Algebraic Geometry

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Chapter 1

Sheaves

Algebraic geometry is built from local algebraic data. A regular function is usually first understood on a sufficiently small open set; local equations are compared on overlaps; and many global objects are assembled from compatible local pieces. Sheaf theory is the language that makes this passage between local and global information systematic.

The purpose of this chapter is deliberately limited. We develop the local-to-global formalism that will be used throughout the rest of the notes, but we do not yet introduce ringed spaces, sheaves of modules over a structure sheaf, or schemes. Those notions become meaningful only after the basic behavior of sheaves has been understood.

1.1 Local Data and Gluing

Let X be a topological space. The most primitive local object is an assignment that associates data to each open subset and allows that data to be restricted to smaller open subsets.

Definition 1.1. A *presheaf of sets* $\mathscr{F}$ on X consists of the following data:

1. for every open subset $U \subseteq X$, a set $\mathscr{F}(U)$;
2. for every inclusion $U \subseteq V$ of open subsets, a restriction map

$$\rho_{V,U} : \mathscr{F}(V) \longrightarrow \mathscr{F}(U),$$

such that

$$\rho_{U,U} = \mathrm{id}_{\mathscr{F}(U)}, \qquad \rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$$

whenever $U \subseteq V \subseteq W$.

The elements of $\mathscr{F}(U)$ are called *sections of* $\mathscr{F}$ *over* U. We usually write $s|_U$ for $\rho_{V,U}(s)$.

Equivalently, if $\mathrm{Open}(X)$ denotes the category of open subsets of X with inclusions as morphisms, a presheaf is a contravariant functor

$$\mathscr{F} : \mathrm{Open}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

We will use this categorical description when it clarifies a construction, but no separate categorical formalism will be needed.

Example 1.2 (Functions)**.** Let X be a topological space. For every open subset $U \subseteq X$, let

$$\mathscr{F}(U) = C^0(U, \mathbb{R}).$$

Restriction of functions gives the restriction maps. The same construction works for smooth functions on a smooth manifold and for holomorphic functions on a complex manifold.

A presheaf tells us how to restrict data, but it does not yet say that compatible local data determine a global object. This is the additional content of the sheaf condition.

Definition 1.3. A presheaf $\mathscr{F}$ on X is a *sheaf* if, for every open subset $U \subseteq X$ and every open covering

$$U = \bigcup_{i \in I} U_i,$$

the following conditions hold.

1. **Locality.** If $s, t \in \mathscr{F}(U)$ and

$$s|_{U_i} = t|_{U_i} \qquad \text{for every } i \in I,$$

then $s = t$.

2. **Gluing.** If sections $s_i \in \mathscr{F}(U_i)$ satisfy

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \qquad \text{for all } i, j,$$

then there exists a section $s \in \mathscr{F}(U)$ such that

$$s|_{U_i} = s_i \qquad \text{for every } i.$$

The section obtained by gluing is unique by locality.

The preceding example of continuous functions is a sheaf: continuous functions that agree on overlaps glue to a unique continuous function. The same statement holds for smooth and holomorphic functions.

Example 1.4 (The constant presheaf)**.** Let A be a set and define a presheaf by assigning A to every nonempty open subset, with all restriction maps equal to the identity. This presheaf is generally not a sheaf. If

$$U = U_1 \sqcup U_2$$

is disconnected, one may choose different elements of A on U_1 and U_2. These local sections are compatible because the overlap is empty, but there is no section of the constant presheaf on U restricting to both choices.

The associated sheaf will later be identified with the sheaf of locally constant A-valued functions.

This example isolates the main point: the sheaf axiom is not merely a compatibility condition on restriction maps. It asserts that local objects which agree wherever they can be compared are precisely the restrictions of a unique global object.

When the sections carry an abelian group structure and the restriction maps are homomorphisms, the sheaf condition can be written algebraically. This formulation will later reappear as the beginning of the Čech complex.

Proposition 1.5 (Equalizer form of the sheaf condition)**.** *Let $\mathscr{F}$ be a presheaf of abelian groups. Then $\mathscr{F}$ is a sheaf if and only if, for every open covering $U = \bigcup_{i \in I} U_i$, the sequence*

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\rho} \prod_{i \in I} \mathscr{F}(U_i) \xrightarrow{\delta} \prod_{(i,j) \in I \times I} \mathscr{F}(U_i \cap U_j)$$

is exact, where

$$\rho(s) = (s|_{U_i})_i,$$

and

$$\delta((s_i)_i) = \left(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}\right)_{i,j}.$$

Proof. The injectivity of ρ is exactly the locality axiom. A family $(s_i)_i$ lies in $\ker\delta$ precisely when the sections agree on every overlap $U_i \cap U_j$. The equality

$$\ker\delta = \operatorname{im}\rho$$

therefore says exactly that every compatible family of local sections is obtained by restricting a global section. This is the gluing axiom. □

Remark 1.6. The exact sequence in proposition 1.5 is the degree-zero part of a construction that will later lead to Čech cohomology. At the present stage, its role is simply to express gluing as an algebraic condition.

1.2 Germs, Stalks, and Localization

The sheaf axiom compares sections on open subsets. Often, however, one wants to retain only the information visible arbitrarily close to a single point. This leads to germs and stalks.

Definition 1.7. Let $\mathscr{F}$ be a presheaf on X and let $x \in X$. The *stalk of $\mathscr{F}$ at x* is the direct limit

$$\mathscr{F}_x := \varinjlim_{x\in U} \mathscr{F}(U),$$

where U runs over the open neighbourhoods of x, ordered by reverse inclusion.

Concretely, an element of $\mathscr{F}_x$ is represented by a pair (U, s) with $x \in U$ and $s \in \mathscr{F}(U)$. Two pairs (U, s) and (V, t) represent the same element if there is an open neighbourhood

$$W \subseteq U \cap V$$

of x such that

$$s|_W = t|_W.$$

The equivalence class of (U, s) is denoted s_x and is called the *germ of s at x*.

Thus a stalk remembers a section only up to shrinking its domain around the chosen point. For example, the stalk of the sheaf of holomorphic functions at a point is the ring of holomorphic germs at that point.

The first fundamental use of stalks is that sections of a sheaf are determined by their germs.

Proposition 1.8. *Let $\mathscr{F}$ be a sheaf on X, let $U \subseteq X$ be open, and let $s, t \in \mathscr{F}(U)$. If*

$$s_x = t_x \qquad \text{for every } x \in U,$$

then $s = t$.

Proof. For every $x \in U$, equality of the germs means that there is an open neighbourhood $U_x \subseteq U$ on which s and t agree. The sets U_x cover U. By the locality axiom, $s = t$. □

Morphisms of sheaves are likewise controlled locally.

Definition 1.9. A *morphism of presheaves*

$$\varphi : \mathscr{F} \longrightarrow \mathscr{G}$$

is a family of maps

$$\varphi_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$$

for open subsets $U \subseteq X$, compatible with all restriction maps. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, we call φ a morphism of sheaves.

Every morphism $\varphi : \mathscr{F} \to \mathscr{G}$ induces, for each $x \in X$, a map on stalks

$$\varphi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x, \qquad \varphi_x(s_x) = \big(\varphi_U(s)\big)_x.$$

Compatibility with restrictions makes this definition independent of the chosen representative.

Theorem 1.10 (Stalk criterion for isomorphisms)**.** *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets on X. Then φ is an isomorphism if and only if*

$$\varphi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x$$

is a bijection for every $x \in X$.

Proof. If φ is an isomorphism, then every induced map on stalks is clearly a bijection.

Conversely, suppose that every φ_x is bijective. Let $U \subseteq X$ be open and let $t \in \mathscr{G}(U)$. For every $x \in U$, surjectivity of φ_x gives a neighbourhood $U_x \subseteq U$ and a section $s_x \in \mathscr{F}(U_x)$ whose image has the same germ as t at x. After shrinking U_x if necessary, we may assume

$$\varphi(s_x) = t|_{U_x}.$$

On an overlap $U_x \cap U_y$, the images of s_x and s_y agree. Since the maps on stalks are injective, the germs of s_x and s_y agree at every point of the overlap. By proposition 1.8, the sections themselves agree on the overlap. The sheaf axiom therefore glues the s_x to a section $s \in \mathscr{F}(U)$ satisfying $\varphi(s) = t$. Hence φ_U is surjective.

If $s, t \in \mathscr{F}(U)$ have the same image under φ_U, then for every $x \in U$ their germs have the same image under the injective map φ_x. Thus $s_x = t_x$ for all x, and proposition 1.8 gives $s = t$. Hence φ_U is injective. Therefore φ is an isomorphism. $\square$

The preceding theorem explains why sheaf theory is intrinsically local. A global morphism of sheaves can be tested entirely on the objects attached to individual points.

A presheaf has stalks even when it fails the gluing axiom. This suggests a natural question: can one repair the failure of gluing without changing the local information encoded by the stalks? The answer is sheafification.

Definition 1.11. Let $\mathscr{P}$ be a presheaf on X. A *sheafification* of $\mathscr{P}$ is a sheaf $\mathscr{P}^{\mathrm{sh}}$ together with a morphism

$$\eta : \mathscr{P} \longrightarrow \mathscr{P}^{\mathrm{sh}}$$

such that, for every sheaf $\mathscr{G}$ and every morphism of presheaves $\alpha : \mathscr{P} \to \mathscr{G}$, there is a unique morphism of sheaves

$$\widetilde{\alpha} : \mathscr{P}^{\mathrm{sh}} \longrightarrow \mathscr{G}$$

with

$$\alpha = \widetilde{\alpha} \circ \eta.$$

The universal property says that $\mathscr{P}^{\mathrm{sh}}$ is the most economical sheaf receiving a morphism from $\mathscr{P}$. It remains to construct it.

For an open subset $U \subseteq X$, let $\mathscr{P}^{\mathrm{sh}}(U)$ be the set of maps

$$\sigma : U \longrightarrow \coprod_{x \in U} \mathscr{P}_x$$

such that $\sigma(x) \in \mathscr{P}_x$ for every x and such that σ is *locally represented by sections of $\mathscr{P}$*: for every $x \in U$, there exist an open neighbourhood $V \subseteq U$ of x and a section $s \in \mathscr{P}(V)$ such that

$$\sigma(y) = s_y \qquad \text{for all } y \in V.$$

Restrictions are obtained by restricting the domain of σ.

Theorem 1.12 (Sheafification)**.** *The presheaf $\mathscr{P}^{\mathrm{sh}}$ constructed above is a sheaf. The maps*

$$\eta_U : \mathscr{P}(U) \longrightarrow \mathscr{P}^{\mathrm{sh}}(U), \qquad s \longmapsto (x \mapsto s_x),$$

define a sheafification of $\mathscr{P}$. Moreover, for every $x \in X$, the induced map

$$\eta_x : \mathscr{P}_x \xrightarrow{\sim} (\mathscr{P}^{\mathrm{sh}})_x$$

is an isomorphism.

Proof. Because elements of $\mathscr{P}^{\mathrm{sh}}(U)$ are functions on U, locality is immediate. If locally representable functions agree on overlaps, they glue as ordinary functions; the resulting function is still locally representable, so the gluing axiom also holds. Hence $\mathscr{P}^{\mathrm{sh}}$ is a sheaf.

The maps η_U commute with restrictions. To see that η_x is surjective, let a germ in $(\mathscr{P}^{\mathrm{sh}})_x$ be represented by a locally representable function σ on a neighbourhood of x. Near x, the function σ is represented by some section s of $\mathscr{P}$, so the germ of σ is the image of s_x. For injectivity, suppose that two germs $s_x, t_x \in \mathscr{P}_x$ have the same image in $(\mathscr{P}^{\mathrm{sh}})_x$. Then the corresponding locally representable functions agree on some neighbourhood of x, and therefore the two germs are equal.

Finally, let $\alpha : \mathscr{P} \to \mathscr{G}$ be a morphism into a sheaf. If $\sigma \in \mathscr{P}^{\mathrm{sh}}(U)$ is represented locally by sections $s_i \in \mathscr{P}(U_i)$, then the sections $\alpha(s_i)$ agree on overlaps because the s_i determine the same local germs. They therefore glue to a section of $\mathscr{G}(U)$. This defines $\widetilde{\alpha}$, and uniqueness follows because every section of $\mathscr{P}^{\mathrm{sh}}$ is locally in the image of η. $\square$

Corollary 1.13. *A sheafification is unique up to a unique isomorphism compatible with the canonical map from the original presheaf.*

Proof. If $(\mathscr{P}^{\mathrm{sh}}, \eta)$ and $(\mathscr{Q}, \eta')$ are two sheafifications, their universal properties produce unique morphisms in both directions compatible with η and η'. The composites are endomorphisms compatible with the canonical maps, so uniqueness forces both composites to be identities. $\square$

Example 1.14 (The constant sheaf)**.** Let A be a set. The sheafification of the constant presheaf with value A is the sheaf

$$U \longmapsto \{\text{locally constant maps } U \to A\}.$$

Thus the failure of the constant presheaf on disconnected open sets is repaired by allowing the value to vary from one connected local piece to another.

1.3 Algebraic Structure and Exactness

The definitions above make sense for sheaves of sets, but algebraic geometry requires sheaves whose sections carry algebraic structure. A presheaf of abelian groups, rings, or modules over a fixed ring is defined by requiring each $\mathscr{F}(U)$ to have the corresponding structure and each restriction map to preserve it. It is a sheaf when the underlying presheaf of sets is a sheaf. Sheafification preserves these algebraic structures because the operations can be defined on germs.

We will use sheaves of abelian groups frequently. Their kernels behave exactly as one might first expect, while images and cokernels reveal why sheafification is an essential operation rather than a technical afterthought.

Let

$$\varphi : \mathscr{F} \longrightarrow \mathscr{G}$$

be a morphism of sheaves of abelian groups. Define

$$(\ker \varphi)(U) = \ker(\varphi_U).$$

This is already a sheaf. By contrast, the sectionwise image

$$U \longmapsto \operatorname{im}(\varphi_U)$$

and the sectionwise cokernel

$$U \longmapsto \operatorname{coker}(\varphi_U)$$

need not be sheaves. We therefore define the image and cokernel sheaves by sheafifying these presheaves.

Proposition 1.15. *For every $x \in X$, there are natural isomorphisms*

$$(\ker \varphi)_x \cong \ker(\varphi_x), \qquad (\operatorname{im} \varphi)_x \cong \operatorname{im}(\varphi_x), \qquad (\operatorname{coker} \varphi)_x \cong \operatorname{coker}(\varphi_x).$$

Proof. The statement for kernels follows directly from the definition of germs: a germ maps to zero if and only if it is represented on some neighbourhood by a section mapping to zero.

For the image, the stalk of the sectionwise image consists exactly of germs that are locally represented by images of sections of $\mathscr{F}$. This is the same as the image of φ_x. Since sheafification does not change stalks by theorem 1.12, the same is true for the image sheaf.

For cokernels, taking a stalk is a filtered direct limit over neighbourhoods of x. We use here the standard algebraic fact that filtered direct limits of abelian groups are exact. In particular they preserve cokernels. Hence the stalk of the sectionwise cokernel is $\operatorname{coker}(\varphi_x)$, and sheafification again leaves the stalk unchanged. This exactness statement is a property of modules (or abelian groups), not an additional sheaf-theoretic input. □

Definition 1.16. A sequence of sheaves of abelian groups

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$$

is *exact at $\mathscr{G}$* if

$$\operatorname{im}(\alpha) = \ker(\beta)$$

as subsheaves of $\mathscr{G}$.

The combination of theorem 1.10 and proposition 1.15 gives the principal local criterion for exactness.

Theorem 1.17 (Stalkwise criterion for exactness)**.** *The sequence*

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$$

is exact at $\mathscr{G}$ if and only if, for every $x \in X$, the sequence

$$\mathscr{F}_x \xrightarrow{\alpha_x} \mathscr{G}_x \xrightarrow{\beta_x} \mathscr{H}_x$$

is exact at $\mathscr{G}_x$.

Proof. By proposition 1.15,

$$(\operatorname{im}\alpha)_x \cong \operatorname{im}(\alpha_x), \qquad (\ker\beta)_x \cong \ker(\beta_x).$$

If the sheaf sequence is exact, the two subsheaves agree and hence have equal stalks, giving exactness at every point.

Conversely, suppose all stalk sequences are exact. The natural inclusion

$$\operatorname{im}\alpha \longrightarrow \ker\beta$$

is then an isomorphism on every stalk. By theorem 1.10, it is an isomorphism of sheaves. Hence the original sequence is exact at $\mathscr{G}$. □

Corollary 1.18. *A morphism of sheaves of abelian groups is a monomorphism, epimorphism, or isomorphism if and only if its maps on all stalks are respectively injective, surjective, or bijective.*

The word "surjective" for sheaves must be interpreted locally. A surjective morphism of sheaves need not be surjective on sections over a fixed open subset.

Remark 1.19 (Local lifting). Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a surjective morphism of sheaves and let $s \in \mathscr{G}(U)$. For every $x \in U$, surjectivity of φ_x gives a neighbourhood $V \subseteq U$ of x such that $s|_V$ lifts to a section of $\mathscr{F}(V)$. These local lifts need not glue to a global lift on U.

For an open subset $U \subseteq X$, write

$$\Gamma(U, \mathscr{F}) := \mathscr{F}(U).$$

In particular, $\Gamma(X, \mathscr{F})$ is the group of global sections.

Proposition 1.20 (Left exactness of sections)**.** *The functor $\Gamma(U, -)$ is left exact. Thus a short exact sequence*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathscr{F}'(U) \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{F}''(U),$$

but the final map need not be surjective.

Proof. Kernels of morphisms of sheaves are computed sectionwise, so taking sections preserves kernels. Surjectivity may fail because a section of $\mathscr{F}''$ can have local lifts without possessing a global lift. □

This failure is the first indication of why cohomology will eventually be needed: higher sheaf cohomology measures, among other things, obstructions to globalizing local lifting data.

Example 1.21 (The exponential sequence)**.** Let X be a complex manifold, let $\mathscr{O}_X$ be the sheaf of holomorphic functions, and let $\mathscr{O}_X^\times$ be the sheaf of nowhere-vanishing holomorphic functions. There is a short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n \mapsto 2\pi i n} \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \longrightarrow 1.$$

Exactness at the final term is local: every nowhere-vanishing holomorphic function admits a holomorphic logarithm near each point.

Nevertheless, the map on global sections need not be surjective. On $X = \mathbb{C}^\times$, the function z has no global holomorphic logarithm. Thus this short exact sequence gives a concrete example in which local lifting exists but global lifting fails.

1.4 Functoriality and Basic Constructions

Sheaves live on topological spaces, so a continuous map between spaces should induce operations on sheaves. These operations will later become part of the basic language of morphisms of schemes.

Let

$$f : X \longrightarrow Y$$

be a continuous map.

If $U \subseteq X$ is open, the restriction of a sheaf $\mathscr{F}$ on X to U is the sheaf $\mathscr{F}|_U$ defined by

$$(\mathscr{F}|_U)(V) = \mathscr{F}(V) \qquad (V \subseteq U \text{ open}).$$

Restriction is the simplest example of changing the underlying space.

