward pointing. Moving it from the first position past the $n-1$ tangential coordinate vectors contributes $(-1)^{n-1}$, and its minus sign contributes another factor -1. Hence

$$\left(-\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{n-1}}\right)$$

has sign $(-1)^n$ relative to the positive coordinate basis of M. The displayed boundary form produces exactly this correction. □

Example 8.38 (The oriented interval)**.** Give $[a, b]$ the standard orientation determined by dx. At b, the outward direction is positive, while at a it is negative. Therefore, as an oriented zero-manifold,

$$\partial[a, b] = \{b\} - \{a\}.$$

Example 8.39 (The orientation of the sphere)**.** Give the closed ball B^{n+1} its standard orientation. The boundary orientation on

$$S^n = \partial B^{n+1}$$

is characterized by the condition that

$$(\nu, v_1, \ldots, v_n)$$

is positive in $\mathbb{R}^{n+1}$ whenever ν is the outward radial vector and $(v_1, \ldots, v_n)$ is a positive tangent basis of S^n.

8.8 Stokes' Theorem

We first prove the local calculation on Euclidean space and half-space.

Lemma 8.40 (Euclidean Stokes formulas)**.** *Let*

$$\eta = \sum_{j=1}^{n} (-1)^{j-1} a_j \, dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

be a compactly supported smooth $(n-1)$-form.

(1) On $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} d\eta = 0.$$

(2) On $\mathbb{H}^n$, with the standard orientation and the induced boundary orientation,

$$\int_{\mathbb{H}^n} d\eta = \int_{\partial \mathbb{H}^n} i^* \eta.$$

Proof. The coordinate formula gives

$$d\eta = \left(\sum_{j=1}^{n} \frac{\partial a_j}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n.$$

For $\mathbb{R}^n$, Fubini's theorem and the fundamental theorem of calculus show that the integral of each compactly supported partial derivative is zero.

For $\mathbb{H}^n$, the same argument makes the terms with $j < n$ vanish. For the normal term,

$$\int_0^\infty \frac{\partial a_n}{\partial x^n}(x', x^n) \, dx^n = -a_n(x', 0).$$

Therefore

$$\int_{\mathbb{H}^n} d\eta = -\int_{\mathbb{R}^{n-1}} a_n(x', 0) \, dx'.$$

Only the $j = n$ term survives under restriction to the boundary:

$$i^* \eta = (-1)^{n-1} a_n(x', 0) \, dx^1 \wedge \cdots \wedge dx^{n-1}.$$

By Proposition 8.37, the positive boundary volume form is

$$(-1)^n \, dx^1 \wedge \cdots \wedge dx^{n-1}.$$

Thus the coefficient of $i^*\eta$ relative to the positive boundary form is $-a_n(x', 0)$, giving the same integral. □

Theorem 8.41 (Stokes' theorem)**.** *Let M be an oriented n-manifold with boundary, and let*

$$i : \partial M \hookrightarrow M$$

be the inclusion. For every $\eta \in \Omega_c^{n-1}(M)$,

$$\int_M d\eta = \int_{\partial M} i^* \eta.$$

If M is compact, the compact-support hypothesis is automatic.

Proof. Choose a locally finite partition of unity $\{\rho_i\}$ subordinate to a cover by interior and boundary charts. Since η is compactly supported, only finitely many forms $\rho_i\eta$ are nonzero. Moreover,

$$\eta = \sum_i \rho_i \eta,$$

so

$$d\eta = \sum_i d(\rho_i \eta)$$

and

$$i^*\eta = \sum_i i^*(\rho_i \eta).$$

It is therefore enough to prove the formula when η is compactly supported in one chart.

If the chart is an interior chart, transport the form to a compactly supported form in an open subset of $\mathbb{R}^n$ and extend it by zero to $\mathbb{R}^n$. The extension is smooth because its support is compactly contained in the chart image. The first part of Lemma 8.40 gives

$$\int_M d\eta = 0.$$

The restriction of η to ∂M is zero, so the boundary integral also vanishes.