Definition 1.22 (Direct image)**.** For a sheaf $\mathscr{F}$ on X, its *direct image* under f is the sheaf $f_*\mathscr{F}$ on Y defined by

$$(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$$

for every open subset $V \subseteq Y$.

The sheaf axiom for $f_*\mathscr{F}$ follows immediately from the sheaf axiom for $\mathscr{F}$, since inverse images preserve unions and intersections of open subsets. Its stalk at $y \in Y$ is

$$(f_*\mathscr{F})_y = \varinjlim_{y \in V} \mathscr{F}(f^{-1}(V)).$$

In general this should not be confused with sections of $\mathscr{F}$ over the set-theoretic fibre $f^{-1}(y)$.

Proposition 1.23. *On sheaves of abelian groups, the functor f_* is left exact.*

Proof. For every open $V \subseteq Y$, evaluation of $f_*\mathscr{F}$ on V is the section functor

$$\Gamma(f^{-1}(V), \mathscr{F}),$$

which is left exact by proposition 1.20. □

The inverse image is slightly subtler. A section of a sheaf on Y can be pulled toward X whenever it is defined on an open set whose inverse image contains the open set under consideration. This first produces a presheaf and then a sheaf by sheafification.

Definition 1.24 (Inverse image)**.** Let $\mathscr{G}$ be a sheaf on Y. Define a presheaf on X by

$$U \longmapsto \varinjlim_{U \subseteq f^{-1}(V)} \mathscr{G}(V),$$

where V ranges over the open subsets of Y satisfying $U \subseteq f^{-1}(V)$. The sheafification of this presheaf is the *inverse image sheaf*, denoted $f^{-1}\mathscr{G}$.

Proposition 1.25. *For every $x \in X$, there is a natural isomorphism*

$$(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}.$$

If $\mathscr{G}$ is a sheaf of abelian groups, the functor f^{-1} is exact.

Proof. Let $\mathscr{P}$ denote the presheaf used in the definition of $f^{-1}\mathscr{G}$. An element of the stalk $\mathscr{P}_x$ is represented by a triple (U, V, s) with

$$x \in U \subseteq f^{-1}(V), \qquad s \in \mathscr{G}(V).$$

Necessarily $f(x) \in V$. Conversely, if V is any neighbourhood of $f(x)$, then taking $U = f^{-1}(V)$ gives such a triple. Thus the indexing system of pairs (U, V) maps cofinally to the system of neighbourhoods V of $f(x)$: every pair already contains such a neighbourhood, and every neighbourhood occurs by the choice $U = f^{-1}(V)$. Therefore

$$\mathscr{P}_x \cong \varinjlim_{f(x)\in V} \mathscr{G}(V) = \mathscr{G}_{f(x)}.$$

Sheafification does not change stalks by theorem 1.12, so

$$(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}.$$

For sheaves of abelian groups, exactness can be checked on stalks by theorem 1.17. Applying f^{-1} to an exact sequence and taking the stalk at x gives exactly the original stalk sequence at $f(x)$, which remains exact. Hence f^{-1} is exact. □

Direct and inverse image are related by a universal property.

Proposition 1.26 (Adjunction)**.** *For a sheaf $\mathscr{G}$ on Y and a sheaf $\mathscr{F}$ on X, there is a natural bijection*

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

Thus f^{-1} is left adjoint to f_*.

Proof. Let $\mathscr{P}$ be the presheaf on X whose sheafification is $f^{-1}\mathscr{G}$. Suppose first that

$$\beta : \mathscr{G} \longrightarrow f_*\mathscr{F}$$

is a morphism of sheaves on Y. If a class in $\mathscr{P}(U)$ is represented by a section $s \in \mathscr{G}(V)$ with $U \subseteq f^{-1}(V)$, define its image in $\mathscr{F}(U)$ by

$$s \longmapsto \beta_V(s)|_U,$$

where $\beta_V(s) \in \mathscr{F}(f^{-1}(V))$. Compatibility of β with restrictions shows that this depends only on the class represented in the colimit, and these maps commute with restriction in U. Hence they define a morphism of presheaves $\mathscr{P} \to \mathscr{F}$, which extends uniquely through sheafification to

$$f^{-1}\mathscr{G} \longrightarrow \mathscr{F}.$$

Conversely, let

$$\alpha : f^{-1}\mathscr{G} \longrightarrow \mathscr{F}$$

be a morphism. For an open $V \subseteq Y$, there is a canonical map

$$\mathscr{G}(V) \longrightarrow (f^{-1}\mathscr{G})(f^{-1}(V))$$

coming from the representative V in the defining colimit. Composing with α gives

$$\mathscr{G}(V) \longrightarrow \mathscr{F}(f^{-1}(V)) = (f_*\mathscr{F})(V).$$

These homomorphisms are compatible with restriction, so they define a morphism $\mathscr{G} \to f_*\mathscr{F}$.

The two constructions are inverse: both are determined before sheafification by the same restriction maps, and the universal property of sheafification gives uniqueness. Naturality in both $\mathscr{G}$ and $\mathscr{F}$ is immediate from the constructions. Hence

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

□

If $j : U \hookrightarrow X$ is the inclusion of an open subset, then the inverse image $j^{-1}\mathscr{F}$ is naturally the ordinary restriction $\mathscr{F}|_U$. Thus restriction is not a separate construction but a special case of inverse image.

We close the chapter with two constructions that will recur later.

Example 1.27 (Constant and locally constant sheaves)**.** For a set or abelian group A, the constant sheaf with value A is the sheaf of locally constant A-valued functions. It is the sheafification of the constant presheaf discussed in section 1.1.

Example 1.28 (Skyscraper sheaves)**.** Let $x \in X$, let A be an abelian group, and let

$$i : \{x\} \longrightarrow X$$

be the inclusion. The *skyscraper sheaf at x with value A* is i_*A. Concretely,

$$(i_*A)(U) = \begin{cases} A, & x \in U, \\ 0, & x \notin U. \end{cases}$$

Its stalks are

$$(i_*A)_y \cong \begin{cases} A, & y \in \overline{\{x\}}, \\ 0, & y \notin \overline{\{x\}}. \end{cases}$$

In particular, if x is closed, then the only nonzero stalk is the stalk at x. This distinction will matter for schemes, whose points need not be closed.

For a sheaf of abelian groups, one can also record where a particular section is nonzero.

Definition 1.29 (Support of a section)**.** Let $\mathscr{F}$ be a sheaf of abelian groups and let $s \in \mathscr{F}(U)$. The *support of s* is

$$\mathrm{Supp}(s) := \{x \in U \mid s_x \neq 0\}.$$

Proposition 1.30. *The support* $\mathrm{Supp}(s)$ *is closed in U.*

Proof. If $x \notin \mathrm{Supp}(s)$, then $s_x = 0$. By the definition of a germ, there is an open neighbourhood $V \subseteq U$ of x such that $s|_V = 0$. Every point of V therefore lies outside $\mathrm{Supp}(s)$. Hence the complement of $\mathrm{Supp}(s)$ is open. □

The constructions of this chapter already exhibit the pattern that will dominate the rest of algebraic geometry: global objects are recovered by gluing local objects, stalks isolate the information near a point, and exactness can be tested locally even when global lifting fails. In the next chapter we place a distinguished sheaf of rings on a topological space and use this local algebra to define locally ringed spaces and schemes.

Chapter 2

Schemes

Sheaves provide a language for local data and gluing, but algebraic geometry requires more than a topological space carrying arbitrary local objects. The local data should themselves be rings of functions, and near each point these rings should arise from commutative algebra. The basic model is the spectrum of a ring. General schemes are then obtained by gluing such affine models.

The purpose of this chapter is to construct schemes and establish the basic geometry already visible before one studies morphisms systematically. We begin with affine schemes, pass to general schemes and their morphisms, discuss subschemes and gluing, construct projective schemes through Proj, and then study generic points, function fields, and dimension. Finiteness conditions on morphisms, base change, separatedness, properness, and related relative notions will be postponed to the next chapter.

2.1 Affine Schemes from Rings

Classically, an affine algebraic set over an algebraically closed field is described by polynomial equations and its coordinate ring. If one retains only maximal ideals of the coordinate ring, however, two important kinds of information are lost. First, irreducible subvarieties are not themselves visible as points. Second, nilpotent elements, which encode infinitesimal structure, disappear after passing to the associated reduced algebra. The spectrum construction keeps both kinds of information by using all prime ideals.

Definition 2.1. Let A be a commutative ring with identity. The *spectrum* of A is the set

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}.$$

For an ideal $I \subseteq A$, define

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subseteq \mathfrak{p}\}.$$

The subsets $V(I)$ are declared to be the closed subsets of $\operatorname{Spec} A$. The resulting topology is the *Zariski topology.*

The identities

$$V(0) = \operatorname{Spec} A, \qquad V(1) = \varnothing,$$

$$V(I) \cup V(J) = V(IJ), \qquad \bigcap_{\lambda} V(I_\lambda) = V\left(\sum_{\lambda} I_\lambda\right)$$

show that this is indeed a topology. Notice also that

$$V(I) = V(\sqrt{I}),$$

so the underlying topological space cannot distinguish an ideal from its radical. Scheme structure will restore precisely this missing information.

For $f \in A$, set

$$D(f) := \operatorname{Spec} A \setminus V(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}.$$

These are called *distinguished open subsets.*

Proposition 2.2. *For $f, g \in A$, one has*

$$D(f) \cap D(g) = D(fg).$$

The distinguished opens form a basis for the Zariski topology. Moreover,

$$D(f) = \varnothing \quad \Longleftrightarrow \quad f \text{ is nilpotent.}$$

Proof. A prime ideal contains fg if and only if it contains f or g, which gives the first identity. If $U \subseteq \operatorname{Spec} A$ is open and $\mathfrak{p} \in U$, write the closed complement as $V(I)$. Since $I \not\subseteq \mathfrak{p}$, there is some $f \in I \setminus \mathfrak{p}$, and then

$$\mathfrak{p} \in D(f) \subseteq U.$$

Thus the $D(f)$ form a basis. Finally, $D(f) = \varnothing$ means that f belongs to every prime ideal, hence to the nilradical $\sqrt{(0)}$; equivalently, f is nilpotent. □

Proposition 2.3. *For every ring A, the topological space* $\operatorname{Spec} A$ *is quasi-compact. More generally, every distinguished open $D(f)$ is quasi-compact.*

Proof. It is enough to consider a cover of $\operatorname{Spec} A$ by distinguished opens, since every open cover admits such a refinement. Suppose

$$\operatorname{Spec} A = \bigcup_{\lambda \in \Lambda} D(f_\lambda).$$

No prime ideal contains all the f_λ, so the ideal they generate is not contained in any maximal ideal. Hence

$$(f_\lambda \mid \lambda \in \Lambda) = A.$$

Thus 1 is already a finite linear combination of finitely many f_λ, and the corresponding finitely many distinguished opens cover $\operatorname{Spec} A$.

For the second assertion, prime ideals of A_f are naturally identified with prime ideals of A not containing f. This gives a homeomorphism

$$\operatorname{Spec} A_f \xrightarrow{\sim} D(f),$$

and the first assertion applied to A_f shows that $D(f)$ is quasi-compact. □

Example 2.4 (The spectrum of a field)**.** If k is a field, then $\operatorname{Spec} k$ consists of one point, corresponding to the zero ideal. This one-point scheme will later play the role of a base space over which varieties are defined.

Example 2.5 (The spectrum of the integers)**.** The points of $\operatorname{Spec}\mathbb{Z}$ are

$$(0), \qquad (p) \quad \text{for primes } p.$$

Every nonempty open subset contains the point (0). Thus (0) behaves like a generic point, while the primes (p) are closed points. This example already shows that points of a scheme need not all have the same geometric character.

The topology alone does not yet tell us which functions should be regarded as regular. On a distinguished open $D(f)$, the element f should be invertible, so the natural ring of functions is the localization A_f. To extend this assignment to all open subsets, we use the fact that a sheaf may be specified on a basis.

Theorem 2.6 (Extension from a basis)**.** *Let X be a topological space and let $\mathcal{B}$ be a basis closed under finite intersections. Suppose rings $\mathscr{F}(B)$ and compatible restriction homomorphisms are given for $B \in \mathcal{B}$, and suppose the sheaf gluing condition holds for coverings by basis elements. Then there exists a sheaf $\mathscr{F}$ on X, unique up to unique isomorphism, whose restriction to $\mathcal{B}$ is the given data.*

Proof. For an open subset $U \subseteq X$, define $\mathscr{F}(U)$ to be the set of compatible families

$$(s_B)_{B\subseteq U,\, B\in\mathcal{B}}, \qquad s_B \in \mathscr{F}(B),$$

such that $s_B|_{B'} = s_{B'}$ whenever $B' \subseteq B \subseteq U$ are basis elements. Restriction to a smaller open subset simply forgets components. Compatible families over an open cover glue by taking their union, so the resulting presheaf is a sheaf. If $U = B$ is itself a basis element, the sheaf axiom on the basis identifies the constructed value with the original $\mathscr{F}(B)$. Uniqueness follows because any section over an arbitrary open set is determined by its compatible restrictions to a basis covering. $\square$

Theorem 2.7 (The structure sheaf on an affine spectrum)**.** *Let A be a ring and put $X = \operatorname{Spec} A$. The assignment*

$$D(f) \longmapsto A_f$$

with the natural localization maps defines a sheaf of rings $\mathscr{O}_X$ on X. It satisfies

$$\Gamma(X, \mathscr{O}_X) \cong A$$

and, for every $\mathfrak{p} \in X$,

$$\mathscr{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}.$$

Proof. By proposition 2.2, the distinguished opens form a basis closed under finite intersections. It remains to verify the sheaf condition on this basis. Suppose

$$D(f) = \bigcup_{i=1}^{r} D(f_i),$$

where we have used proposition 2.3 to pass to a finite subcover. Put

$$R := A_f, \qquad g_i := f_i/1 \in R.$$

Then $\operatorname{Spec} R = \bigcup_i D(g_i)$. Equivalently, the ideal $(g_1, \dots, g_r)$ has radical equal to R. The same is therefore true after replacing all g_i by a common positive power. Since each $D(f_i)$ is contained in $D(f)$, the element f is invertible in A_{f_i}; consequently

$$R_{g_i} = A_{ff_i} \cong A_{f_i}, \qquad R_{g_ig_j} \cong A_{f_if_j}.$$

Thus it is enough to prove that

$$R \longrightarrow \prod_i R_{g_i} \rightrightarrows \prod_{i,j} R_{g_i g_j}$$

is an equalizer. First suppose $s \in R$ maps to zero in every R_{g_i}. For each i there is an integer $N_i \geq 0$ such that $g_i^{N_i} s = 0$. Taking a common N, we have $g_i^N s = 0$ for all i. Since $(g_1^N, \ldots, g_r^N) = R$, choose $c_i \in R$ with

$$\sum_i c_i g_i^N = 1.$$

Multiplying by s gives $s = 0$, so the first map is injective.

Now let $s_i \in R_{g_i}$ be a compatible family. First choose N_0 so that

$$g_i^{N_0} s_i \in R$$

for every i. Compatibility says that the images of s_i and s_j agree in $R_{g_i g_j}$. Therefore, in R_{g_j}, the element

$$g_i^{N_0} s_i - g_i^{N_0} s_j$$

becomes zero after localizing further at g_i. Hence some power of g_i annihilates it. Since there are only finitely many pairs (i, j), after replacing N_0 by a larger common integer N we may arrange that

$$g_i^N s_i = g_i^N s_j \qquad \text{in } R_{g_j}$$

for every i, j, with $g_i^N s_i$ viewed through its representative in R. Again the elements g_i^N generate the unit ideal, so choose $c_i \in R$ such that

$$\sum_i c_i g_i^N = 1.$$

Define

$$s := \sum_i c_i g_i^N s_i \in R,$$

where each $g_i^N s_i$ has already been viewed as an element of R. In R_{g_j} we obtain

$$s - s_j = \sum_i c_i (g_i^N s_i - g_i^N s_j) = 0,$$

so s restricts to the prescribed section s_j on every $D(g_j)$. This proves the gluing statement and hence the equalizer property. By theorem 2.6, the assignment $D(f) \mapsto A_f$ therefore extends to a sheaf.

Since $X = D(1)$, we have $\Gamma(X, \mathscr{O}_X) = A_1 \cong A$. Finally,

$$\mathscr{O}_{X,\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in D(f)} A_f = \varinjlim_{f \notin \mathfrak{p}} A_f \cong A_{\mathfrak{p}}.$$

Indeed, the last colimit inverts precisely the elements of $A \backslash \mathfrak{p}$, which is the universal property of $A_{\mathfrak{p}}$. $\square$

The stalk computation explains why local rings are built into the definition of a scheme.

Definition 2.8. A *ringed space* is a pair $(X, \mathscr{O}_X)$ consisting of a topological space and a sheaf of rings. It is a *locally ringed space* if every stalk $\mathscr{O}_{X,x}$ is a local ring.

For a locally ringed space and a point $x \in X$, let $\mathfrak{m}_x \subseteq \mathscr{O}_{X,x}$ be the maximal ideal. The field

$$\kappa(x) := \mathscr{O}_{X,x}/\mathfrak{m}_x$$

is called the *residue field* at x.

Because $A_\mathfrak{p}$ is local with maximal ideal $\mathfrak{p}A_\mathfrak{p}$, theorem 2.7 makes $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ a locally ringed space. Its residue field at $\mathfrak{p}$ is

$$\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong \operatorname{Frac}(A/\mathfrak{p}).$$

Definition 2.9. An *affine scheme* is a locally ringed space isomorphic to

$$(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$$

for some ring A.

Thus an affine scheme is not merely the set of prime ideals of a ring. Its geometry is carried by the pair consisting of the Zariski space and the sheaf of localizations.

2.2 Schemes and Their Morphisms

A general scheme is obtained by allowing different affine coordinate rings on different open pieces. Before defining it, we must identify the correct notion of morphism. A continuous map alone is insufficient: regular functions on the target must pull back to regular functions on the source, and this pullback must respect the local nature of the stalks.

Definition 2.10. A *morphism of ringed spaces*

$$(f, f^\#) : (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$$

consists of a continuous map $f : X \to Y$ together with a morphism of sheaves of rings

$$f^\# : \mathscr{O}_Y \longrightarrow f_*\mathscr{O}_X.$$

Equivalently, for every open subset $V \subseteq Y$, there is a ring homomorphism

$$f^\#(V) : \mathscr{O}_Y(V) \longrightarrow \mathscr{O}_X(f^{-1}V)$$

compatible with restrictions.

For each $x \in X$, such a morphism induces a homomorphism on stalks

$$f^\#_x : \mathscr{O}_{Y,f(x)} \longrightarrow \mathscr{O}_{X,x}.$$

Definition 2.11. A morphism of locally ringed spaces is a morphism of ringed spaces such that every $f^\#_x$ is a local homomorphism, equivalently

$$(f^\#_x)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}.$$

The local condition guarantees that a function vanishing at $f(x)$ pulls back to one vanishing at x, and that units remain units locally.