If the chart is a boundary chart, transport the form to a relatively open subset of $\mathbb{H}^n$ and extend it by zero to $\mathbb{H}^n$. The second part of Lemma 8.40 gives the desired equality. Coordinate independence of the integrals and the definition of the boundary orientation transport the equality back to M. □

Corollary 8.42 (Fundamental theorem of calculus)**.** *For $f \in C^\infty([a,b])$,*

$$\int_a^b df = f(b) - f(a).$$

Equivalently,

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

Proof. Apply Stokes' theorem to the oriented interval and use Example 8.38. □

Corollary 8.43. *Let M be an oriented n-manifold without boundary. Then*

$$\int_M d\eta = 0$$

for every $\eta \in \Omega_c^{n-1}(M)$.

Proof. The boundary integral is zero because $\partial M = \varnothing$. □

Corollary 8.44. *Let M be a compact oriented manifold with boundary. If $\theta \in \Omega^{n-2}(M)$, then*

$$\int_{\partial M} d(i^*\theta) = 0.$$

Proof. Naturality of the exterior derivative gives

$$d(i^*\theta) = i^*(d\theta).$$

Apply Stokes' theorem to the closed oriented manifold ∂M. □

8.9 Cohomological Consequences and Periods

Let M be a compact oriented n-manifold without boundary. By Corollary 8.43, the integral of an exact top form vanishes. Hence integration descends to de Rham cohomology.

Proposition 8.45 (Integration on top-degree cohomology)**.** *There is a well-defined linear map*

$$\int_M : H^n_{\mathrm{dR}}(M) \longrightarrow \mathbb{R}, \qquad [\omega] \longmapsto \int_M \omega.$$

It changes sign when the orientation of M is reversed.

Proof. If $\omega' = \omega + d\eta$, then

$$\int_M \omega' - \int_M \omega = \int_M d\eta = 0.$$

Thus the integral depends only on the cohomology class. Reversing orientation changes the sign of every local integral. □

We do not yet assert that this map is an isomorphism for connected M; that statement belongs to Poincaré duality, or to an independent computation of top-degree cohomology.

Definition 8.46. Let S be a compact oriented k-manifold without boundary and let

$$f : S \longrightarrow M$$

be smooth. For a closed k-form ω on M, the number

$$\int_S f^*\omega$$

is called the *period* of ω over the oriented smooth cycle (S, f).

Proposition 8.47 (Periods depend only on cohomology classes)**.** *For every compact oriented k-manifold S without boundary and every smooth map $f : S \to M$, integration defines a linear functional*

$$\mathrm{per}_{S,f} : H^k_{\mathrm{dR}}(M) \longrightarrow \mathbb{R},$$
$$\mathrm{per}_{S,f}([\omega]) = \int_S f^*\omega.$$

Proof. If $\omega' = \omega + d\eta$, then naturality of d and Stokes' theorem give

$$\int_S f^*\omega' - \int_S f^*\omega = \int_S d(f^*\eta) = \int_{\partial S} f^*\eta = 0.$$

□

Example 8.48 (A nontrivial period around the origin)**.** On $\mathbb{R}^2 \setminus \{0\}$, consider

$$\alpha = \frac{-y\,dx + x\,dy}{x^2 + y^2}.$$

A direct calculation gives $d\alpha = 0$. Let

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2 \setminus \{0\}, \qquad \gamma(t) = (\cos t, \sin t).$$

The line integral of α along this closed loop is defined by pullback to the oriented interval. Since

$$\gamma^*\alpha = dt,$$

one obtains

$$\int_\gamma \alpha = \int_{[0,2\pi]} \gamma^* \alpha = \int_0^{2\pi} dt = 2\pi.$$

If $\alpha = df$, the fundamental theorem of calculus would give

$$\int_\gamma \alpha = f(\gamma(2\pi)) - f(\gamma(0)) = 0,$$

because $\gamma(0) = \gamma(2\pi)$. Consequently α is not exact and

$$[\alpha] \neq 0$$

in $H^1_{\mathrm{dR}}(\mathbb{R}^2 \setminus \{0\})$. This line integral agrees with the period over the positively oriented unit circle.

Chapter 9

Mayer–Vietoris and Cohomological Computations

The preceding chapters constructed de Rham cohomology, proved its homotopy invariance, and related its classes to integration and periods. We now develop a computational principle which reconstructs the cohomology of a manifold from an open cover. The sheaf property of differential forms gives exactness at the level of local data, while partitions of unity provide the surjectivity needed to pass from local forms to a short exact sequence of complexes. The resulting Mayer–Vietoris sequence reduces global calculations to simpler open subsets and their intersection.

We first construct the sequence and describe its connecting homomorphism explicitly. We then compute the de Rham cohomology of spheres, punctured Euclidean spaces, and the two-dimensional torus. These examples also show how periods and products distinguish cohomology classes.

9.1 The de Rham Sequence of an Open Cover

Let M be a smooth manifold and suppose that

$$M = U \cup V$$

for open subsets $U, V \subseteq M$. For every degree k, define

$$r^k : \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V)$$

by

$$r^k(\omega) = (\omega|_U, \omega|_V)\,,$$

and define

$$s^k : \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U \cap V)$$

by

$$s^k(\alpha, \beta) = \beta|_{U\cap V} - \alpha|_{U\cap V}.$$

Restriction commutes with the exterior derivative, so

$$r^{k+1}d = dr^k, \qquad s^{k+1}d = ds^k.$$

Thus r and s are morphisms of cochain complexes.

Theorem 9.1 (The de Rham sequence of an open cover)**.** *For every open cover $M = U \cup V$, the sequence*

$$0 \longrightarrow \Omega^\bullet(M) \xrightarrow{r} \Omega^\bullet(U) \oplus \Omega^\bullet(V) \xrightarrow{s} \Omega^\bullet(U \cap V) \longrightarrow 0$$

is a short exact sequence of cochain complexes.

Proof. A differential form is determined by its restrictions to an open cover, so r is injective.

Suppose that
$$s(\alpha, \beta) = 0.$$
Then
$$\alpha|_{U\cap V} = \beta|_{U\cap V}.$$
The sheaf property of differential forms gives a unique
$$\omega \in \Omega^k(M)$$
whose restrictions to U and V are α and β. Hence
$$\ker s = \operatorname{im} r.$$

It remains to prove that s is surjective. Choose a smooth partition of unity subordinate to the cover $\{U, V\}$. By summing members assigned to the same open set, we may write it as two functions
$$\rho_U + \rho_V = 1,$$
with
$$\operatorname{supp}(\rho_U) \subseteq U, \qquad \operatorname{supp}(\rho_V) \subseteq V.$$
Let
$$\gamma \in \Omega^k(U \cap V).$$
On $U \cap V$, form the products
$$-\rho_V\gamma \qquad \text{and} \qquad \rho_U\gamma.$$
The support of the first product, considered relative to U, is contained in $U \cap V$, because $\operatorname{supp}(\rho_V) \subseteq V$. It therefore extends by zero to a smooth form
$$\alpha \in \Omega^k(U).$$
Similarly, $\rho_U\gamma$ extends by zero to a smooth form
$$\beta \in \Omega^k(V).$$
On the intersection,
$$\beta - \alpha = (\rho_U + \rho_V)\gamma = \gamma.$$
Thus $s(\alpha, \beta) = \gamma$, proving surjectivity. □

Remark 9.2. The only genuinely global input in the proof is the partition of unity. The identity
$$\ker s = \operatorname{im} r$$
is the ordinary sheaf-gluing property, whereas surjectivity of s expresses a special feature of the smooth category.

9.2 The Mayer–Vietoris Sequence

A short exact sequence of cochain complexes induces a long exact sequence on cohomology. Applying this general theorem to Theorem 9.1 gives the principal computational tool of this chapter.