Definition 2.12. A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ admitting an open covering

$$X = \bigcup_i U_i$$

such that each $(U_i, \mathscr{O}_X|_{U_i})$ is an affine scheme. A morphism of schemes is a morphism of locally ringed spaces.

This definition is local by construction: every point has an affine neighbourhood, and all questions that can be checked on such neighbourhoods reduce to commutative algebra.

The contravariance of Spec is fundamental.

Theorem 2.13 (Affine duality)**.** *For rings A and B, there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} B, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathrm{Ring}}(A, B).$$

Thus the category of affine schemes is anti-equivalent to the category of commutative rings.

Proof. A ring homomorphism $\varphi : A \to B$ defines a map

$$\varphi^* : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A, \qquad \mathfrak{q} \longmapsto \varphi^{-1}(\mathfrak{q}).$$

Since

$$(\varphi^*)^{-1}(D(f)) = D(\varphi(f)),$$

the map is continuous. On distinguished opens, localization gives compatible homomorphisms

$$A_f \longrightarrow B_{\varphi(f)},$$

and these determine a morphism of sheaves

$$\mathscr{O}_{\operatorname{Spec} A} \longrightarrow (\varphi^*)_* \mathscr{O}_{\operatorname{Spec} B}.$$

At a point $\mathfrak{q} \in \operatorname{Spec} B$, the induced homomorphism is

$$A_{\varphi^{-1}(\mathfrak{q})} \longrightarrow B_{\mathfrak{q}}.$$

Its inverse image of the maximal ideal $\mathfrak{q}B_{\mathfrak{q}}$ is $\varphi^{-1}(\mathfrak{q})A_{\varphi^{-1}(\mathfrak{q})}$, so it is local. Thus φ determines a morphism of affine schemes.

Conversely, let

$$g : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

be a morphism of schemes, and let

$$\varphi := g^{\#}(\operatorname{Spec} A) : A \longrightarrow B$$

be the induced homomorphism on global sections. We claim first that

$$g(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q})$$

for every $\mathfrak{q} \in \operatorname{Spec} B$. Write $\mathfrak{p} = g(\mathfrak{q})$. The local map

$$g^{\#}_{\mathfrak{q}} : A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$$

extends φ. Hence, for $a \in A$,

$$a \in \mathfrak{p} \iff a/1 \in \mathfrak{p}A_{\mathfrak{p}} \iff \varphi(a)/1 \in \mathfrak{q}B_{\mathfrak{q}} \iff \varphi(a) \in \mathfrak{q},$$

where the middle equivalence uses that $g^\#_{\mathfrak{q}}$ is local. Thus the underlying continuous map of g is exactly φ^*.

It remains to identify the map of structure sheaves. On a distinguished open $D(a) \subseteq \operatorname{Spec} A$, its inverse image is $D(\varphi(a))$. The restriction of $g^\#$ gives a homomorphism

$$A_a \longrightarrow B_{\varphi(a)}$$

whose composite with $A \to A_a$ is φ. Since $\varphi(a)$ is invertible in $B_{\varphi(a)}$, the universal property of localization shows that this homomorphism is uniquely the localization of φ. Therefore the original morphism g is precisely the one constructed from φ.

Starting with a ring homomorphism and taking global sections clearly recovers the same homomorphism, since $D(1)$ is the whole spectrum. The two constructions are therefore inverse and natural. □

There is a useful non-affine version of the same statement.

Theorem 2.14. *For every scheme X and every ring A, there is a natural bijection*

$$\operatorname{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} A) \cong \operatorname{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathscr{O}_X)).$$

Proof. A morphism $g : X \to \operatorname{Spec} A$ gives, by applying $g^\#$ to the whole target, a homomorphism

$$A = \Gamma(\operatorname{Spec} A, \mathscr{O}) \longrightarrow \Gamma(X, \mathscr{O}_X).$$

Conversely, let

$$\alpha : A \longrightarrow \Gamma(X, \mathscr{O}_X)$$

be a ring homomorphism. For $x \in X$, compose α with the germ map to obtain

$$\alpha_x : A \longrightarrow \mathscr{O}_{X,x},$$

and define

$$f(x) := \alpha_x^{-1}(\mathfrak{m}_x) \in \operatorname{Spec} A.$$

For $a \in A$, we have

$$f^{-1}(D(a)) = \{x \in X \mid \alpha(a)_x \notin \mathfrak{m}_x\} = \{x \in X \mid \alpha(a)_x \text{ is a unit in } \mathscr{O}_{X,x}\}.$$

Invertibility is local: if a germ is a unit, a local inverse is represented on some neighbourhood and the product equals 1 after shrinking. Thus $f^{-1}(D(a))$ is open, so f is continuous.

On $f^{-1}(D(a))$, the section $\alpha(a)$ is invertible. Hence the restriction of α factors uniquely through localization:

$$A_a \longrightarrow \Gamma(f^{-1}(D(a)), \mathscr{O}_X).$$

These maps are compatible with further localization and therefore, because the $D(a)$ form a basis, define a morphism of sheaves

$$f^\# : \mathscr{O}_{\operatorname{Spec} A} \longrightarrow f_*\mathscr{O}_X.$$

Let $x \in X$ and put $\mathfrak{p} = f(x)$. The induced map on stalks is the localization

$$A_{\mathfrak{p}} \longrightarrow \mathscr{O}_{X,x}$$

of α_x. If $b/c \in A_{\mathfrak{p}}$ with $c \notin \mathfrak{p}$, then $\alpha(c)_x$ is a unit. Consequently

$$\frac{b}{c} \in \mathfrak{p}A_{\mathfrak{p}} \iff b \in \mathfrak{p} \iff \alpha(b)_x \in \mathfrak{m}_x \iff \alpha(b)_x\alpha(c)_x^{-1} \in \mathfrak{m}_x.$$

Thus the stalk map is local, and $(f, f^{\#})$ is a morphism of schemes.

We now check that the constructions are inverse. Start with a morphism $g : X \to \operatorname{Spec} A$ and let α be its map on global sections. For $x \in X$, if $\mathfrak{p} = g(x)$, the local homomorphism

$$A_{\mathfrak{p}} \longrightarrow \mathscr{O}_{X,x}$$

shows that $\alpha_x^{-1}(\mathfrak{m}_x) = \mathfrak{p}$. Hence the reconstructed continuous map is g. On each $D(a)$, both sheaf morphisms are the unique localization of the same map $A \to \Gamma(X, \mathscr{O}_X)$, so they agree. Conversely, beginning with α, the resulting morphism induces α again on global sections by construction. This proves the bijection. □

Definition 2.15. Let S be a scheme. An S*-scheme* is a scheme X equipped with a specified morphism

$$X \longrightarrow S,$$

called its *structure morphism*.

When $S = \operatorname{Spec} k$ for a field k, one usually says that X is a scheme over k.

Residue fields describe morphisms from one-point schemes.

Proposition 2.16 (Field-valued points)**.** *Let X be a scheme and let K be a field. To give a morphism*

$$\operatorname{Spec} K \longrightarrow X$$

is equivalent to giving a point $x \in X$ together with a field homomorphism

$$\kappa(x) \longrightarrow K.$$

Proof. The unique point of $\operatorname{Spec} K$ maps to some $x \in X$. The corresponding local homomorphism

$$\mathscr{O}_{X,x} \longrightarrow K$$

sends $\mathfrak{m}_x$ to zero and therefore factors uniquely through a field homomorphism

$$\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x \longrightarrow K.$$

Conversely, suppose $x \in X$ and a field homomorphism $\kappa(x) \to K$ are given. Choose an affine neighbourhood

$$U = \operatorname{Spec} A$$

of x, and let $\mathfrak{p} \subseteq A$ be the corresponding prime. The composite

$$A \longrightarrow A_{\mathfrak{p}} = \mathscr{O}_{X,x} \longrightarrow \kappa(x) \longrightarrow K$$

induces by affine duality a morphism

$$\operatorname{Spec} K \longrightarrow U \hookrightarrow X$$

whose unique point maps to x.

This construction is independent of the chosen affine neighbourhood. Indeed, if V is another affine neighbourhood of x, choose an affine open $W \subseteq U \cap V$ containing x. Both morphisms factor through W, since their topological image is the point x, and on W both are induced by the same local homomorphism $\mathscr{O}_{X,x} \to K$. Affine duality therefore identifies them. The two procedures are inverse, because the local homomorphism attached to the constructed morphism is exactly the original composite $\mathscr{O}_{X,x} \to \kappa(x) \to K$. □

The proposition explains why a topological point and a rational point are not the same notion. A point $x \in X$ records a prime-like geometric position, while a K-valued point also chooses an embedding of its residue field into K.

2.3 Subschemes, Gluing, and Affine-Local Methods

Schemes are local objects, so one should be able both to pass to smaller pieces and to reconstruct a global scheme from compatible pieces. Open subschemes correspond to localization, while closed subschemes correspond to quotient rings. These two operations already exhibit the fundamental geometric contrast between inverting a function and imposing an equation.

Proposition 2.17. *Let $X = \operatorname{Spec} A$ and let $f \in A$. Then the open subspace $D(f)$, with the restricted structure sheaf, is affine and there is a natural isomorphism*

$$D(f) \cong \operatorname{Spec} A_f.$$

Proof. Prime ideals of A_f are in bijection with prime ideals of A not containing f, so the underlying spaces are naturally homeomorphic. Under this correspondence a distinguished open $D(a/f^n) \subseteq \operatorname{Spec} A_f$ corresponds to $D(a) \cap D(f)$. Their rings of sections agree because

$$(A_f)_{a/f^n} \cong A_{af}.$$

Thus the structure sheaves agree on a basis. □

Definition 2.18. Let X be a scheme and $U \subseteq X$ open. The locally ringed space

$$(U, \mathscr{O}_X|_U)$$

is called the *open subscheme* associated to U. A morphism $j : Y \to X$ is an *open immersion* if it identifies Y isomorphically with such an open subscheme.

For closed subsets, simply restricting the structure sheaf is not the right operation. One must quotient by the equations defining the closed subset.

Proposition 2.19. *Let A be a ring and let $I \subseteq A$ be an ideal. The quotient homomorphism $A \to A/I$ induces a closed immersion*

$$\operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec} A$$

whose image is $V(I)$. On the stalk at a prime $\mathfrak{p} \supseteq I$, the induced map is

$$A_{\mathfrak{p}} \longrightarrow A_{\mathfrak{p}}/IA_{\mathfrak{p}}.$$

Proof. Prime ideals of A/I correspond exactly to prime ideals of A containing I, giving a homeomorphism onto $V(I)$. On distinguished opens the map on rings is

$$A_f \longrightarrow (A/I)_f \cong A_f/IA_f,$$

which is surjective. Hence the map of sheaves is surjective as well. □

Definition 2.20. A morphism $i : Y \to X$ is a *closed immersion* if the underlying map is a homeomorphism onto a closed subset of X and the morphism

$$i^\# : \mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y$$

is surjective. Its kernel $\mathscr{I}$ is the *ideal sheaf* of the immersion, and

$$i_*\mathscr{O}_Y \cong \mathscr{O}_X/\mathscr{I}.$$

The distinction between a closed subset and a closed subscheme is essential. Since

$$V(I) = V(\sqrt{I}),$$

the ideals I and $\sqrt{I}$ define the same closed set, but generally

$$\operatorname{Spec}(A/I) \ncong \operatorname{Spec}(A/\sqrt{I}).$$

Nilpotent functions therefore record geometric structure invisible to the underlying topology.

Example 2.21 (A nonreduced point)**.** Let

$$X = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2).$$

The underlying topological space consists of a single point, exactly as for $\operatorname{Spec} k$, but the structure sheaf contains the nonzero nilpotent ε. Thus the two schemes are not isomorphic. The extra nilpotent may be regarded as a first-order infinitesimal thickening of the point.

The converse operation to cutting a scheme into open subsets is gluing.

Theorem 2.22 (Gluing schemes)**.** *Suppose schemes X_i are given, together with open subschemes $U_{ij} \subseteq X_i$ and isomorphisms*

$$\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$$

satisfying

$$U_{ii} = X_i, \qquad \varphi_{ii} = \mathrm{id}, \qquad \varphi_{ji} = \varphi_{ij}^{-1},$$

and the cocycle condition

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$$

on triple overlaps, after the natural identifications. Then there exists a scheme X, unique up to unique isomorphism, obtained by gluing the X_i along the φ_{ij}. Each $X_i \to X$ is an open immersion and the images cover X.

Proof. As a topological space, take the quotient of the disjoint union $\coprod_i X_i$ by the equivalence relation generated by

$$x \sim \varphi_{ij}(x), \qquad x \in U_{ij}.$$

The cocycle condition makes the identifications consistent, and the images of the X_i are open and cover the quotient.

For an open subset $V \subseteq X$, define $\mathscr{O}_X(V)$ to be the families

$$(s_i), \qquad s_i \in \mathscr{O}_{X_i}(V \cap X_i),$$

whose restrictions agree under the transition isomorphisms on every overlap. The sheaf axioms follow from those on the X_i. By construction the restriction of $(X, \mathscr{O}_X)$ to the image of each X_i is isomorphic to X_i, so X is locally affine and hence a scheme. Uniqueness follows because both the topological space and the structure sheaf are determined by the given local identifications. □

Example 2.23 (The projective line as a gluing)**.** Take two copies

$$U_0 = \operatorname{Spec} k[t], \qquad U_1 = \operatorname{Spec} k[u].$$

Their distinguished opens $D(t)$ and $D(u)$ are both isomorphic to $\operatorname{Spec} k[t, t^{-1}]$. Glue them by the isomorphism $u = t^{-1}$. The resulting scheme is the projective line $\mathbb{P}^1_k$, which will be recovered systematically from Proj in the next section.

Example 2.24 (The affine line with doubled origin)**.** Take two copies of $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ and glue them along the common open subset $D(t)$ by the identity. The resulting scheme has two distinct origins but is otherwise identical to the affine line. This construction shows that the gluing theorem produces schemes that need not behave like ordinary Hausdorff spaces. Later, separatedness will isolate the scheme-theoretic analogue of the Hausdorff condition.

The gluing theorem tells us how to build schemes from affine pieces. In practice one also needs to move information from one affine chart to another. The next proposition is the basic geometric mechanism behind this "communication" between affine opens.

Proposition 2.25 (Common distinguished refinements)**.** *Let*

$$U = \operatorname{Spec} A, \qquad V = \operatorname{Spec} B$$

be affine open subschemes of a scheme X. For every point $x \in U \cap V$, there exists an open neighbourhood W of x which is distinguished in both U and V.

Proof. Choose a distinguished open $D_U(f) \subseteq U$ containing x and contained in $U \cap V$. Inside V, choose a distinguished open

$$D_V(g) \subseteq D_U(f)$$

containing x. Since $D_U(f) = \operatorname{Spec} A_f$, the restriction of the function g to $D_U(f)$ is represented by some element

$$\frac{a}{f^n} \in A_f.$$

The locus on which this element is invertible is

$$D_{A_f}\left(\frac{a}{f^n}\right),$$

which corresponds inside U to the distinguished open $D_U(af)$. On the other hand this locus is exactly $D_V(g)$, because $D_V(g) \subseteq D_U(f)$. Hence

$$W := D_V(g) = D_U(af)$$

is distinguished in both affine charts. $\square$

This proposition leads to a useful local-to-global principle. Its value is not the particular property under consideration, but the method: once a statement survives localization and can be reconstructed from finitely many distinguished opens, checking it on one affine cover is enough.

Lemma 2.26 (Affine Communication Lemma)**.** *Let P be a property of affine open subschemes of a scheme X. Assume that:*

1. *if $U = \operatorname{Spec} A$ has property P, then every distinguished open $D(f) \subseteq U$ has property P;*
2. *if $f_1, \ldots, f_r \in A$ generate the unit ideal and every $D(f_i) \subseteq \operatorname{Spec} A$ has property P, then $\operatorname{Spec} A$ has property P.*

If X admits an affine open cover whose members have property P, then every affine open subscheme of X has property P.

Proof. Let $U = \operatorname{Spec} A \subseteq X$ be any affine open. By proposition 2.25, every point of U has a neighbourhood which is distinguished in U and distinguished in one of the affine opens from the given cover. The latter affine open has property P, so the common distinguished neighbourhood has property P by condition (1).

Since U is quasi-compact by proposition 2.3, finitely many of these neighbourhoods cover U. Write them as

$$D(f_1), \ldots, D(f_r) \subseteq U.$$

Because they cover $\operatorname{Spec} A$, the elements $f_1, \ldots, f_r$ generate the unit ideal. Condition (2) therefore implies that U has property P. □

A first application shows that the usual definition of a locally Noetherian scheme does not depend on the chosen affine cover.

Definition 2.27. A scheme X is *locally Noetherian* if it admits an affine open cover by spectra of Noetherian rings. It is *Noetherian* if it is locally Noetherian and quasi-compact, equivalently if it admits a finite affine open cover by spectra of Noetherian rings.

Proposition 2.28. *A scheme X is locally Noetherian if and only if, for every affine open subscheme*

$$U = \operatorname{Spec} A \subseteq X,$$

the ring A is Noetherian.

Proof. Only one implication requires proof. Apply lemma 2.26 to the property

$$P(U): \quad \Gamma(U, \mathscr{O}_X) \text{ is Noetherian.}$$

Localization preserves Noetherian rings, so condition (1) holds. For condition (2), suppose $f_1, \ldots, f_r$ generate the unit ideal of A and each A_{f_i} is Noetherian. Let $I \subseteq A$ be an ideal. For each i, choose finitely many elements of I whose images generate I_{f_i}, and let $J \subseteq I$ be the ideal generated by all these finitely many elements. Then

$$(I/J)_{f_i} = 0$$

for every i. If $m \in I/J$, then for each i some power $f_i^{N_i}$ annihilates m. The elements $f_i^{N_i}$ still generate the unit ideal up to radical, hence generate an ideal whose radical is A; therefore 1 belongs to that ideal and $m = 0$. Thus $I = J$ is finitely generated, so A is Noetherian. □

The same local philosophy applies to morphisms. For example, if $f : X \to Y$ is a morphism and $Y = \bigcup_i V_i$ is an open cover such that every restriction

$$f^{-1}(V_i) \longrightarrow V_i$$

is an isomorphism, then the local inverses agree on overlaps and glue to a global inverse. Thus being an isomorphism is local on the target.

There is another form of localization which starts from a global function rather than from an affine chart. Let X be a scheme and let

$$s \in \Gamma(X, \mathscr{O}_X).$$

Define its *invertibility locus* by

$$X_s := \{x \in X \mid s_x \in \mathscr{O}_{X,x}^\times\}.$$

Proposition 2.29. *The subset X_s is open. If $X = \operatorname{Spec} A$ and s corresponds to $f \in A$, then*

$$X_s = D(f)$$

and

$$\Gamma(X_s, \mathscr{O}_X) \cong A_f.$$

Proof. If s_x is a unit, choose a germ $t_x \in \mathscr{O}_{X,x}$ with $s_x t_x = 1$. Represent t_x by a section t on a neighbourhood of x. After shrinking, the equality

$$st = 1$$

holds on that neighbourhood, so every nearby germ of s is invertible. Hence X_s is open.

In the affine case, s_x is invertible in $A_\mathfrak{p}$ exactly when $f \notin \mathfrak{p}$, so $X_s = D(f)$. The last assertion follows from proposition 2.17. □

For a non-affine scheme, localization of global sections needs a mild finiteness hypothesis. We call a scheme *quasi-separated* if the intersection of any two affine open subschemes is quasi-compact. This condition will later reappear as a property of the diagonal morphism.

Theorem 2.30 (Localization by a global section)**.** *Let X be a quasi-compact and quasi-separated scheme, and let $s \in \Gamma(X, \mathscr{O}_X)$. Then the canonical homomorphism*

$$\Gamma(X, \mathscr{O}_X)_s \longrightarrow \Gamma(X_s, \mathscr{O}_X)$$

is an isomorphism.

Proof. Put

$$R := \Gamma(X, \mathscr{O}_X).$$

Choose a finite affine open cover

$$X = U_1 \cup \cdots \cup U_r, \qquad U_i = \operatorname{Spec} A_i.$$

Because X is quasi-separated, every intersection $U_i \cap U_j$ is quasi-compact. By proposition 2.25, it admits a finite cover by opens W_{ijk} which are affine and distinguished in both U_i and U_j.

A family of sections on the U_i glues on X if and only if the two restrictions agree on every W_{ijk}: agreement on the W_{ijk} implies agreement on all of $U_i \cap U_j$ by the locality axiom. Hence R is the kernel of the difference map

$$\prod_i \Gamma(U_i, \mathscr{O}_X) \longrightarrow \prod_{i,j,k} \Gamma(W_{ijk}, \mathscr{O}_X).$$

All terms are naturally R-modules through restriction of global sections. Localization

$$M \longmapsto M_s$$

is an exact functor on R-modules, so it preserves this kernel. Moreover, because the products above are finite, localization commutes with them. We therefore obtain an equalizer diagram

$$R_s \longrightarrow \prod_i \Gamma(U_i, \mathscr{O}_X)_s \rightrightarrows \prod_{i,j,k} \Gamma(W_{ijk}, \mathscr{O}_X)_s.$$

On U_i, let $s_i \in A_i$ be the restriction of s. By proposition 2.29,

$$U_i \cap X_s = D(s_i), \qquad \Gamma(U_i \cap X_s, \mathscr{O}_X) = (A_i)_{s_i}.$$

The localization $\Gamma(U_i, \mathscr{O}_X)_s$ is exactly $(A_i)_{s_i}$, since the action of the global element s on A_i is multiplication by s_i. The same argument applies to every affine W_{ijk}. Consequently the localized equalizer is precisely the sheaf equalizer for the cover $\{U_i \cap X_s\}$ of X_s. Therefore

$$\Gamma(X, \mathscr{O}_X)_s \xrightarrow{\sim} \Gamma(X_s, \mathscr{O}_X).$$

□

The preceding theorem gives a practical criterion for recognizing when a scheme is affine.

Theorem 2.31 (Affineness criterion)**.** *Let X be a quasi-compact, quasi-separated scheme. Put*

$$A := \Gamma(X, \mathscr{O}_X).$$

Suppose there exist $f_1, \dots, f_r \in A$ generating the unit ideal such that every invertibility locus X_{f_i} is affine. Then the canonical morphism

$$X \longrightarrow \operatorname{Spec} A$$

is an isomorphism.

Proof. By theorem 2.14, the identity homomorphism of A determines a canonical morphism

$$\Phi : X \longrightarrow \operatorname{Spec} A.$$

Since the f_i generate the unit ideal, the distinguished opens $D(f_i)$ cover $\operatorname{Spec} A$, and the opens X_{f_i} cover X. By theorem 2.30,

$$\Gamma(X_{f_i}, \mathscr{O}_X) \cong A_{f_i}.$$

Because X_{f_i} is affine, affine duality identifies the restriction of Φ with an isomorphism

$$X_{f_i} \xrightarrow{\sim} \operatorname{Spec} A_{f_i} = D(f_i).$$

These local isomorphisms come from the single morphism Φ, so they agree on overlaps and glue to an isomorphism $X \cong \operatorname{Spec} A$. □

2.4 Projective Schemes and the Proj Construction

Affine schemes come directly from ordinary rings. Projective geometry is governed instead by graded rings. The construction Proj is the projective analogue of Spec: homogeneous prime ideals determine the points, while regular functions on projective charts are represented by homogeneous fractions of degree zero.

Let

$$S = \bigoplus_{d \geq 0} S_d$$

be a graded ring, and let

$$S_+ := \bigoplus_{d > 0} S_d$$

be the irrelevant ideal.

Definition 2.32. The *projective spectrum* of S is

$$\operatorname{Proj} S := \{\mathfrak{p} \subseteq S \mid \mathfrak{p} \text{ is a homogeneous prime ideal and } S_+ \not\subseteq \mathfrak{p}\}.$$

For a homogeneous ideal $I \subseteq S$, define

$$V_+(I) := \{\mathfrak{p} \in \operatorname{Proj} S \mid I \subseteq \mathfrak{p}\}.$$

These sets define the Zariski topology on $\operatorname{Proj} S$. For a homogeneous element $f \in S_+$, set

$$D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}.$$

As in the affine case,

$$D_+(f) \cap D_+(g) = D_+(fg),$$

and the $D_+(f)$ form a basis.

If f is homogeneous, the localization S_f inherits a $\mathbb{Z}$-grading. Its degree-zero part is denoted

$$S_{(f)} := (S_f)_0.$$

Thus a typical element of $S_{(f)}$ is represented by a homogeneous fraction

$$\frac{a}{f^n}$$

with $\deg a = n \deg f$.

Proposition 2.33. *For every homogeneous $f \in S_+$, there is a natural homeomorphism*

$$D_+(f) \xrightarrow{\sim} \operatorname{Spec} S_{(f)}.$$

These homeomorphisms identify intersections of standard opens with distinguished opens in the corresponding affine spectra.

Proof. Write $d = \deg f > 0$. If $\mathfrak{p} \in D_+(f)$, then $f \notin \mathfrak{p}$, so the localization $\mathfrak{p}S_f$ is a homogeneous prime ideal of S_f. Its degree-zero part

$$\mathfrak{q} := \mathfrak{p}S_f \cap S_{(f)}$$

is therefore a prime ideal of $S_{(f)}$. This defines a map

$$D_+(f) \longrightarrow \operatorname{Spec} S_{(f)}.$$

We construct the inverse explicitly. Let $\mathfrak{q} \subseteq S_{(f)}$ be prime. For every degree $e \geq 0$, define $P_e \subseteq S_e$ by declaring that a homogeneous element $a \in S_e$ lies in P_e if, for some positive integer N such that $d \mid Ne$,

$$\frac{a^N}{f^{Ne/d}} \in \mathfrak{q}.$$

If this holds for one such N, it also holds after replacing N by any positive multiple. Set

$$\mathfrak{p}_{\mathfrak{q}} := \bigoplus_{e \geq 0} P_e.$$

We verify that this is a homogeneous prime ideal. Suppose $a \in P_e$ and $c \in S_h$ are homogeneous. Choose N divisible enough that $d \mid Ne$, $d \mid Nh$, and

$$a^N / f^{Ne/d} \in \mathfrak{q}.$$

Then

$$\frac{(ac)^N}{f^{N(e+h)/d}} = \left(\frac{a^N}{f^{Ne/d}}\right)\left(\frac{c^N}{f^{Nh/d}}\right) \in \mathfrak{q},$$

so $ac \in P_{e+h}$. For addition, suppose $a, b \in P_e$. Choose a common N with $d \mid Ne$ such that

$$a^N/f^{Ne/d}, \quad b^N/f^{Ne/d} \in \mathfrak{q}.$$

Every term in the expansion of $(a+b)^{2N}$ contains either a^N or b^N. After division by $f^{2Ne/d}$, a term containing a^N, for example, factors as

$$\frac{a^N}{f^{Ne/d}} \cdot \frac{a^{r-N}b^{2N-r}}{f^{Ne/d}},$$

with both factors of degree zero; hence it lies in $\mathfrak{q}$. The same holds for terms containing b^N, so

$$\frac{(a+b)^{2N}}{f^{2Ne/d}} \in \mathfrak{q},$$

and $a + b \in P_e$. Thus $\mathfrak{p}_\mathfrak{q}$ is a homogeneous ideal.

Finally, suppose homogeneous $a \in S_e$ and $b \in S_h$ satisfy $ab \in \mathfrak{p}_\mathfrak{q}$. Choose N divisible enough that $d \mid Ne$, $d \mid Nh$, and the defining condition for ab holds at the exponent N. Then

$$\left(\frac{a^N}{f^{Ne/d}}\right)\left(\frac{b^N}{f^{Nh/d}}\right) = \frac{(ab)^N}{f^{N(e+h)/d}} \in \mathfrak{q}.$$

Since $\mathfrak{q}$ is prime, one factor lies in $\mathfrak{q}$, hence $a \in \mathfrak{p}_\mathfrak{q}$ or $b \in \mathfrak{p}_\mathfrak{q}$. For a homogeneous ideal in an $\mathbb{N}$-graded ring, it is enough to test primality on homogeneous elements: if two nonzero classes had zero product, the product of their lowest-degree nonzero homogeneous parts would also vanish. Thus $\mathfrak{p}_\mathfrak{q}$ is prime. Also $f \notin \mathfrak{p}_\mathfrak{q}$, because the normalized fraction associated to f is $1 \notin \mathfrak{q}$. Hence $\mathfrak{p}_\mathfrak{q} \in D_+(f)$.

The two constructions are inverse. If $\mathfrak{p} \in D_+(f)$ and $\mathfrak{q} = \mathfrak{p}S_f \cap S_{(f)}$, then for homogeneous $a \in S_e$,

$$a \in \mathfrak{p} \iff \frac{a^N}{f^{Ne/d}} \in \mathfrak{q}$$

for any sufficiently divisible N: the forward implication is clear. For the reverse implication, membership of the fraction in $\mathfrak{p}S_f$ means that $f^M a^N \in \mathfrak{p}$ for some $M \geq 0$. Since $f \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime, this forces $a \in \mathfrak{p}$. Conversely, if $\mathfrak{q} \subseteq S_{(f)}$ and $\mathfrak{p} = \mathfrak{p}_\mathfrak{q}$, then for a degree-zero fraction $a/f^n \in S_{(f)}$ one has

$$\frac{a}{f^n} \in \mathfrak{q} \iff a \in \mathfrak{p},$$

because $a \in \mathfrak{p}$ implies that some power $(a/f^n)^N$ lies in the prime ideal $\mathfrak{q}$, and hence $a/f^n \in \mathfrak{q}$. Thus contraction recovers $\mathfrak{q}$.

It remains to identify the topology. If $g \in S$ is homogeneous, then the degree-zero element

$$\frac{g^d}{f^{\deg g}} \in S_{(f)}$$

has the property that

$$D_+(f) \cap D_+(g)$$

corresponds exactly to the distinguished open where this element is invertible. Hence the bijection carries a basis to a basis and is a homeomorphism. $\square$

The preceding proposition suggests the structure sheaf immediately: each standard open should carry the affine scheme structure of $\operatorname{Spec} S_{(f)}$, and these structures agree on overlaps.

Theorem 2.34. *There is a unique sheaf of rings $\mathscr{O}_{\operatorname{Proj} S}$ such that*

$$D_+(f) \cong \operatorname{Spec} S_{(f)}$$

as locally ringed spaces for every homogeneous $f \in S_+$. With this sheaf, $\operatorname{Proj} S$ *is a scheme.*

Proof. Transport the affine structure sheaf from $\operatorname{Spec} S_{(f)}$ to each standard open $D_+(f)$ using proposition 2.33. On an overlap $D_+(f) \cap D_+(g)$, both restrictions are obtained by localizing the corresponding degree-zero rings at the same homogeneous fraction, so the induced locally ringed spaces are canonically isomorphic. These isomorphisms satisfy the cocycle condition. The gluing theorem therefore produces a sheaf of rings on $\operatorname{Proj} S$, and the standard opens form an affine cover. □

Example 2.35 (Projective space)**.** Let

$$S = A[x_0, \dots, x_n]$$

with the standard grading $\deg x_i = 1$. Define

$$\mathbb{P}^n_A := \operatorname{Proj} A[x_0, \dots, x_n].$$

For every i,

$$D_+(x_i) \cong \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right] \cong \mathbb{A}^n_A.$$

Thus projective space is obtained by gluing $n+1$ affine n-spaces along explicit distinguished opens.

Example 2.36 (A useful degenerate case)**.** Let $S = A[T]$ with $\deg T = 1$. Then every point of $\operatorname{Proj} S$ lies in $D_+(T)$, and

$$S_{(T)} = (A[T, T^{-1}])_0 = A.$$

Hence

$$\operatorname{Proj} A[T] \cong \operatorname{Spec} A.$$

This is the projective 0-space over A.

Homogeneous ideals define projective closed subschemes just as ordinary ideals define affine closed subschemes.

Proposition 2.37. *Let $I \subseteq S$ be a homogeneous ideal. The quotient map $S \to S/I$ induces a closed immersion*

$$\operatorname{Proj}(S/I) \hookrightarrow \operatorname{Proj} S$$

with image $V_+(I)$.

Proof. On the standard open $D_+(f)$, the map is the affine closed immersion associated to

$$S_{(f)} \longrightarrow (S/I)_{(f)} \cong S_{(f)}/I_{(f)}.$$

These affine closed immersions agree on overlaps and therefore glue. □

Definition 2.38. A scheme over A is called a *projective scheme over A* if it is isomorphic to a closed subscheme of $\mathbb{P}^n_A$ for some n.

At this stage we use projective schemes mainly as a basic source of non-affine examples. The relative notion of a projective morphism, its behavior under base change, and its relation with properness will be developed later. Likewise, twisting sheaves and the relation between graded modules and sheaves on Proj belong to the later study of coherent sheaves and line bundles.

2.5 The Topology of Schemes and Function Fields

The topology of a scheme is coarse compared with classical metric topology, but it is highly structured. In particular, irreducible closed subsets have canonical generic points, and specialization records inclusion relations among prime ideals. These features make the topology closely reflect the algebra of the local rings.

Definition 2.39. A nonempty topological space X is *irreducible* if it cannot be written as the union of two proper closed subsets. A maximal irreducible closed subset is called an *irreducible component*.

Proposition 2.40. *For a nonempty topological space X, the following are equivalent:*

1. *X is irreducible;*
2. *any two nonempty open subsets of X intersect;*
3. *every nonempty open subset of X is dense.*

In particular, every irreducible space is connected.

Proof. If two nonempty opens U, V were disjoint, then

$$X = (X \setminus U) \cup (X \setminus V)$$

would express X as the union of two proper closed subsets. This proves (1) $\Rightarrow$ (2). Condition (2) says precisely that every nonempty open meets every nonempty open, hence every nonempty open is dense, giving (2) $\Rightarrow$ (3). If $X = Z_1 \cup Z_2$ with both Z_i proper closed, then $X \setminus Z_1$ and $X \setminus Z_2$ are nonempty opens that cannot both be dense, so (3) implies (1). A disconnected space is a union of two disjoint nonempty opens, hence cannot be irreducible. □

Definition 2.41. Let $x, y \in X$. We say that x *specializes to y*, or that y is a specialization of x, if

$$y \in \overline{\{x\}}.$$

A point η is a *generic point* of an irreducible closed subset Z if

$$\overline{\{\eta\}} = Z.$$

Proposition 2.42. *Let $X = \operatorname{Spec} A$, and let $x, y \in X$ correspond to prime ideals $\mathfrak{p}, \mathfrak{q}$. Then*

$$\overline{\{x\}} = V(\mathfrak{p}),$$

and

$$x \text{ specializes to } y \iff \mathfrak{p} \subseteq \mathfrak{q}.$$

Moreover, the irreducible components of X are the sets $V(\mathfrak{p})$ for minimal prime ideals $\mathfrak{p}$.

Proof. The closed subsets containing x are precisely the $V(I)$ with $I \subseteq \mathfrak{p}$, whose intersection is $V(\mathfrak{p})$. Thus $y \in \overline{\{x\}}$ exactly when $\mathfrak{p} \subseteq \mathfrak{q}$. Finally, $V(\mathfrak{p})$ is irreducible when $\mathfrak{p}$ is prime, and maximal irreducible closed subsets correspond to minimal primes. $\square$

The local ring at a point has a geometric spectrum of its own. It records precisely the points that can move generically toward the chosen point.

Proposition 2.43 (The local scheme at a point). *Let $x \in X$ be a point of a scheme. Choose an affine neighbourhood*

$$U = \operatorname{Spec} A$$

of x, and let $\mathfrak{p} \subseteq A$ correspond to x. Then

$$\operatorname{Spec} \mathscr{O}_{X,x} \cong \operatorname{Spec} A_{\mathfrak{p}},$$

and the canonical morphism

$$\operatorname{Spec} \mathscr{O}_{X,x} \longrightarrow X$$

has as its image exactly the generizations of x, namely the points $y \in X$ such that y specializes to x.

Proof. The first assertion follows from

$$\mathscr{O}_{X,x} \cong A_{\mathfrak{p}}.$$

Prime ideals of $A_{\mathfrak{p}}$ correspond to prime ideals $\mathfrak{q} \subseteq \mathfrak{p}$. Under the affine specialization criterion, these are exactly the points $y \in U$ satisfying

$$y \rightsquigarrow x.$$

Any generization of x belongs to every open neighbourhood of x: if an open neighbourhood U contained x but not y, then the closed set $X \setminus U$ would contain y and hence its closure, which contains x, a contradiction. Thus the affine description already gives all generizations in X. $\square$

In this sense $\operatorname{Spec} \mathscr{O}_{X,x}$ is the scheme-theoretic germ of X at x. At a generic point it is the spectrum of a field, while at a more special point it retains the chains of generizations leading toward that point.

Theorem 2.44 (Generic points of schemes). *Every irreducible closed subset of a scheme has a unique generic point. In particular, irreducible components of a scheme are in bijection with their generic points.*

Proof. Let $Z \subseteq X$ be irreducible and closed. Choose an affine open $U \subseteq X$ meeting Z. Then $Z \cap U$ is irreducible and closed in $U = \operatorname{Spec} A$, so

$$Z \cap U = V(\mathfrak{p})$$

for a unique prime ideal $\mathfrak{p} \subseteq A$. The corresponding point $\eta \in U$ has closure containing $Z \cap U$. Since Z is irreducible and $Z \cap U$ is a nonempty open subset of Z, it is dense in Z, hence $\overline{\{\eta\}} = Z$. Uniqueness follows because two generic points of the same closed subset lie in exactly the same open subsets, and in an affine neighbourhood this forces the corresponding prime ideals to coincide. $\square$

A useful finiteness condition on the topology comes from Noetherian rings.