Theorem 9.3 (Mayer–Vietoris sequence)**.** *Let $M = U \cup V$ be an open cover. There is a natural long exact sequence*

$$\begin{aligned} 0 \longrightarrow H^0_{\mathrm{dR}}(M) &\longrightarrow H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \longrightarrow H^0_{\mathrm{dR}}(U \cap V) \\ \xrightarrow{\delta} H^1_{\mathrm{dR}}(M) &\longrightarrow H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V) \longrightarrow H^1_{\mathrm{dR}}(U \cap V) \\ \xrightarrow{\delta} H^2_{\mathrm{dR}}(M) &\longrightarrow \cdots . \end{aligned}$$

In general, its degree-k portion is

$$\begin{aligned} H^{k-1}_{\mathrm{dR}}(U) \oplus H^{k-1}_{\mathrm{dR}}(V) \longrightarrow H^{k-1}_{\mathrm{dR}}(U \cap V) &\xrightarrow{\delta} H^k_{\mathrm{dR}}(M) \\ &\longrightarrow H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \longrightarrow H^k_{\mathrm{dR}}(U \cap V). \end{aligned}$$

The nonconnecting maps are induced by

$$[\omega] \longmapsto ([\omega|_U], [\omega|_V])$$

and

$$([\alpha], [\beta]) \longmapsto [\beta|_{U\cap V} - \alpha|_{U\cap V}].$$

Proof. Theorem 9.1 is a short exact sequence of cochain complexes. The general long-exact-sequence theorem in homological algebra gives the displayed sequence. The formulas for the ordinary maps are obtained by passing the cochain maps r and s to cohomology. □

Remark 9.4. Naturality means that a smooth map respecting two chosen open covers induces a morphism between the corresponding long exact sequences. In the calculations below, only exactness and the explicit form of the maps will be required.

9.3 The Connecting Homomorphism

The connecting homomorphism admits a direct description in terms of differential forms. This description will also produce explicit global classes from forms on an overlap.

Let

$$\gamma \in \Omega^{k-1}(U \cap V), \qquad d\gamma = 0.$$

By surjectivity in Theorem 9.1, choose

$$\alpha \in \Omega^{k-1}(U), \qquad \beta \in \Omega^{k-1}(V)$$

such that

$$\beta|_{U\cap V} - \alpha|_{U\cap V} = \gamma.$$

Then

$$d\beta|_{U\cap V} - d\alpha|_{U\cap V} = d\gamma = 0.$$

Consequently, $d\alpha$ and $d\beta$ glue to a unique form

$$\eta \in \Omega^k(M)$$

satisfying

$$\eta|_U = d\alpha, \qquad \eta|_V = d\beta.$$

Since $d\eta$ vanishes on both members of the cover, η is closed.

Proposition 9.5 (Explicit connecting homomorphism)**.** *With the notation above, the connecting homomorphism is*

$$\delta : H^{k-1}_{\mathrm{dR}}(U \cap V) \longrightarrow H^k_{\mathrm{dR}}(M), \qquad \delta[\gamma] = [\eta].$$

The class $[\eta]$ is independent of all choices.

Proof. Suppose that another pair (α', β') also lifts γ. Then

$$(\beta' - \beta) - (\alpha' - \alpha) = 0$$

on $U \cap V$. Exactness in the middle of Theorem 9.1 gives a global form

$$\theta \in \Omega^{k-1}(M)$$

with

$$\theta|_U = \alpha' - \alpha, \qquad \theta|_V = \beta' - \beta.$$

If η' is obtained from the second lift, then

$$\eta' - \eta = d\theta.$$

Thus $[\eta'] = [\eta]$.

In degree $k = 1$, there are no nonzero exact 0-forms, so dependence on the representative requires no further verification. Suppose $k \geq 2$ and replace γ by a cohomologous form

$$\gamma + d\zeta.$$

Choose

$$\mu \in \Omega^{k-2}(U), \qquad \nu \in \Omega^{k-2}(V)$$

with

$$\nu - \mu = \zeta$$

on $U \cap V$. Then

$$(\alpha + d\mu, \beta + d\nu)$$

is a lift of $\gamma + d\zeta$, while its exterior derivatives are still $d\alpha$ and $d\beta$. Hence the resulting class is unchanged. This proves that the construction depends only on $[\gamma]$.