Definition 2.45. A topological space is *Noetherian* if every descending chain of closed subsets stabilizes.

For schemes, the Noetherian conditions were introduced in proposition 2.28. The point here is that their underlying topology inherits strong finiteness properties.

Proposition 2.46. *If A is Noetherian, then* $\operatorname{Spec} A$ *is a Noetherian topological space. Every Noetherian topological space is quasi-compact and has only finitely many irreducible components.*

Proof. A descending chain of closed subsets of $\operatorname{Spec} A$ corresponds, after taking radicals, to an ascending chain of ideals, so it stabilizes when A is Noetherian. In any Noetherian space, an open cover with no finite subcover would produce a strictly increasing chain of open subsets, hence a strictly decreasing chain of closed complements. Finally, by induction on closed subsets, every nonempty closed subset is a finite union of irreducible closed subsets; taking maximal ones gives finitely many irreducible components. □

We now return from topology to the structure sheaf.

Definition 2.47. A scheme X is *reduced* if every local ring $\mathscr{O}_{X,x}$ is reduced. It is *integral* if it is nonempty, reduced, and irreducible.

Proposition 2.48. *For a scheme X, the following are equivalent:*

1. *X is reduced;*
2. *every affine open $U = \operatorname{Spec} A \subseteq X$ has A reduced;*
3. *for every open subset $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is reduced.*

Proof. Reducedness of a ring is local: a ring A is reduced if and only if every localization $A_{\mathfrak{p}}$ is reduced. Therefore (1) and (2) are equivalent. If every stalk is reduced and $s \in \mathscr{O}_X(U)$ is nilpotent, then every germ s_x is zero; since sections of a sheaf are determined by their germs, $s = 0$. This gives (1)⇒(3), while (3)⇒(2) is immediate. □

Proposition 2.49. *For a nonempty scheme X, the following are equivalent:*

1. *X is integral;*
2. *every nonempty affine open $U = \operatorname{Spec} A \subseteq X$ has A an integral domain;*
3. *for every nonempty open subset $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is an integral domain.*

Proof. Assume first that X is reduced and irreducible. Let $U \subseteq X$ be nonempty and suppose $s, t \in \mathscr{O}_X(U)$ satisfy $st = 0$. Consider the invertibility loci

$$U_s := \{x \in U \mid s_x \in \mathscr{O}_{X,x}^{\times}\}, \qquad U_t := \{x \in U \mid t_x \in \mathscr{O}_{X,x}^{\times}\}.$$

They are open by proposition 2.29. They are disjoint: if $x \in U_s \cap U_t$, then $(st)_x$ would be a unit, contradicting $st = 0$. Since every nonempty open subset of the irreducible space X is irreducible, the open set U is irreducible; hence one of U_s, U_t is empty. Suppose $U_s = \varnothing$. Then for every $x \in U$, the image of s_x in the residue field $\kappa(x)$ is zero.

We claim that this forces $s = 0$. Cover U by affine opens $V = \operatorname{Spec} A$. The restriction of s corresponds to an element $a \in A$. Its image in every residue field is zero, so a lies in every prime ideal of A, hence in the nilradical. Since X is reduced, A is reduced by proposition 2.48; thus $a = 0$. Therefore s vanishes on an affine cover of U, and hence $s = 0$. The same argument applies if $U_t = \varnothing$. Thus $\mathscr{O}_X(U)$ is a domain.

Conversely, suppose every nonempty affine open $\operatorname{Spec} A \subseteq X$ has A an integral domain. Every stalk is then a localization of a domain, so X is reduced. If X were reducible, proposition 2.40 would give disjoint nonempty open subsets $U, V \subseteq X$. Choose nonempty affine opens

$$U' \subseteq U, \qquad V' \subseteq V.$$

Their disjoint union is an affine open subscheme of X: if $U' = \operatorname{Spec} A$ and $V' = \operatorname{Spec} B$, then

$$U' \sqcup V' \cong \operatorname{Spec}(A \times B).$$

But $A \times B$ has nontrivial zero divisors, contradicting the hypothesis on nonempty affine opens. Hence X is irreducible, and therefore integral.

The implication from (3) to (2) is immediate, while the first part of the proof shows that (1) implies (3). Thus all three conditions are equivalent. □

For an integral scheme, the generic point turns the local rings into subrings of a single field.

Definition 2.50. Let X be an integral scheme with generic point η. The field

$$K(X) := \mathscr{O}_{X,\eta}$$

is called the *function field* of X. Its elements are the *rational functions* on X.

Proposition 2.51. *Let X be an integral scheme with generic point η.*

1. *If $U = \operatorname{Spec} A \subseteq X$ is a nonempty affine open, then*

$$K(X) \cong \operatorname{Frac}(A).$$

2. *For every nonempty open $U \subseteq X$ and every $x \in U$, the natural maps*

$$\mathscr{O}_X(U) \longrightarrow \mathscr{O}_{X,x} \longrightarrow K(X)$$

are injective.

3. *Inside $K(X)$,*

$$\mathscr{O}_X(U) = \bigcap_{x \in U} \mathscr{O}_{X,x}.$$

Proof. On a nonempty affine open $U = \operatorname{Spec} A$, the generic point corresponds to the zero prime because A is a domain. Hence

$$\mathscr{O}_{X,\eta} = A_{(0)} = \operatorname{Frac}(A),$$

proving (1).

We first record a useful fact. If $V \subseteq U$ are nonempty open subsets of an integral scheme, then the restriction map

$$\mathscr{O}_X(U) \longrightarrow \mathscr{O}_X(V)$$

is injective. Indeed, suppose $s \in \mathscr{O}_X(U)$ restricts to zero on V. Cover U by affine opens $W = \operatorname{Spec} A$. Since X is irreducible, every nonempty W meets V. The restriction of s to W corresponds to an element $a \in A$ that becomes zero on the nonempty open $W \cap V$. Choose a distinguished open $D(f) \subseteq W \cap V$. Then $a/1 = 0$ in the domain A_f, so some power of f annihilates a in A. Since $f \neq 0$ and A is a domain, $a = 0$. Thus s vanishes on every affine open W, hence $s = 0$.

Now let $x \in U$. If a section $s \in \mathscr{O}_X(U)$ has zero germ at x, then it vanishes on some nonempty neighbourhood $V \subseteq U$ of x; the injectivity just proved gives $s = 0$. Hence

$$\mathscr{O}_X(U) \longrightarrow \mathscr{O}_{X,x}$$

is injective. Likewise, if a germ $s_x \in \mathscr{O}_{X,x}$ maps to zero in $K(X) = \mathscr{O}_{X,\eta}$, represent it by a section $s \in \mathscr{O}_X(W)$ on a nonempty neighbourhood W of x. Vanishing at the generic stalk means that s vanishes on some nonempty open subset of W, so the same restriction argument gives $s = 0$, and hence $s_x = 0$. This proves both injections in (2).

For (3), the inclusion from left to right follows from (2). Conversely, suppose $r \in K(X)$ lies in every $\mathscr{O}_{X,x}$ for $x \in U$. For each x, choose a neighbourhood $U_x \subseteq U$ and a section $s_x \in \mathscr{O}_X(U_x)$ representing r. On an overlap $U_x \cap U_y$, both restrictions have the same image r in $K(X)$. By (2), the map from sections on the nonempty overlap to $K(X)$ is injective, so the two restrictions agree. The sheaf axiom therefore glues the s_x to a section of $\mathscr{O}_X(U)$ representing r. Hence

$$\mathscr{O}_X(U) = \bigcap_{x \in U} \mathscr{O}_{X,x}$$

inside $K(X)$. □

A rational function is said to be *regular at x* precisely when it belongs to $\mathscr{O}_{X,x}$. The preceding proposition says that being regular on an open subset is exactly the same as being regular at every point of that subset.

2.6 Dimension

The topology of a scheme carries an intrinsic notion of dimension. Instead of measuring distances, one measures the possible lengths of chains of irreducible closed subsets. In the affine case this is exactly Krull dimension, so the geometric and algebraic notions coincide.

Definition 2.52. Let X be a topological space. Its *dimension* is

$$\dim X := \sup\{n \mid Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \text{ are irreducible closed subsets of } X\}.$$

The dimension may be infinite.

Proposition 2.53. *For every ring A,*

$$\dim \operatorname{Spec} A = \dim A,$$

where the right-hand side is the Krull dimension of A.

Proof. Every irreducible closed subset of $\operatorname{Spec} A$ is of the form $V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$. Moreover,

$$V(\mathfrak{p}) \subsetneq V(\mathfrak{q}) \iff \mathfrak{q} \subsetneq \mathfrak{p}.$$

Thus chains of irreducible closed subsets correspond, with the order reversed, to chains of prime ideals of the same length. □

Definition 2.54. Let $x \in X$ be a point of a scheme. The *local dimension at x* is

$$\dim_x X := \dim \mathscr{O}_{X,x}.$$

If x corresponds to $\mathfrak{p}$ in an affine neighbourhood $\operatorname{Spec} A$, then

$$\dim_x X = \dim A_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p}).$$

For an irreducible closed subset $Z \subseteq X$ with generic point η_Z, we define its codimension in X by

$$\operatorname{codim}_X(Z) := \dim \mathscr{O}_{X,\eta_Z}.$$

Proposition 2.55. *Let $x \in X$. The number $\dim \mathscr{O}_{X,x}$ is the supremum of the lengths of chains*

$$\overline{\{x\}} = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

of irreducible closed subsets of X. Equivalently, it measures the codimension of $\overline{\{x\}}$ inside the components of X containing it.

Proof. Work in an affine neighbourhood $U = \operatorname{Spec} A$ of x, with $x \leftrightarrow \mathfrak{p}$. By proposition 2.43, prime ideals of $\mathscr{O}_{X,x} \cong A_{\mathfrak{p}}$ correspond to primes

$$\mathfrak{q} \subseteq \mathfrak{p}.$$

Every chain of primes in $A_{\mathfrak{p}}$ may be extended, if necessary, by the maximal prime $\mathfrak{p}A_{\mathfrak{p}}$; hence the dimension is the supremum of chains ending at $\mathfrak{p}$. A chain

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n = \mathfrak{p}$$

corresponds, with the order reversed, to a chain

$$V_U(\mathfrak{p}) \subsetneq V_U(\mathfrak{q}_{n-1}) \subsetneq \cdots \subsetneq V_U(\mathfrak{q}_0)$$

of irreducible closed subsets of U, all containing x.

Such chains are equivalent to chains of irreducible closed subsets of X beginning with $\overline{\{x\}}$. Indeed, intersecting a chain in X with U preserves strict inclusions: if $Z \subsetneq Z'$ are irreducible closed subsets containing x and $Z \cap U = Z' \cap U$, then the nonempty open subset $Z' \cap U$ of Z' would lie in the closed subset Z, forcing $Z' = Z$. Conversely, taking closures in X of irreducible closed subsets of U preserves strict inclusions for the same reason. Therefore the two kinds of chains have the same possible lengths, and their supremum is

$$\dim A_{\mathfrak{p}} = \dim \mathscr{O}_{X,x}.$$

□

Thus an irreducible closed subset of codimension one is controlled locally by a one-dimensional local ring. This observation will later be the bridge between divisors and discrete valuation rings.

Proposition 2.56. *If X is a Noetherian scheme and $X = \bigcup_{i=1}^r U_i$ is a finite affine open cover, then*

$$\dim X = \max_i \dim U_i.$$

Consequently, the dimension of a Noetherian scheme can be computed on affine charts.

Proof. Let

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

be a chain of irreducible closed subsets of X. Choose a point $x \in Z_0$, and choose an affine chart U_i containing x. Then every $Z_j \cap U_i$ is a nonempty irreducible closed subset of U_i. Moreover the inclusions remain strict. Indeed, if for some j one had

$$Z_j \cap U_i = Z_{j+1} \cap U_i,$$

then the nonempty open subset $Z_{j+1} \cap U_i$ of the irreducible space Z_{j+1} would be contained in the closed subset Z_j, hence its closure Z_{j+1} would be contained in Z_j, a contradiction. Thus

$$n \leq \dim U_i \leq \max_i \dim U_i,$$

and therefore $\dim X \leq \max_i \dim U_i$.

Conversely, fix i and take a chain

$$Y_0 \subsetneq \cdots \subsetneq Y_m$$

of irreducible closed subsets of U_i. Their closures $\overline{Y_j}^X$ are irreducible closed subsets of X. The inclusions remain strict: if $\overline{Y_j}^X = \overline{Y_{j+1}}^X$, then intersecting this common closure with U_i and using that Y_j is closed in U_i would give $Y_j = Y_{j+1}$. Hence $m \leq \dim X$. Taking the supremum over chains in every U_i gives

$$\max_i \dim U_i \leq \dim X.$$

Combining the two inequalities proves the result. □

Before turning to the general relative notion in Chapter 3, we use one absolute piece of terminology. If k is a field, a k-scheme X is said to be *of finite type over k* if it admits a finite affine open cover

$$X = \bigcup_{i=1}^{r} \operatorname{Spec} A_i$$

with every A_i a finitely generated k-algebra. Chapter 3 will show that this is exactly the special case, for the structure morphism $X \to \operatorname{Spec} k$, of the general notion of a morphism of finite type.

For schemes of finite type over a field, dimension has an especially concrete algebraic interpretation.

Theorem 2.57 (Dimension and transcendence degree)**.** *Let k be a field and let A be a finitely generated integral k-algebra. Then*

$$\dim A = \operatorname{trdeg}_k \operatorname{Frac}(A).$$

Equivalently, if X is an integral affine scheme of finite type over k, then

$$\dim X = \operatorname{trdeg}_k K(X).$$

Proof. This is the dimension theorem obtained from Noether normalization. There exist algebraically independent elements $t_1, \ldots, t_d \in A$ such that A is integral over the polynomial ring

$$k[t_1, \ldots, t_d].$$

Integral extensions preserve Krull dimension, while the polynomial ring has dimension d. Moreover, integrality implies that $\operatorname{Frac}(A)$ is algebraic over $k(t_1, \ldots, t_d)$, so its transcendence degree is also d. □

Example 2.58 (Affine and projective space)**.** For a field k,

$$\dim \mathbb{A}^n_k = \dim k[x_1, \ldots, x_n] = n.$$

The standard affine charts of $\mathbb{P}^n_k$ are copies of $\mathbb{A}^n_k$, and $\mathbb{P}^n_k$ is irreducible. Therefore

$$\dim \mathbb{P}^n_k = n.$$

Example 2.59 (Curves)**.** An integral scheme of finite type over a field with dimension one has function field of transcendence degree one. In such a scheme the generic point has codimension zero, while every proper irreducible closed subset has dimension zero. This is the basic topological reason that divisors on a curve are supported at closed points.

The constructions of this chapter establish the basic objects of scheme theory. Rings give affine schemes, affine schemes glue to arbitrary schemes, quotients and localizations produce closed and open subschemes, graded rings produce projective schemes, and the underlying topology records generic points, specialization, and dimension. The next step is to study schemes relative to one another. Fiber products and base change will provide the ambient language, after which properties such as finite type, separatedness, properness, and projectivity can be treated systematically.

Chapter 3

Morphisms and Relative Geometry

The first two chapters described schemes as spaces assembled from affine pieces. From this point onward, the geometry is increasingly relative: rather than studying a scheme in isolation, we study a morphism

$$f : X \longrightarrow S$$

and ask how the geometry of X varies over the points of the base S. Fibered products provide the basic language for changing the base; fibres record the geometric objects seen over individual points; constructibility and dimension measure the behavior of images and fibres; separatedness and properness control specialization; and projective morphisms supply the principal concrete class of proper morphisms.

The chapter is organized around these four themes. The categorical formalism is kept to the amount that is repeatedly used later, while the emphasis remains on the geometry of schemes and on the interaction between local algebra and global constructions.

3.1 Fiber Products, Fibres, and Base Change

Suppose that two schemes X and Y are equipped with morphisms to a common base S:

$$X \xrightarrow{f} S \xleftarrow{g} Y.$$

The object that parametrizes pairs of points of X and Y having the same image in S is not the ordinary cartesian product of their underlying sets. The correct construction must remember the scheme structures and the residue fields, and is characterized by a universal property.

Definition 3.1. A *fibered product* of X and Y over S is a scheme $X \times_S Y$ together with morphisms p_X and p_Y fitting into a cartesian diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S \end{array}$$

such that for every scheme T and every pair of morphisms $a : T \to X$ and $b : T \to Y$ satisfying $f \circ a = g \circ b$, there exists a unique morphism $u : T \to X \times_S Y$ for which

$$p_X \circ u = a, \qquad p_Y \circ u = b.$$

Equivalently,

$$\mathrm{Hom}(T, X \times_S Y) \cong \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, Y)$$

naturally in T.

The universal property immediately gives uniqueness up to unique isomorphism, symmetry

$$X \times_S Y \cong Y \times_S X,$$

and the expected associativity isomorphisms. More importantly, it allows cartesian diagrams to be manipulated without choosing coordinates.

Proposition 3.2 (Pasting of cartesian squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z'. \end{tikzcd}$$

If both squares are cartesian, then the outer rectangle is cartesian. Conversely, if the outer rectangle and the right-hand square are cartesian, then the left-hand square is cartesian.

Proof. For the first assertion, let T be a test scheme. A morphism $T \to X$ is equivalent, by the left square, to a compatible pair of morphisms $T \to X'$ and $T \to Y$. By the right square, the latter morphism is equivalent to compatible morphisms $T \to Y'$ and $T \to Z$. Combining these conditions gives exactly the universal property of the outer rectangle.

For the converse, the universal property of the outer rectangle identifies maps $T \to X$ with pairs consisting of a map $T \to X'$ and a map $T \to Z$ satisfying the compatibility over Z'. The right cartesian square identifies the compatible map $T \to Z$ with a map $T \to Y$ compatible with $T \to Y'$. This is precisely the universal property required for the left square. □

The affine case reveals the algebra behind the construction.

Theorem 3.3 (Affine fiber products)**.** *Let $A \to B$ and $A \to C$ be ring homomorphisms. Then*

$$\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C \cong \operatorname{Spec}(B \otimes_A C).$$

Proof. For every scheme T, theorem 2.14 and the universal property of the tensor product give natural bijections

$$\begin{aligned} \mathrm{Hom}(T, \operatorname{Spec}(B \otimes_A C)) &\cong \mathrm{Hom}_{\mathbf{Ring}}(B \otimes_A C, \Gamma(T, \mathscr{O}_T)) \\ &\cong \mathrm{Hom}_{\mathbf{Ring}}(B, \Gamma(T, \mathscr{O}_T)) \\ &\qquad \times_{\mathrm{Hom}_{\mathbf{Ring}}(A, \Gamma(T,\mathscr{O}_T))} \mathrm{Hom}_{\mathbf{Ring}}(C, \Gamma(T, \mathscr{O}_T)) \\ &\cong \mathrm{Hom}(T, \operatorname{Spec} B) \times_{\mathrm{Hom}(T, \operatorname{Spec} A)} \mathrm{Hom}(T, \operatorname{Spec} C). \end{aligned}$$

Thus $\operatorname{Spec}(B \otimes_A C)$ represents the functor defining the fibered product. □

Example 3.4. For a field k,

$$\mathbb{A}_k^m \times_k \mathbb{A}_k^n \cong \mathbb{A}_k^{m+n},$$

because

$$k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n] \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n].$$

The affine construction glues.