The construction is precisely the standard connecting morphism for a short exact sequence of cochain complexes: lift a cocycle, apply the differential, and identify the result with an element of the left-hand complex. □

The lift can be made explicit from a partition of unity. Let

$$\rho_U + \rho_V = 1$$

be subordinate to $\{U, V\}$. As in the proof of Theorem 9.1, take

$$\alpha = -\widetilde{\rho_V \gamma}, \qquad \beta = \widetilde{\rho_U \gamma},$$

where each product is extended by zero to the appropriate open set. Since $d\gamma = 0$, the two pieces of the resulting global form satisfy

$$d\alpha = -d\rho_V \wedge \gamma = d\rho_U \wedge \gamma = d\beta$$

on $U \cap V$. Thus the connecting class is represented locally by $d\rho_U \wedge \gamma$.

9.4 The de Rham Cohomology of Spheres

Let $n \geq 1$, and write a point of $S^n \subseteq \mathbb{R}^{n+1}$ as

$$(x', x^{n+1}), \qquad x' \in \mathbb{R}^n.$$

Let N and S denote the north and south poles, and set

$$U = S^n \setminus \{S\}, \qquad V = S^n \setminus \{N\}.$$

Stereographic projection shows that U and V are diffeomorphic to $\mathbb{R}^n$ and hence are smoothly contractible.

The intersection is the sphere with both poles removed. It is diffeomorphic to

$$S^{n-1} \times \mathbb{R}.$$

Indeed, the map

$$U \cap V \longrightarrow S^{n-1} \times \mathbb{R},$$
$$(x', x^{n+1}) \longmapsto \left(\frac{x'}{\|x'\|}, \operatorname{artanh}(x^{n+1}) \right)$$

has inverse

$$(u, t) \longmapsto (\operatorname{sech}(t)u, \tanh(t)).$$

Consequently,

$$U \cap V \simeq S^{n-1}$$

by smooth homotopy equivalence.

We begin with the circle. When $n = 1$, the intersection $U \cap V$ has two connected components. Therefore

$$H^0_{\mathrm{dR}}(U \cap V) \simeq \mathbb{R}^2,$$

whereas

$$H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \simeq \mathbb{R}^2.$$

Under these identifications, the difference-of-restrictions map is

$$D : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad D(a, b) = (b - a, b - a).$$

Its image is the diagonal line, so its cokernel is one-dimensional. Since U and V have no positive-degree cohomology, Mayer–Vietoris gives

$$H^1_{\mathrm{dR}}(S^1) \simeq \mathbb{R}.$$

For $n \geq 2$, the intersection $U \cap V$ is connected. Hence the map

$$H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \longrightarrow H^0_{\mathrm{dR}}(U \cap V)$$

is

$$\mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad (a, b) \longmapsto b - a,$$

and is surjective. It follows that

$$H^1_{\mathrm{dR}}(S^n) = 0$$

for $n \geq 2$. In every degree $k \geq 2$, exactness and the vanishing of the positive-degree cohomology of U and V give an isomorphism

$$\delta : H^{k-1}_{\mathrm{dR}}(U \cap V) \xrightarrow{\simeq} H^k_{\mathrm{dR}}(S^n).$$

Using the homotopy equivalence $U \cap V \simeq S^{n-1}$, we obtain

$$H^k_{\mathrm{dR}}(S^n) \simeq H^{k-1}_{\mathrm{dR}}(S^{n-1})$$

for $k \geq 2$.

Theorem 9.6 (Cohomology of spheres)**.** *For every $n \geq 1$,*

$$H^k_{\mathrm{dR}}(S^n) \simeq \begin{cases} \mathbb{R}, & k = 0, n, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The sphere is connected, so

$$H^0_{\mathrm{dR}}(S^n) \simeq \mathbb{R}.$$

The case $n = 1$ was established above. For $n \geq 2$, the preceding Mayer–Vietoris calculation gives

$$H^1_{\mathrm{dR}}(S^n) = 0$$

and, for $k \geq 2$,

$$H^k_{\mathrm{dR}}(S^n) \simeq H^{k-1}_{\mathrm{dR}}(S^{n-1}).$$

Induction on n now gives the stated groups. □

Equip S^n with its standard orientation as the boundary of the unit ball. The integration map constructed in Chapter 8 becomes completely explicit in top degree.