Theorem 3.5 (Existence of fiber products)**.** *The category of schemes has fibered products.*

Proof. Let $X \to S$ and $Y \to S$ be morphisms. Choose an affine open covering $S = \bigcup_\lambda V_\lambda$. Cover $f^{-1}(V_\lambda)$ and $g^{-1}(V_\lambda)$ by affine opens

$$U_{\lambda i} = \operatorname{Spec} B_{\lambda i}, \qquad W_{\lambda j} = \operatorname{Spec} C_{\lambda j},$$

with $V_\lambda = \operatorname{Spec} A_\lambda$. By theorem 3.3, each pair gives an affine scheme

$$Z_{\lambda ij} := U_{\lambda i} \times_{V_\lambda} W_{\lambda j} \cong \operatorname{Spec}(B_{\lambda i} \otimes_{A_\lambda} C_{\lambda j}).$$

We next construct the transition maps between two affine pieces $Z_{\lambda ij}$ and $Z_{\mu i'j'}$. Cover the open intersection $V_\lambda \cap V_\mu$ by affine opens V. Over such a V, the intersections

$$U_{\lambda i} \cap U_{\mu i'} \cap f^{-1}(V), \qquad W_{\lambda j} \cap W_{\mu j'} \cap g^{-1}(V)$$

are open subschemes of X and Y. By the common distinguished-refinement result of proposition 2.25, they are covered by affine opens U and W which are simultaneously distinguished in the relevant affine charts. For every such pair the affine fibre product

$$U \times_V W$$

comes with open immersions into both $Z_{\lambda ij}$ and $Z_{\mu i'j'}$: on coordinate rings these maps are localizations of the tensor products in theorem 3.3. These common affine products cover precisely the portions of the two pieces that must be identified. On further common refinements, the two induced identifications coincide by uniqueness in the universal property of a fibre product. Hence the transition maps satisfy the cocycle condition, and the gluing theorem from Chapter 2 produces a scheme Z from the $Z_{\lambda ij}$.

The affine projections glue to morphisms

$$\begin{tikzcd} Z \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d] \\ X \arrow[r] & S. \end{tikzcd}$$

To verify the universal property, let T carry compatible maps to X and Y. The inverse images of the affine opens $U_{\lambda i}$ and $W_{\lambda j}$ cover T. On every common inverse image, the affine universal property gives a unique map to $Z_{\lambda ij}$. These maps agree on overlaps by uniqueness and therefore glue to a unique morphism $T \to Z$. Thus $Z \cong X \times_S Y$. □

Once fiber products exist, a morphism can be transported to a new base.

Definition 3.6 (Base change)**.** Let $X \to S$ be an S-scheme and let $S' \to S$ be a morphism. The scheme

$$X_{S'} := X \times_S S'$$

is the *base change* of X from S to S'. If $f : X \to Y$ is an S-morphism, its base change is the induced morphism

$$f_{S'} : X_{S'} \longrightarrow Y_{S'}.$$

It is characterized by the cartesian square

$$\begin{tikzcd} X_{S'} \arrow[r] \arrow[d, "f_{S'}"'] & X \arrow[d, "f"] \\ Y_{S'} \arrow[r] & Y. \end{tikzcd}$$

The universal property gives canonical identifications

$$(X \times_S Y) \times_S S' \cong X_{S'} \times_{S'} Y_{S'},$$

$$(X \times_S S') \times_{S'} S'' \cong X \times_S S'',$$

and shows that base change is functorial.

Proposition 3.7. *Open immersions and closed immersions are stable under arbitrary base change.*

Proof. For an open immersion $U \hookrightarrow X$, the pullback to $Y \to X$ is the open subscheme corresponding to the inverse image of U in Y.

For a closed immersion the assertion is local on the target. Write

$$X = \operatorname{Spec} A, \qquad Z = \operatorname{Spec}(A/I), \qquad Y = \operatorname{Spec} B.$$

Then

$$Z \times_X Y \cong \operatorname{Spec}((A/I) \otimes_A B) \cong \operatorname{Spec}(B/IB),$$

which is a closed subscheme of Y. □

Surjectivity is also stable under arbitrary base change. Indeed, if $X \to Y$ is surjective, $Y' \to Y$ is any morphism, and $y' \in Y'$ maps to $y \in Y$, choose $x \in X$ over y. The fibre of $X \times_Y Y' \to Y'$ over y' contains

$$\operatorname{Spec}(\kappa(x) \otimes_{\kappa(y)} \kappa(y')),$$

which is nonempty because the tensor product of two field extensions of $\kappa(y)$ is a nonzero ring.

Definition 3.8 (Scheme-theoretic fibre)**.** Let $f : X \to Y$ be a morphism and let $y \in Y$. The *fibre of f over y* is

$$X_y := X \times_Y \operatorname{Spec} \kappa(y),$$

where $\operatorname{Spec} \kappa(y) \to Y$ is the canonical morphism associated with the point y.

If $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and y corresponds to $\mathfrak{p} \in \operatorname{Spec} A$, then

$$X_y \cong \operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p})).$$

This formula explains why fibres contain more information than set-theoretic inverse images: the tensor product remembers nilpotents, residue-field extensions, and multiplicities.

Proposition 3.9. *The projection $X_y \to X$ identifies the underlying topological space of X_y homeomorphically with the set-theoretic fibre $f^{-1}(y)$ endowed with the subspace topology.*

Proof. The assertion is local on X and Y. In the affine situation above, prime ideals of

$$B \otimes_A \kappa(\mathfrak{p})$$

correspond to prime ideals $\mathfrak{q} \subseteq B$ satisfying $\mathfrak{q} \cap A = \mathfrak{p}$. These are exactly the points of $f^{-1}(y)$. If $b \in B$, the distinguished open defined by the image of b in the fibre corresponds to

$$D(b) \cap f^{-1}(y).$$

These sets form bases for both topologies, proving the assertion. □

Definition 3.10 (Geometric fibre)**.** Let $\overline{\kappa(y)}$ be an algebraic closure of the residue field. The *geometric fibre* at y is

$$X_{\overline{y}} := X_y \times_{\operatorname{Spec}\kappa(y)} \operatorname{Spec}\overline{\kappa(y)}.$$

Geometric fibres will become essential when smoothness is studied: regularity of X_y over $\kappa(y)$ need not survive an extension of the residue field, whereas smoothness is designed to be stable under arbitrary base change.

3.2 Images, Generic Finiteness, and Dimension of Fibres

The preceding constructions are formal. We now ask geometric questions. How complicated can the image of a morphism be? How does the dimension of the fibre vary? What does it mean for a morphism to be finite only near the generic point? These questions require finiteness hypotheses on the morphism.

We first fix two topological finiteness conditions for morphisms. A morphism $f : X \to Y$ is *quasi-compact* if $f^{-1}(V)$ is quasi-compact for every affine open subset $V \subseteq Y$. It is *quasi-separated* if, for every affine open $V \subseteq Y$ and every pair of affine opens

$$U_1, U_2 \subseteq f^{-1}(V),$$

the intersection $U_1 \cap U_2$ is quasi-compact. When $Y = \operatorname{Spec}\mathbb{Z}$, these reduce to the notions of quasi-compact and quasi-separated schemes introduced in Chapter 2. Later, after the diagonal morphism has been introduced, quasi-separatedness can equivalently be expressed by requiring $\Delta_{X/Y}$ to be quasi-compact.

Definition 3.11. A morphism $f : X \to Y$ is *locally of finite type* if for every affine open $V = \operatorname{Spec} A \subseteq Y$ and every affine open $U = \operatorname{Spec} B \subseteq f^{-1}(V)$, the A-algebra B is finitely generated. It is *of finite type* if it is locally of finite type and quasi-compact.

A morphism is *locally of finite presentation* if, in the same affine situation, B can be written as

$$B \cong A[x_1, \dots, x_n]/(g_1, \dots, g_m)$$

with finitely many generators and finitely many relations. It is *of finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated. Over a locally Noetherian target, morphisms locally of finite type are locally of finite presentation, and morphisms of finite type are of finite presentation.

A morphism is *affine* if the inverse image of every affine open subscheme of the target is affine. It is *finite* if it is affine and, for every affine open $V = \operatorname{Spec} A$ with $f^{-1}(V) = \operatorname{Spec} B$, the A-module B is finite.

A morphism of finite type is *quasi-finite* if every fibre has only finitely many points.

For a morphism $X \to \operatorname{Spec} k$ to the spectrum of a field, the definition of finite type agrees with the absolute terminology introduced before theorem 2.57: equivalently, X admits a finite affine cover by spectra of finitely generated k-algebras.

These properties are local enough to reduce quickly to algebra.

Proposition 3.12. *Morphisms locally of finite type, morphisms of finite type, affine morphisms, and finite morphisms are stable under composition and arbitrary base change. Every finite morphism is of finite type and quasi-finite.*

Proof. The statements about composition follow from the corresponding algebraic statements together with stability of quasi-compactness under composition. For base change, the local algebraic assertions may be checked affinely, while quasi-compactness is stable under base change. If $A \to B$ is a finitely generated algebra and $A \to C$ is arbitrary, then $B \otimes_A C$ is generated as a C-algebra by the images of any finite set of A-algebra generators of B. If B is finite as an A-module, then the same finite set of module generators generates $B \otimes_A C$ over C. The affine assertion follows directly from theorem 3.3.

If B is finite over A, then it is certainly finitely generated as an A-algebra. For $\mathfrak{p} \in \operatorname{Spec} A$, the fibre ring

$$B \otimes_A \kappa(\mathfrak{p})$$

is finite-dimensional over $\kappa(\mathfrak{p})$ and hence Artinian. An Artinian ring has only finitely many prime ideals, so every fibre has finitely many points. □

Images of finite morphisms are especially well behaved.

Proposition 3.13. *Every finite morphism is closed. Moreover, every base change of a finite morphism is closed.*

Proof. The assertion is local on the target. Let $A \to B$ be finite and let $V(I) \subseteq \operatorname{Spec} B$ be closed. We claim

$$f(V(I)) = V(I \cap A).$$

One inclusion is immediate. Conversely, if $\mathfrak{p} \supseteq I \cap A$, then B/I is integral over the image of $A/(I \cap A)$. By lying over there is a prime of B/I over the prime induced by $\mathfrak{p}$, giving a prime $\mathfrak{q} \supseteq I$ with $\mathfrak{q} \cap A = \mathfrak{p}$. Hence the image is closed. Since finite morphisms remain finite after base change, the same closedness statement holds after every base change. □

For general finite-type morphisms, images need not be open or closed. The correct topological class is that of constructible sets.

Definition 3.14. A subset C of a topological space X is *locally closed* if it is the intersection of an open subset and a closed subset. It is *constructible* if it is a finite union of locally closed subsets.

Lemma 3.15. *Let Y be an irreducible Noetherian topological space and let $C \subseteq Y$ be constructible and dense. Then C contains a nonempty open subset of Y.*

Proof. Write C as a finite union of locally closed subsets $U_i \cap Z_i$, with U_i open and Z_i closed. Since C is dense and Y is irreducible, at least one Z_i must equal Y; otherwise the closure of C would be contained in the finite union of the proper closed subsets Z_i. For such an i, the set $U_i \cap Z_i = U_i$ is a nonempty open subset contained in C. □

The algebraic input for Chevalley's theorem is the following relative form of Noether normalization.

Lemma 3.16 (Relative Noether normalization)**.** *Let $A \subseteq B$ be domains such that B is a finitely generated A-algebra, and put*

$$r := \operatorname{trdeg}_{\operatorname{Frac}(A)} \operatorname{Frac}(B).$$

Then there exists a nonzero $a \in A$ and elements $z_1, \ldots, z_r \in B_a$ such that

$$A_a[z_1, \ldots, z_r] \subseteq B_a$$

is a finite integral extension.

Proof. Let $K = \mathrm{Frac}(A)$. The K-algebra

$$B_K := B \otimes_A K$$

is a finitely generated integral K-algebra with fraction field of transcendence degree r over K. By Noether normalization, there are algebraically independent elements $z_1, \ldots, z_r \in B_K$ such that B_K is finite over $K[z_1, \ldots, z_r]$.

Choose finitely many generators of B as an A-algebra and express the z_i and the coefficients of monic integral equations for these generators with denominators in A. After inverting the product of finitely many nonzero denominators, all z_i lie in B_a and the chosen algebra generators are integral over $A_a[z_1, \ldots, z_r]$. Thus B_a is finite over that polynomial algebra. Algebraic independence over K implies that the map $A_a[T_1, \ldots, T_r] \to B_a$ sending T_i to z_i is injective. □

Theorem 3.17 (Chevalley's Constructibility Theorem, Noetherian form)**.** *Let $f : X \to Y$ be a morphism of finite type between Noetherian schemes. Then the image of every constructible subset of X is constructible in Y.*

Proof. Because a constructible subset is a finite union of locally closed subsets, it is enough to prove that the image of a locally closed subset is constructible. Giving a locally closed subset its reduced induced scheme structure reduces the problem to proving that $f(X)$ is constructible.

Since Y is Noetherian and f is of finite type, X is Noetherian. We argue by Noetherian induction on closed subsets of X. Replacing X by X_{red} does not change its underlying set or its image. If this reduced scheme is reducible, it is a finite union of proper irreducible closed subschemes; the induction hypothesis applies to each of them, and the image of X is the finite union of their constructible images. We may therefore assume that X is integral.

Let

$$Z := \overline{f(X)} \subseteq Y$$

with its reduced induced structure. Because X is reduced and its image is contained in Z, the morphism factors through Z. Replacing Y by Z, we may suppose that Y is integral and that f is dominant.

Choose an affine open $V = \operatorname{Spec} A \subseteq Y$ containing the generic point of Y, and then an affine open $U = \operatorname{Spec} B \subseteq f^{-1}(V)$ containing the generic point of X. The rings A and B are domains, and since the generic point of X maps to the generic point of Y, the homomorphism $A \to B$ is injective. By lemma 3.16, after replacing V by a distinguished nonempty open $D(a)$ and U by its inverse image, there is a finite integral inclusion

$$A_a[z_1, \ldots, z_r] \subseteq B_a.$$

Consequently the morphism

$$\operatorname{Spec} B_a \longrightarrow \operatorname{Spec} A_a[z_1, \ldots, z_r]$$

is surjective by lying over, and the projection

$$\operatorname{Spec} A_a[z_1, \ldots, z_r] \longrightarrow \operatorname{Spec} A_a$$

is surjective. Hence the nonempty open $D(a) \subseteq Y$ is contained in $f(X)$.

Let $W = D(a)$. The closed subset

$$X_1 := X \setminus f^{-1}(W)$$

is proper in X, because the generic point of X maps into W. By the induction hypothesis, $f(X_1)$ is constructible. Since

$$f(X) = W \cup f(X_1),$$

the image $f(X)$ is constructible. $\square$

Remark 3.18. There is a more general form of Chevalley's theorem: a quasi-compact morphism locally of finite presentation maps locally constructible subsets to locally constructible subsets. Its proof uses approximation by finitely presented schemes and is logically broader than the Noetherian argument above. We record the general statement for orientation, but the arguments below that require constructibility use the Noetherian form proved here.

Example 3.19 (A constructible image that is neither open nor closed)**.** Consider

$$f : \mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k, \qquad (x, y) \longmapsto (u, v) = (x, xy).$$

The induced map

$$k[u, v] \longrightarrow k[x, y], \qquad u \mapsto x, \quad v \mapsto xy,$$

is injective, so f is dominant. Its image is

$$D(u) \cup \{(0, 0)\}.$$

Indeed, when $u \neq 0$ one has $y = v/u$, while for $u = 0$ necessarily $v = 0$. The image is constructible but is neither open nor closed.

We now turn to generic fibres. We first record the link between the global dimension introduced in Chapter 2 and the function field.

Lemma 3.20. *Let X be an integral scheme of finite type over a field k. Then every nonempty affine open $U \subseteq X$ has*

$$\dim U = \operatorname{trdeg}_k K(X),$$

and consequently

$$\dim X = \operatorname{trdeg}_k K(X).$$

In particular, every nonempty open subset of X has the same dimension as X.

Proof. If $U = \operatorname{Spec} A$ is nonempty and affine, proposition 2.51 identifies $\operatorname{Frac}(A)$ with $K(X)$. Hence theorem 2.57 gives

$$\dim U = \operatorname{trdeg}_k K(X).$$

Because X is quasi-compact, choose a finite affine open cover. Every nonempty member of this cover has the same dimension by the preceding computation, and proposition 2.56 therefore gives the same value for $\dim X$. Applying this argument to any nonempty open subscheme of X proves the final assertion. $\square$

Function fields also encode maps that are defined only on a dense open subset. This is the natural language for comparing schemes which agree generically but may differ on lower-dimensional closed subsets.

Definition 3.21 (Rational maps and birational morphisms)**.** Let X be an integral scheme and let Y be a scheme. A *rational map*

$$\varphi : X \dashrightarrow Y$$

is an equivalence class of morphisms $f : U \to Y$ defined on nonempty open subsets $U \subseteq X$. Two representatives

$$f : U \to Y, \qquad g : V \to Y$$

are equivalent if there exists a nonempty open subset $W \subseteq U \cap V$ on which

$$f|_W = g|_W.$$

If Y is integral, the rational map is *dominant* if one, hence every, representative has dense image in Y.

If $f : X \to Y$ is a dominant morphism of integral schemes, then f is called *birational* if the induced homomorphism of function fields

$$f^* : K(Y) \longrightarrow K(X)$$

is an isomorphism.

A dominant rational map $\varphi : X \dashrightarrow Y$ between integral schemes is called a *birational map* if there exists a dominant rational map $\psi : Y \dashrightarrow X$ such that

$$\psi \circ \varphi = \mathrm{id}_X, \qquad \varphi \circ \psi = \mathrm{id}_Y$$

as rational maps. In this case X and Y are said to be *birational.*

If X and Y are schemes over a base S, a rational map *over* S is one represented by S-morphisms. Birationality *over* S means that the birational map and its rational inverse are both over S.

The relation on representatives is indeed an equivalence relation. Only transitivity needs comment: if two equalities hold on nonempty opens W_1 and W_2, then $W_1 \cap W_2$ is again nonempty because every nonempty open subset of an integral scheme contains the generic point. The same observation shows that dominance is independent of the chosen representative.

The composition in the last definition is well defined for dominant rational maps. Indeed, if φ is represented by $f : U \to Y$ and ψ by $g : V \to Z$, with X, Y, Z integral and φ dominant, then $f^{-1}(V)$ is a nonempty open subset of U and

$$g \circ f : f^{-1}(V) \longrightarrow Z$$

represents the composite. Replacing either representative by an equivalent one changes this morphism only after shrinking to another nonempty open subset.