Corollary 9.7 (Integration on the sphere)**.** *For every $n \geq 1$, integration is an isomorphism*

$$\int_{S^n} : H^n_{\mathrm{dR}}(S^n) \xrightarrow{\simeq} \mathbb{R}.$$

Consequently, there is a unique class

$$u_n \in H^n_{\mathrm{dR}}(S^n)$$

satisfying

$$\int_{S^n} u_n = 1.$$

Moreover,

$$H^\bullet_{\mathrm{dR}}(S^n) \simeq \mathbb{R} \oplus \mathbb{R}u_n,$$

with u_n in degree n and

$$u_n^2 = 0.$$

Proof. Choose a positive smooth top form ω on the compact oriented manifold S^n. By Proposition 8.23,

$$\int_{S^n} \omega > 0.$$

If ω were exact, its integral would vanish by Stokes' theorem. Thus $[\omega]$ is nonzero. Since

$$\dim H^n_{\mathrm{dR}}(S^n) = 1,$$

the integration map is a nonzero linear map between one-dimensional vector spaces and hence is an isomorphism. The normalized class u_n follows. Finally, u_n^2 lies in degree $2n > n$ and is therefore zero. □

9.5 Punctured Euclidean Spaces and Angular Forms

Let $n \geq 2$. The inclusion

$$i : S^{n-1} \longrightarrow \mathbb{R}^n \setminus \{0\}$$

and radial projection

$$p : \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}, \qquad p(x) = \frac{x}{\|x\|},$$

are smooth homotopy inverses. Indeed,

$$p \circ i = \mathrm{id}_{S^{n-1}},$$

and

$$H(x,t) = \left((1-t) + \frac{t}{\|x\|}\right) x$$

is a smooth homotopy from the identity of $\mathbb{R}^n \setminus \{0\}$ to $i \circ p$. Homotopy invariance and Theorem 9.6 therefore give the following.

Corollary 9.8 (Cohomology of punctured Euclidean space)**.** *For $n \geq 2$,*

$$H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\}) \simeq \begin{cases} \mathbb{R}, & k = 0, n-1, \\ 0, & \textit{otherwise.} \end{cases}$$

The nonzero class in degree $n-1$ has a canonical explicit representative. Let

$$\varrho = \left((x^1)^2 + \cdots + (x^n)^2\right)^{1/2}$$

on $\mathbb{R}^n \setminus \{0\}$, and define

$$\Theta = \sum_{i=1}^{n} (-1)^{i-1} x^i \, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

Set

$$\eta_{n-1} = \varrho^{-n} \Theta.$$

Proposition 9.9 (The angular form)**.** *The form η_{n-1} is closed. Its restriction to the standardly oriented unit sphere S^{n-1} is a positive top form. Consequently,*

$$[\eta_{n-1}] \neq 0$$

in

$$H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\}),$$

and this class generates that one-dimensional vector space.

Proof. Let

$$\mathrm{vol} = dx^1 \wedge \cdots \wedge dx^n.$$

Differentiating Θ gives

$$d\Theta = n \,\mathrm{vol}\,.$$

Moreover, all terms with repeated differentials vanish, and the remaining terms give

$$\left(\sum_{j=1}^{n} x^j \, dx^j\right) \wedge \Theta = \varrho^2 \,\mathrm{vol}\,.$$

Since

$$d(\varrho^{-n}) = -n\varrho^{-n-2}\sum_{j=1}^{n} x^j\,dx^j,$$

the graded Leibniz rule yields

$$\begin{aligned} d\eta_{n-1} &= d(\varrho^{-n}) \wedge \Theta + \varrho^{-n} d\Theta \\ &= -n\varrho^{-n}\operatorname{vol} + n\varrho^{-n}\operatorname{vol} \\ &= 0. \end{aligned}$$

At a point $u \in S^{n-1}$, the outward normal vector is u. If

$$(v_1, \ldots, v_{n-1})$$

is a positive basis of T_uS^{n-1} for the boundary orientation, then

$$(u, v_1, \ldots, v_{n-1})$$

is a positive basis of $\mathbb{R}^n$. Since $\varrho = 1$ on the unit sphere,

$$(i^*\eta_{n-1})_u(v_1, \ldots, v_{n-1}) = \operatorname{vol}_u(u, v_1, \ldots, v_{n-1}) > 0.$$

Thus $i^*\eta_{n-1}$ is positive, and therefore

$$\int_{S^{n-1}} i^*\eta_{n-1} > 0.$$

An exact form has zero period over the closed oriented sphere, so η_{n-1} is not exact. Its class is nonzero and hence generates the one-dimensional group from Corollary 9.8. □

Let

$$c_n = \int_{S^{n-1}} i^*\eta_{n-1} > 0$$

and put

$$\omega_{n-1} = \frac{1}{c_n}\eta_{n-1}.$$

Then

$$\int_{S^{n-1}} i^*\omega_{n-1} = 1.$$

When $n = 2$,

$$\eta_1 = \frac{-y\,dx + x\,dy}{x^2 + y^2},$$

and $c_2 = 2\pi$. Thus the form considered at the end of Chapter 8 is precisely the one-dimensional angular form.

9.6 The Two-Dimensional Torus

The sphere calculation has at most one nonzero class in each positive degree. The torus provides the first example in which the first cohomology has rank greater than one and the multiplication of positive-degree classes is nontrivial.

Write

$$T^2 = S^1 \times S^1.$$

Choose connected proper open arcs $A, B \subseteq S^1$ such that

$$A \cup B = S^1$$

and $A \cap B$ has exactly two connected components. Put

$$U = S^1 \times A, \qquad V = S^1 \times B.$$

Both U and V are smoothly homotopy equivalent to S^1, while $U \cap V$ is the disjoint union of two spaces smoothly homotopy equivalent to S^1. Therefore

$$H^0_{\mathrm{dR}}(U) \simeq H^0_{\mathrm{dR}}(V) \simeq \mathbb{R},$$

$$H^1_{\mathrm{dR}}(U) \simeq H^1_{\mathrm{dR}}(V) \simeq \mathbb{R},$$

and

$$H^0_{\mathrm{dR}}(U \cap V) \simeq \mathbb{R}^2, \qquad H^1_{\mathrm{dR}}(U \cap V) \simeq \mathbb{R}^2.$$

All higher groups of U, V, and $U \cap V$ vanish.

Under compatible choices of generators, the difference-of-restrictions maps in both degrees zero and one are represented by

$$D : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad D(a, b) = (b - a, b - a).$$

Indeed, a constant function restricts to the same constant on both components of the intersection. In degree one, choose the generator pulled back from the first S^1 factor. Its restriction to each component of $U \cap V$ is again the same generator. Thus

$$\ker D \simeq \mathbb{R}, \qquad \operatorname{coker} D \simeq \mathbb{R}.$$

The Mayer–Vietoris sequence becomes

$$\begin{aligned} 0 \longrightarrow H^0_{\mathrm{dR}}(T^2) \longrightarrow \mathbb{R}^2 \xrightarrow{D} \mathbb{R}^2 \longrightarrow H^1_{\mathrm{dR}}(T^2) \\ \longrightarrow \mathbb{R}^2 \xrightarrow{D} \mathbb{R}^2 \longrightarrow H^2_{\mathrm{dR}}(T^2) \longrightarrow 0. \end{aligned}$$

Exactness gives the following groups.

Theorem 9.10 (Cohomology groups of the two-torus)**.** *The de Rham cohomology of T^2 is*

$$H^k_{\mathrm{dR}}(T^2) \simeq \begin{cases} \mathbb{R}, & k = 0, 2, \\ \mathbb{R}^2, & k = 1, \\ 0, & k > 2. \end{cases}$$

Proof. The torus is connected, so its degree-zero cohomology is $\mathbb{R}$. In the long exact sequence above, exactness gives a short exact sequence

$$0 \longrightarrow \operatorname{coker} D \longrightarrow H^1_{\mathrm{dR}}(T^2) \longrightarrow \ker D \longrightarrow 0.$$