Proposition 3.22 (Rational maps and function fields)**.** *Let X and Y be integral schemes of finite type over a field k. Sending a dominant rational map over k to its map on generic stalks gives a natural bijection*

$$\{\textit{dominant rational maps over } k\ X \dashrightarrow Y\} \xleftrightarrow{\sim} \{k\textit{-embeddings } K(Y) \hookrightarrow K(X)\}.$$

Proof. Let $\varphi : X \dashrightarrow Y$ be represented by a dominant morphism $f : U \to Y$ from a nonempty open subset of X. Since X and Y are integral, U contains the generic point η_X, and dominance implies

$$f(\eta_X) = \eta_Y.$$

The morphism on local rings at the generic points therefore gives a k-embedding

$$K(Y) = \mathscr{O}_{Y,\eta_Y} \longrightarrow \mathscr{O}_{X,\eta_X} = K(X).$$

Equivalent representatives agree after restriction to a nonempty open subset and hence induce the same map on the generic stalks.

Conversely, let

$$\sigma : K(Y) \hookrightarrow K(X)$$

be a k-embedding. Choose a nonempty affine open

$$V = \operatorname{Spec} A \subseteq Y.$$

Because Y is of finite type over k, the k-algebra A is finitely generated; choose generators $a_1, \dots, a_m$. Each rational function $\sigma(a_i) \in K(X)$ is regular on some nonempty open subset $U_i \subseteq X$. Since X is irreducible, the intersection

$$U := \bigcap_{i=1}^{m} U_i$$

is nonempty. The assignments

$$a_i \longmapsto \sigma(a_i)|_U$$

induce a k-algebra homomorphism

$$A \longrightarrow \Gamma(U, \mathscr{O}_X).$$

To see that the defining relations of A vanish, map both sides into $K(X)$: by proposition 2.51, the restriction homomorphism

$$\Gamma(U, \mathscr{O}_X) \hookrightarrow K(X)$$

is injective, and the relations vanish there because σ is a field homomorphism. The resulting ring map defines a morphism

$$U \longrightarrow V \subseteq Y.$$

At the generic point η_X, the corresponding prime of A is the kernel of the composite

$$A \longrightarrow \Gamma(U, \mathscr{O}_X) \longrightarrow K(X),$$

which is zero because it is the restriction of the field embedding σ. Hence η_X maps to the generic point of V, so the morphism is dominant; its induced map on function fields is exactly σ.

It remains to prove uniqueness. Suppose two dominant rational maps induce the same field embedding. Choose an affine open $V = \operatorname{Spec} A \subseteq Y$ containing η_Y. After shrinking both domains, we may assume that both representatives land in V. On a common nonempty open subset $W \subseteq X$, each $a \in A$ is pulled back to the same element of $K(X)$; injectivity of

$$\Gamma(W, \mathscr{O}_X) \hookrightarrow K(X)$$

shows that the two pullbacks are equal as regular sections. Hence the two morphisms $W \to V$ agree, so the rational maps are equal. $\square$

Proposition 3.23 (Equivalent forms of birationality)**.** *Let X and Y be integral schemes of finite type over a field k. The following are equivalent:*

1. *X and Y are birational over k;*
2. *there is a k-isomorphism of function fields*
$$K(X) \cong K(Y);$$
3. *there exist nonempty open subsets $U \subseteq X$ and $V \subseteq Y$ and a k-isomorphism*
$$U \xrightarrow{\sim} V.$$

Moreover, if $f : X \to Y$ is a dominant k-morphism of integral schemes of finite type over k, then f is birational in the sense of definition 3.21 if and only if, after shrinking X and Y to nonempty open subsets, the morphism f becomes an isomorphism.

Proof. A birational map and its rational inverse induce inverse embeddings of function fields by proposition 3.22, so (1) implies (2). Conversely, a field isomorphism and its inverse correspond by the same proposition to dominant rational maps in opposite directions. Their composites induce the identity maps on $K(X)$ and $K(Y)$, and uniqueness in proposition 3.22 shows that the composites are the identity rational maps. Thus (2) implies (1).

Assume (1), represented by dominant morphisms
$$f : U_0 \to Y, \qquad g : V_0 \to X$$
on nonempty open subsets. Because $\psi \circ \varphi = \mathrm{id}_X$ as rational maps, after shrinking inside $U_0 \cap f^{-1}(V_0)$ there is a nonempty open $U_1 \subseteq X$ on which
$$g \circ f = \mathrm{id}_{U_1}.$$
Similarly, there is a nonempty open $V_1 \subseteq V_0 \cap g^{-1}(U_0)$ on which
$$f \circ g = \mathrm{id}_{V_1}.$$
Since f is dominant, $f^{-1}(V_1)$ is a nonempty open subset of U_0. The irreducibility of X therefore implies that
$$U := U_1 \cap f^{-1}(V_1)$$
is nonempty. Since g is dominant, the open subset
$$V := V_1 \cap g^{-1}(U)$$
is nonempty as well. If $x \in U$, then $f(x) \in V_1$ and
$$g(f(x)) = x \in U,$$
so $f(x) \in V$. If $y \in V$, then $g(y) \in U \subseteq U_1$ and
$$f(g(y)) = y$$
because $y \in V_1$. Thus the restrictions
$$f|_U : U \longrightarrow V, \qquad g|_V : V \longrightarrow U$$
are inverse isomorphisms, proving (3). Conversely, an isomorphism of nonempty opens is itself a rational map with rational inverse, so (3) implies (1).

For the final assertion, a dominant morphism $f : X \to Y$ is birational precisely when its map on function fields is an isomorphism. By the equivalence just proved, its rational-map class contains an isomorphism $h : U \xrightarrow{\sim} V$ between nonempty opens. Since h and the global morphism f represent the same rational map, there is a nonempty open $W \subseteq U$ on which $f|_W = h|_W$. The image $h(W)$ is open in V, and therefore

$$f|_W : W \xrightarrow{\sim} h(W)$$

is an isomorphism obtained by restricting f. The converse follows immediately from the induced isomorphism of generic local rings. □

Birational geometry studies what may change on strict closed subsets while the common function field remains fixed.

Let $f : X \to Y$ be dominant with X and Y integral, and let η be the generic point of Y. The inclusion of function fields

$$K(Y) \hookrightarrow K(X)$$

comes from the morphism on the generic stalks.

Proposition 3.24 (Dimension of the generic fibre)**.** *Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over a field k. Then*

$$\dim X_\eta = \operatorname{trdeg}_{K(Y)} K(X) = \dim X - \dim Y.$$

Proof. Choose nonempty affine opens $V = \operatorname{Spec} A \subseteq Y$ and $U = \operatorname{Spec} B \subseteq f^{-1}(V)$ containing the respective generic points; then $A \to B$ is injective. Base change to $\eta = \operatorname{Spec} K(Y)$ gives the nonempty affine open

$$U_\eta = \operatorname{Spec}(B \otimes_A K(Y)) \subseteq X_\eta.$$

The ring $B \otimes_A K(Y)$ is a localization of the domain B, hence is a domain, and its fraction field is $K(X)$. The whole generic fibre X_η is integral by the same affine-local argument. Applying lemma 3.20 over the field $K(Y)$ gives

$$\dim X_\eta = \dim U_\eta = \operatorname{trdeg}_{K(Y)} K(X).$$

The tower formula for transcendence degree gives

$$\operatorname{trdeg}_k K(X) = \operatorname{trdeg}_{K(Y)} K(X) + \operatorname{trdeg}_k K(Y).$$

Using lemma 3.20 for X and Y gives the stated dimension formula. □

Definition 3.25. A dominant morphism $f : X \to Y$ of integral schemes is *generically finite* if the field extension

$$K(Y) \subseteq K(X)$$

is finite. Its degree is

$$\deg(f) := [K(X) : K(Y)].$$

By definition 3.21, every birational morphism of integral schemes is therefore generically finite of degree one.

Corollary 3.26. *For a dominant morphism $f : X \to Y$ of integral finite-type k-schemes, the following are equivalent:*

1. *f is generically finite;*
2. $\dim X_\eta = 0$*;*
3. $\dim X = \dim Y$*.*

Proof. A finitely generated field extension has transcendence degree zero exactly when it is algebraic, and a finitely generated algebraic field extension is finite. The equivalences now follow from proposition 3.24. □

The generic fibre controls a dense open set of ordinary fibres.

Theorem 3.27 (Dimension of general fibres)**.** *Let $f : X \to Y$ be a dominant morphism of integral schemes of finite type over a field k, and put*

$$r := \dim X - \dim Y.$$

Then there exists a nonempty open subset $V \subseteq Y$ such that

$$\dim X_y = r$$

for every $y \in V$.

Proof. We argue by induction on $\dim X$; every proper irreducible closed subset of the integral finite-type scheme X has strictly smaller dimension. Choose affine opens $V = \operatorname{Spec} A \subseteq Y$ and $U = \operatorname{Spec} B \subseteq X$ containing the generic points and with $f(U) \subseteq V$. Then $A \subseteq B$. By lemma 3.16, after shrinking V to a distinguished nonempty open $D(a)$, there is a finite integral inclusion

$$A_a[z_1, \dots, z_r] \subseteq B_a.$$

Thus

$$U_{D(a)} = \operatorname{Spec} B_a \longrightarrow \mathbb{A}^r_{D(a)}$$

is finite and surjective. After any base change to a point $y \in D(a)$ we obtain a finite surjective morphism

$$(U_{D(a)})_y \longrightarrow \mathbb{A}^r_{\kappa(y)},$$

so

$$\dim(U_{D(a)})_y = r.$$

Indeed, a finite surjective morphism is induced by an integral extension of coordinate rings, and integral extensions preserve Krull dimension.

Let $C = X \setminus U$. It has finitely many irreducible components $C_1, \dots, C_m$, each of dimension strictly smaller than $\dim X$. If C_i does not dominate Y, then $\overline{f(C_i)}$ is a proper closed subset of Y, so after removing it the fibre has no points coming from C_i. If C_i dominates Y, apply the induction hypothesis to the integral reduction of C_i: after shrinking the base,

$$\dim(C_i)_y = \dim C_i - \dim Y \leq r - 1.$$

Intersecting the finitely many resulting dense open subsets with $D(a)$ gives a nonempty open $V' \subseteq Y$ on which the affine part U_y has dimension r and every component coming from C has dimension at most $r - 1$. Since

$$X_y = U_y \cup C_y$$

as topological spaces, every irreducible component of the Noetherian fibre X_y either lies in C_y or has generic point in the open subset U_y. In the second case its intersection with U_y is a dense open subset of that component and has the same dimension. Hence

$$\dim X_y = \max\{\dim U_y, \dim C_y\} = r$$

for every $y \in V'$. $\square$

Corollary 3.28. *A dominant generically finite morphism of integral schemes of finite type over a field is quasi-finite over some nonempty open subset of the target.*

Proof. By corollary 3.26 and theorem 3.27, after shrinking the target every fibre has dimension zero. Each fibre is a Noetherian scheme of dimension zero. Its irreducible components are therefore single closed points, and a Noetherian space has only finitely many irreducible components. Hence every fibre has finitely many points, so the restricted morphism is quasi-finite. $\square$

Example 3.29 (Jumping fibre dimension)**.** For the morphism of example 3.19, the generic fibre has dimension zero. Over a point (u, v) with $u \neq 0$, the fibre is a single point. Over $(0, 0)$, however,

$$X_{(0,0)} \cong \operatorname{Spec} k[y] = \mathbb{A}^1_k,$$

so the fibre dimension jumps to one. Thus the theorem on general fibres cannot be extended to every point of the base.

A stronger and very useful principle is that the function measuring fibre dimension is upper semicontinuous for proper morphisms. We postpone this statement until properness has been introduced, where the closed-map property makes its geometric meaning transparent.

3.3 Separated and Proper Morphisms

In ordinary topology, the Hausdorff condition can be expressed by saying that the diagonal is closed. This formulation survives in algebraic geometry, where the diagonal carries a scheme structure and must be a closed immersion rather than merely a closed subset.

Definition 3.30. A morphism $f : X \to Y$ is *separated* if the diagonal morphism

$$\Delta_{X/Y} : X \longrightarrow X \times_Y X$$

is a closed immersion. A scheme X is separated over S if its structure morphism $X \to S$ is separated.

The diagonal is characterized by

$$\begin{tikzcd}
X \arrow[rr, "\Delta_{X/Y}"] \arrow[dr, "\mathrm{id}_X"] \arrow[ddr, bend right, "f"'] & & X \times_Y X \arrow[dl, "p_1"'] \arrow[ddl, bend left, "p_2"] \\
& X \arrow[d, "f"] & \\
& Y. &
\end{tikzcd}$$

Proposition 3.31. *Every morphism of affine schemes is separated. Consequently every affine morphism is separated.*

Proof. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then

$$X \times_Y X \cong \operatorname{Spec}(B \otimes_A B).$$

The diagonal corresponds to the multiplication homomorphism

$$B \otimes_A B \longrightarrow B, \qquad b \otimes b' \longmapsto bb'.$$

This map is surjective because b is the image of $b \otimes 1$, so the diagonal is a closed immersion. The affine-morphism statement follows by checking the diagonal over an affine cover of the target. □

Proposition 3.32. *For every morphism $f : X \to Y$, the diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is an immersion.*

Proof. The assertion is local on the target $X \times_Y X$. Choose affine opens $U, V \subseteq X$ whose images lie in a common affine open $W \subseteq Y$. The inverse image of $U \times_W V$ under the diagonal is $U \cap V$. Cover $U \cap V$ by affine opens T. For each such T, the map

$$T \longrightarrow U \times_W V$$

factors as the diagonal $T \to T \times_W T$, which is a closed immersion by the affine case, followed by the open immersion $T \times_W T \to U \times_W V$. Hence the diagonal is locally a composition of an open immersion and a closed immersion, i.e. an immersion. □

Proposition 3.33 (Graph criterion)**.** *Let X and Y be schemes over S and let $f : X \to Y$ be an S-morphism. If $Y \to S$ is separated, then the graph*

$$\Gamma_f : X \longrightarrow X \times_S Y$$

is a closed immersion.

Proof. The graph is the base change of the diagonal of Y/S. More precisely, the square

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "(f \circ p_X, p_Y)"] \\ Y \arrow[r, "\Delta_{Y/S}"'] & Y \times_S Y \end{tikzcd}$$

is cartesian. Since $\Delta_{Y/S}$ is a closed immersion and closed immersions are stable under base change, Γ_f is a closed immersion. □

Proposition 3.34. *Separated morphisms are stable under composition and arbitrary base change. Open immersions and closed immersions are separated. If $X \to Z$ and $Y \to Z$ are separated, then $X \times_Z Y \to Z$ is separated.*

Proof. Open and closed immersions are monomorphisms, and the diagonal of a monomorphism is an isomorphism, hence a closed immersion.

For base change, the diagonal of the base-changed morphism is the base change of the original diagonal. For composition

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

the diagonal $\Delta_{X/Z}$ factors as

$$X \xrightarrow{\Delta_{X/Y}} X \times_Y X \longrightarrow X \times_Z X,$$

where the second arrow is a base change of $\Delta_{Y/Z}$. Both arrows are closed immersions. The product assertion follows because $X \times_Z Y \to Y$ is a base change of $X \to Z$ and its composition with $Y \to Z$ is separated. □

The doubled-origin example from Chapter 2 now has a precise diagnosis.

Example 3.35 (The doubled origin is not separated)**.** Let X be obtained by gluing two copies U and V of $\mathbb{A}^1_k$ along $D(t)$ by the identity. The two origins define a point

$$(0_U, 0_V) \in U \times_k V \subseteq X \times_k X$$

which is not on the diagonal. Every open neighbourhood of $(0_U, 0_V)$ meets the diagonal along points (a, a) with $a \neq 0$. Hence the image of the diagonal is not closed, so X is not separated over k.

Separatedness also forces uniqueness from dense open subsets.

Proposition 3.36. *Let X be a reduced S-scheme, let Y be separated over S, and let $f, g : X \to Y$ be S-morphisms. If f and g agree on a dense open subset of X, then $f = g$.*

Proof. Consider

$$h = (f, g) : X \longrightarrow Y \times_S Y.$$

The equalizer of f and g is the inverse image

$$Z = h^{-1}(\Delta_{Y/S}),$$

which is a closed subscheme because Y/S is separated. Its underlying closed subset contains a dense open subset and therefore equals X. Affine locally write

$$X = \operatorname{Spec} A, \qquad Z = \operatorname{Spec}(A/I).$$

Since Z and X have the same underlying space, every element of I belongs to every prime ideal of A and is nilpotent. Reducedness gives $I = 0$, so $Z = X$ and $f = g$. □

Valuation rings translate separatedness into a uniqueness statement for specializations.

Definition 3.37. Let K be a field. A subring $R \subseteq K$ is a *valuation ring of* K if for every $0 \neq a \in K$,

$$a \in R \qquad \text{or} \qquad a^{-1} \in R.$$

Equivalently, the ideals of R are totally ordered by inclusion. A valuation ring R *dominates* a local domain $(A, \mathfrak{m})$ if

$$A \subseteq R, \qquad \mathfrak{m}_R \cap A = \mathfrak{m}.$$

Lemma 3.38 (Existence of dominating valuation rings)**.** *Let $(A, \mathfrak{m})$ be a local domain and let K be a field containing* $\operatorname{Frac}(A)$*. Then there exists a valuation ring $R \subseteq K$ dominating A.*

Proof. Order the local subrings of K dominating A by domination. The union of a totally ordered family is again a local subring dominating every member, so Zorn's lemma gives a maximal element R.

First, R has fraction field K. Otherwise choose $t \in K \setminus \mathrm{Frac}(R)$. If t is transcendental over R, then

$$R[t]_{(\mathfrak{m}_R, t)}$$

is a strictly larger local subring of K dominating R. If t is algebraic over $\mathrm{Frac}(R)$, then after multiplying by a nonzero element of R we may arrange that at is integral over R. The finite extension $R[at]$ has a prime lying over $\mathfrak{m}_R$; localizing at such a prime again gives a strictly larger local ring dominating R. Both alternatives contradict maximality.

Next, R is integrally closed in K. If $x \in K$ is integral over R, then a prime of the finite extension $R[x]$ lies over $\mathfrak{m}_R$, and localization would dominate R; maximality forces $x \in R$.

Finally let $x \in K^\times$ with $x \notin R$. There can be no prime of $R[x]$ lying over $\mathfrak{m}_R$, for otherwise localization would strictly dominate R. Hence

$$\mathfrak{m}_R R[x] = R[x].$$

Write $1 = \sum_{i=0}^{d} a_i x^i$ with $a_i \in \mathfrak{m}_R$. Multiplying by x^{-d} gives a monic integral equation for x^{-1} after dividing by the unit $1 - a_0$. Thus x^{-1} is integral over R, hence belongs to R. Therefore for every nonzero $x \in K$, either $x \in R$ or $x^{-1} \in R$, so R is a valuation ring dominating A. □

Chevalley's theorem now supplies the specialization statement needed for the valuative criteria.