Both outer vector spaces are one-dimensional, so

$$\dim H^1_{\mathrm{dR}}(T^2) = 2.$$

Exactness at the next stage gives

$$H^2_{\mathrm{dR}}(T^2) \simeq \operatorname{coker} D \simeq \mathbb{R}.$$

There are no forms in degrees greater than two. $\square$

We now determine the ring structure. Regard S^1 as the unit circle in $\mathbb{R}^2$, and let

$$\lambda = x\,dy - y\,dx$$

be the induced 1-form on S^1. Under the positive parametrization

$$t \longmapsto (\cos t, \sin t),$$

one has

$$\lambda = dt,$$

and hence

$$\int_{S^1} \lambda = 2\pi.$$

Let

$$p_1, p_2 : T^2 \longrightarrow S^1$$

be the two projections, and define

$$\alpha = p_1^*\lambda, \qquad \beta = p_2^*\lambda.$$

Both forms are closed.

Fix $q \in S^1$ and consider the two oriented coordinate circles

$$\iota_1 : S^1 \longrightarrow T^2, \qquad \iota_1(z) = (z, q),$$

and

$$\iota_2 : S^1 \longrightarrow T^2, \qquad \iota_2(z) = (q, z).$$

Their periods are

$$\int_{S^1} \iota_1^*\alpha = 2\pi, \qquad \int_{S^1} \iota_1^*\beta = 0,$$
$$\int_{S^1} \iota_2^*\alpha = 0, \qquad \int_{S^1} \iota_2^*\beta = 2\pi.$$

It follows that $[\alpha]$ and $[\beta]$ are linearly independent. By Theorem 9.10, they form a basis of $H^1_{\mathrm{dR}}(T^2)$.

We shall use the following product-integration formula.

Lemma 9.11 (Integration on products)**.** *Let M and N be compact oriented manifolds of dimensions m and n. For $\mu \in \Omega^m(M)$ and $\nu \in \Omega^n(N)$,*

$$\int_{M\times N} \mathrm{pr}_M^*\,\mu \wedge \mathrm{pr}_N^*\,\nu = \left(\int_M \mu\right)\left(\int_N \nu\right),$$

where $M \times N$ has the product orientation.

Proof. Choose finite oriented coordinate covers of M and N with subordinate partitions of unity $\{\rho_i\}$ and $\{\sigma_j\}$. Then

$$\{(\rho_i \circ \mathrm{pr}_M)(\sigma_j \circ \mathrm{pr}_N)\}_{i,j}$$

is a partition of unity on $M \times N$ subordinate to the corresponding product charts. On each product chart, the form in the statement is the product of the local coefficient of μ and the local coefficient of ν, multiplied by the standard product volume form. The Euclidean Fubini theorem gives the asserted factorization for each pair of chart-supported terms. Summing over i and j proves the formula. □

Applying the lemma to the circle gives

$$\int_{T^2} \alpha \wedge \beta = \left(\int_{S^1} \lambda \right)^2 = (2\pi)^2.$$

Therefore $[\alpha \wedge \beta]$ is nonzero and hence generates the one-dimensional space $H^2_{\mathrm{dR}}(T^2)$.

Theorem 9.12 (The de Rham algebra of the two-torus)**.** *There is an isomorphism of graded-commutative algebras*

$$H^\bullet_{\mathrm{dR}}(T^2) \simeq \Lambda_{\mathbb{R}}\left([\alpha], [\beta]\right),$$

where both generators have degree one.

Proof. The classes $[\alpha]$ and $[\beta]$ form a basis of degree one, and their product $[\alpha \wedge \beta]$ generates degree two. Since the wedge product is alternating,

$$[\alpha]^2 = [\beta]^2 = 0,$$

and graded commutativity gives

$$[\beta][\alpha] = -[\alpha][\beta].$$

The dimensions in Theorem 9.10 now show that the induced homomorphism from the exterior algebra is an isomorphism in every degree. □

Remark 9.13. The calculations in this chapter exhibit the two complementary roles of de Rham theory. Mayer–Vietoris reconstructs global cohomology from local pieces, while integration and periods identify concrete differential forms as nontrivial classes. For spheres the unique top-degree class is detected by its total integral; for the torus, periods detect two independent degree-one classes and their wedge product generates the top degree.