Lemma 3.39 (Specialization lemma)**.** *Let $f : X \to Y$ be a morphism of finite type between locally Noetherian schemes, and let*

$$y \in \overline{f(X)}.$$

Then there exist a field K, a valuation ring $R \subseteq K$, and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

such that the closed point of $\operatorname{Spec} R$ *maps to y.*

Proof. Choose an affine open neighbourhood $Y_0 \subseteq Y$ of y. Since f is of finite type, $X_0 := f^{-1}(Y_0)$ is quasi-compact; because Y_0 is Noetherian, X_0 is Noetherian. Thus Chevalley's theorem applies to $X_0 \to Y_0$, and replacing (X, Y) by (X_0, Y_0) does not change the assertion.

Write $\overline{f(X)} = Z_1 \cup \cdots \cup Z_m$ as its finite decomposition into irreducible components. Since $f(X)$ is dense in this union, for every component Z_i the set $f(X) \cap Z_i$ is dense in Z_i: otherwise the union of the other components together with the proper closed subset $\overline{f(X) \cap Z_i}$ would still contain $f(X)$ and hence its closure, contradicting maximality of Z_i. Choose a component Z containing y. By Chevalley's theorem and lemma 3.15, $f(X) \cap Z$ contains a nonempty open subset of Z, and therefore contains the generic point η of Z. Choose $x \in X$ with $f(x) = \eta$, and put $K = \kappa(x)$.

The local ring $\mathscr{O}_{Z,y}$ is a local domain with fraction field $\kappa(\eta)$, and the map on residue fields induced by f gives an inclusion

$$\kappa(\eta) \hookrightarrow K.$$

By lemma 3.38 above, there is a valuation ring $R \subseteq K$ dominating $\mathscr{O}_{Z,y}$. The induced morphism $\operatorname{Spec} R \to Z \hookrightarrow Y$ and the canonical K-valued point associated with x fit into the commutative square

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & Y. \end{tikzcd}$$

Commutativity on the generic point is exactly the equality of the two inclusions $\kappa(\eta) \hookrightarrow K$. The closed point of $\operatorname{Spec} R$ maps to y because R dominates $\mathscr{O}_{Z,y}$. This is the required specialization diagram. $\square$

Theorem 3.40 (Valuative criterion for separatedness)**.** *Let $f : X \to Y$ be a morphism of finite type between locally Noetherian schemes. Then f is separated if and only if, for every valuation ring R with fraction field K and every commutative solid diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there is at most one dotted arrow making the diagram commute.

Proof. Assume first that f is separated and let $a, b : \operatorname{Spec} R \to X$ be two extensions. The morphism

$$(a, b) : \operatorname{Spec} R \longrightarrow X \times_Y X$$

maps the generic point into the diagonal. Since the diagonal is a closed immersion, its inverse image is a closed subscheme of $\operatorname{Spec} R$ whose underlying closed subset contains the generic point. The generic point is dense because R is a domain, so this closed subset is all of $\operatorname{Spec} R$. The defining ideal of the inverse-image subscheme is therefore contained in the nilradical of the reduced ring R, hence is zero. Thus (a, b) factors scheme-theoretically through the diagonal and $a = b$.

Conversely, suppose uniqueness holds. By proposition 3.32, the diagonal is an immersion, so it suffices to prove that its image is closed. Let

$$z \in \overline{\Delta_{X/Y}(X)}.$$

Because f is of finite type over a locally Noetherian target, it is of finite presentation. Its diagonal is therefore of finite presentation, in particular of finite type. Moreover X is locally Noetherian and the projection

$$X \times_Y X \longrightarrow X$$

is a base change of f, hence is of finite type; consequently $X \times_Y X$ is locally Noetherian. Thus lemma 3.39 applies to the diagonal and gives a valuation ring R with fraction field K and a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "\Delta_{X/Y}"] \\ \operatorname{Spec} R \arrow[r] & X \times_Y X \end{tikzcd}$$

whose closed point maps to z. Composing the lower map with the two projections gives morphisms

$$a, b : \operatorname{Spec} R \longrightarrow X$$

that agree on $\operatorname{Spec} K$. By the uniqueness hypothesis, $a = b$. Therefore the map $\operatorname{Spec} R \to X \times_Y X$ factors through the diagonal, so its closed point z belongs to the image of the diagonal. Thus the diagonal image is closed and f is separated. □

Properness adds existence to this uniqueness phenomenon.

Definition 3.41. A morphism $f : X \to Y$ is *universally closed* if every base change

$$X \times_Y Y' \longrightarrow Y'$$

is a closed map on underlying topological spaces. It is *proper* if it is of finite type, separated, and universally closed.

Proposition 3.42. *The following statements hold.*

1. *Closed immersions are proper.*
2. *Proper morphisms are stable under composition and arbitrary base change.*
3. *If $X \to Z$ and $Y \to Z$ are proper, then $X \times_Z Y \to Z$ is proper.*
4. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ with g separated and $g \circ f$ proper, then f is proper.*
5. *Every finite morphism is proper.*
6. *Properness is local on the target: if $Y = \bigcup_i V_i$ is an open cover and every $f^{-1}(V_i) \to V_i$ is proper, then f is proper.*

Proof. Closed immersions are finite type and separated, and remain closed immersions after arbitrary base change, hence are proper. Finite type, separatedness, and universal closedness are each stable under composition and base change, proving (2).

For (3), the projection $X \times_Z Y \to Y$ is the base change of $X \to Z$ and is proper; its composition with $Y \to Z$ is proper.

For (4), the graph

$$\Gamma_f : X \longrightarrow X \times_Z Y$$

is a closed immersion because g is separated. The projection $X \times_Z Y \to Y$ is the base change of $g \circ f$ and is proper. Since f is their composition, it is proper.

A finite morphism is of finite type and separated by proposition 3.31, and it is universally closed by proposition 3.13.

For the local-on-the-target assertion, finite type and separatedness are local on the target. To check universal closedness, make an arbitrary base change $Y' \to Y$ and let $Z \subseteq X_{Y'}$ be closed. The inverse images $V_i' = Y' \times_Y V_i$ cover Y', and

$$f_{Y'}(Z) \cap V_i' = f_{V_i'}(Z \cap f_{Y'}^{-1}(V_i'))$$

is closed in V_i' because the restricted morphism is proper. Thus $f_{Y'}(Z)$ is closed locally on the open cover $\{V_i'\}$, hence closed in Y'. □

Proposition 3.43 (Properness descends to a separated finite-type image)**.** *Let $X \to S$ be proper and let $f : X \to Y$ be a surjective S-morphism. If $Y \to S$ is separated and of finite type, then $Y \to S$ is proper.*

Proof. Only universal closedness remains. After any base change $S' \to S$, the morphism $X_{S'} \to Y_{S'}$ remains surjective (surjectivity is stable under base change, as can be checked on the nonemptiness of fibres). If $Z \subseteq Y_{S'}$ is closed, its inverse image in $X_{S'}$ is closed. Since $X_{S'} \to S'$ is closed, the image of this inverse image is closed in S'. Surjectivity shows that this image equals the image of Z. Hence $Y_{S'} \to S'$ is closed for every S', proving universal closedness. □

Theorem 3.44 (Valuative criterion for universal closedness)**.** *Let $h : Z \to T$ be a quasi-compact morphism of schemes. Then h is universally closed if and only if every commutative solid diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & Z \arrow[d, "h"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

with R a valuation ring and $K = \operatorname{Frac}(R)$ admits a dotted arrow.

Proof. Assume first that h is universally closed. After base change along the lower horizontal map, we may replace T by $\operatorname{Spec} R$ and Z by Z_R. Let $\xi \in Z_R$ be the image of the generic point. Since ξ lies over the generic point of $\operatorname{Spec} R$, the structure morphism gives an inclusion $K \hookrightarrow \kappa(\xi)$. The given K-valued point gives a K-algebra homomorphism $\kappa(\xi) \to K$. A homomorphism of fields is injective, and its image both contains and is contained in K, so $\kappa(\xi) = K$. Let Γ be the reduced closure of $\{\xi\}$ in Z_R. It is integral with generic point ξ and function field $\kappa(\xi) = K$. Since h_R is closed, the image of Γ is a closed subset of $\operatorname{Spec} R$ containing the generic point, hence is all of $\operatorname{Spec} R$. Choose $\gamma \in \Gamma$ over the closed point. The local ring at γ therefore embeds in the common function field K, and the structural morphism gives local inclusions

$$R \subseteq \mathscr{O}_{\Gamma,\gamma} \subseteq K.$$

The middle local ring dominates R. The valuation property forces equality: if $a \in \mathscr{O}_{\Gamma,\gamma} \setminus R$, then $a^{-1} \in \mathfrak{m}_R \subseteq \mathfrak{m}_{\Gamma,\gamma}$, although a^{-1} is a unit in $\mathscr{O}_{\Gamma,\gamma}$. Thus $\mathscr{O}_{\Gamma,\gamma} = R$, and

$$\operatorname{Spec} R = \operatorname{Spec} \mathscr{O}_{\Gamma,\gamma} \longrightarrow \Gamma \hookrightarrow Z_R$$

is the required extension.

Conversely, assume the existence condition. It is preserved by arbitrary base change by the universal property of the fibre product, so it is enough to show that h itself is closed. We first show that specializations lift along h. Let $z' \in Z$ lie over $t' \in T$, and let t be a specialization of t'. Put $K = \kappa(z')$. The local domain

$$\mathscr{O}_{\overline{\{t'\}},t}$$

has fraction field $\kappa(t')$, which embeds into K. By lemma 3.38, choose a valuation ring $R \subseteq K$ dominating this local domain. We obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "z'"] \arrow[d] & Z \arrow[d, "h"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & T \end{tikzcd}$$

whose closed point maps to t. A dotted lift therefore supplies a specialization of z' lying over t.

It remains to use quasi-compactness. We claim that the image of a quasi-compact morphism of schemes which is stable under specialization is closed. This can be checked after restricting the target to an affine open, so write $T = \operatorname{Spec} A$. Since Z is quasi-compact, choose a finite affine open cover $Z = \bigcup_{i=1}^r \operatorname{Spec} B_i$ and put $B = \prod_i B_i$. The image of $\operatorname{Spec} B \to \operatorname{Spec} A$ is exactly $h(Z)$. Let

$$I = \ker(A \longrightarrow B).$$

The closure of the image is $V(I)$. Indeed, every point in the image contains I, while if $a \notin \sqrt{I}$, then its image in B is not nilpotent and is avoided by some prime ideal of B; hence the image meets $D(a)$. If $\mathfrak{p}$ is minimal over I, then $B_{\mathfrak{p}} \neq 0$: otherwise some $a \notin \mathfrak{p}$ would annihilate 1 in B, hence would lie in $I \subseteq \mathfrak{p}$. Choose a prime of $B_{\mathfrak{p}}$. Its contraction to A is a prime contained in $\mathfrak{p}$ and containing I, hence equals $\mathfrak{p}$ by minimality. Thus every minimal point of $V(I)$ belongs to $h(Z)$. Since $h(Z)$ is stable under specialization, it contains every prime containing such a minimal prime, and therefore all of $V(I)$. Hence $h(Z) = V(I)$ is closed.

Now let $C \subseteq Z$ be closed. The restriction $C \to T$ remains quasi-compact. It also satisfies the existence condition. Give C its reduced induced closed-scheme structure. Any lift $\operatorname{Spec} R \to Z$ whose generic point factors through C pulls the ideal sheaf of C back to an ideal of the valuation ring R whose vanishing set is all of $\operatorname{Spec} R$. Since a valuation ring is reduced, that ideal is zero; hence the lift factors scheme-theoretically through C. Applying the preceding argument to $C \to T$ shows that $h(C)$ is closed. Thus h is closed, and the same reasoning after every base change proves that h is universally closed. □

Theorem 3.45 (Valuative criterion for properness). *Let $f : X \to Y$ be a morphism of finite type between locally Noetherian schemes. Then f is proper if and only if, for every valuation ring R with fraction field K and every commutative solid diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there exists a unique dotted arrow making the diagram commute.

Proof. If f is proper, then it is separated and quasi-compact. Uniqueness of a lift follows from theorem 3.40, while existence follows from universal closedness and theorem 3.44.

Conversely, assume that every valuation-ring diagram admits a unique lift. The uniqueness part implies that f is separated by theorem 3.40. Since f is of finite type, it is quasi-compact. The existence part therefore implies that f is universally closed by theorem 3.44. Hence f is finite type, separated, and universally closed, so it is proper. □

Remark 3.46 (General form of the valuative criterion). The Noetherian assumptions in theorem 3.45 were used only to derive separatedness from uniqueness via theorem 3.40. In the usual general form, a finite-type quasi-separated morphism is proper if and only if every valuation-ring diagram admits a unique lift. For the applications below we will not need to import that stronger separatedness criterion: projective space will already be known to be separated, and theorem 3.44 will supply universal closedness from existence.

The two valuative criteria can be remembered schematically as

separated	at most one extension
proper	exactly one extension.

They express uniqueness and existence of specialization in purely local algebraic terms.

Lemma 3.47 (Integral closure as an intersection of valuation rings). *Let A be a subring of a field K. The integral closure of A in K is the intersection of all valuation rings of K containing A.*

Proof. Every valuation ring is integrally closed, so every element of K integral over A belongs to every valuation ring containing A.

Conversely, let $x \in K$ be not integral over A. Then x^{-1} is not a unit in $A[x^{-1}]$: otherwise an identity

$$x^{-1}(a_0 + a_1 x^{-1} + \cdots + a_m x^{-m}) = 1$$

with $a_i \in A$ would give, after multiplication by a suitable power of x, a monic equation for x over A. Choose a maximal ideal $\mathfrak{m} \subset A[x^{-1}]$ containing x^{-1} and apply lemma 3.38 to the local domain $A[x^{-1}]_{\mathfrak{m}}$. We obtain a valuation ring $V \subseteq K$ containing A and satisfying $x^{-1} \in \mathfrak{m}_V$. If x belonged to V, then x^{-1} would be a unit of V, a contradiction. Thus x is excluded from at least one valuation ring containing A, proving the reverse inclusion. □

Proposition 3.48 (Affine proper morphisms)**.** *Let $f : X \to Y$ be an affine morphism of integral schemes of finite type. If f is proper, then f is finite.*

Proof. The assertion is local on Y, so write

$$Y = \operatorname{Spec} A, \qquad X = \operatorname{Spec} B.$$

First assume $A \to B$ is injective and put $K = \operatorname{Frac}(B)$. Let V be any valuation ring of K containing A. The inclusions define a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \operatorname{Spec} V & \longrightarrow & \operatorname{Spec} A. \end{array}$$

Properness extends the generic map to $\operatorname{Spec} V \to \operatorname{Spec} B$, which on rings says $B \subseteq V$. Hence B is contained in every valuation ring of K containing A. By lemma 3.47, the integral closure of A in K is the intersection of all such valuation rings, so every element of B is integral over A. Since B is finitely generated as an A-algebra, it is finite as an A-module.

In general let $I = \ker(A \to B)$. Then

$$\operatorname{Spec} B \longrightarrow \operatorname{Spec}(A/I) \longrightarrow \operatorname{Spec} A$$

factors f. The first map is dominant, and it is proper by proposition 3.42(4), since the second map is a separated closed immersion and their composite is f. The dominant case shows that B is finite over A/I, hence finite over A. □

Properness also improves the behaviour of fibre dimension.

Remark 3.49 (Fibre dimension under properness). A standard local dimension theorem says that for a finite-type morphism the locus of points $x \in X$ where the local dimension of the fibre is at least a fixed integer is closed. Combining this with the fact that a proper morphism is closed shows that, for a proper morphism of Noetherian schemes,

$$y \longmapsto \dim X_y$$

is upper semicontinuous. We record the phenomenon here but postpone its systematic use until flatness is available: flatness supplies the complementary lower-semicontinuity statement, and proper flat families have locally constant fibre dimension.

3.4 Projective Morphisms and Chow's Lemma

Projective schemes were constructed in Chapter 2 from graded rings. We now place that construction over a variable base. The key compatibility is that Proj commutes with extension of scalars.

Proposition 3.50 (Base change for Proj). *Let $A \to B$ be a ring homomorphism and let*

$$S = \bigoplus_{n \geq 0} S_n$$

be a graded A-algebra. Give $S \otimes_A B$ the grading in which B has degree zero. Then there is a canonical isomorphism

$$\operatorname{Proj} S \times_{\operatorname{Spec} A} \operatorname{Spec} B \cong \operatorname{Proj}(S \otimes_A B).$$

Proof. For a homogeneous $f \in S_+$, the standard affine open of $\operatorname{Proj} S$ is

$$D_+(f) = \operatorname{Spec} S_{(f)}.$$

After base change to B it becomes

$$\operatorname{Spec}(S_{(f)} \otimes_A B).$$

Since B is concentrated in degree zero, localization and taking the degree-zero part commute with tensoring by B:

$$S_{(f)} \otimes_A B \cong (S \otimes_A B)_{(f \otimes 1)}.$$

Hence the base change of $D_+(f)$ is canonically the standard open $D_+(f \otimes 1)$ in $\operatorname{Proj}(S \otimes_A B)$. The opens obtained in this way cover the right-hand side as well: if a homogeneous prime contained every element $f \otimes 1$ with $f \in S_+$ homogeneous, it would contain every positive-degree pure tensor and hence the whole irrelevant ideal. The identifications agree on overlaps because all restriction maps are induced by localization. They therefore glue to the stated isomorphism. □

Corollary 3.51. *For every ring homomorphism $A \to B$,*

$$\mathbb{P}^n_A \times_{\operatorname{Spec} A} \operatorname{Spec} B \cong \mathbb{P}^n_B.$$

Proof. Apply proposition 3.50 to the standard graded algebra $S = A[T_0, \ldots, T_n]$. Then

$$S \otimes_A B \cong B[T_0, \ldots, T_n]$$

as graded rings. □

Definition 3.52. Let Y be a scheme. The *relative projective n-space* over Y is the scheme $\mathbb{P}^n_Y$ obtained by gluing the schemes $\mathbb{P}^n_A$ over affine opens $\operatorname{Spec} A \subseteq Y$. The ring-level base-change isomorphisms above identify these local pieces on overlaps. Equivalently,

$$\mathbb{P}^n_Y \cong Y \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}.$$

Corollary 3.53. *For every morphism $Y' \to Y$, there is a canonical isomorphism*

$$\mathbb{P}^n_Y \times_Y Y' \cong \mathbb{P}^n_{Y'}.$$

Proof. The assertion is local on Y and Y'. On affine opens it is exactly corollary 3.51, and the affine isomorphisms agree on overlaps because they are induced by the canonical base-change identifications. Hence they glue. □

Definition 3.54. A morphism $f : X \to Y$ is *projective* if there exist $n \geq 0$ and a factorization

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

with i a closed immersion. It is *quasi-projective* if i may be taken to be a locally closed immersion.

Proposition 3.55. *Projective and quasi-projective morphisms are stable under arbitrary base change.*

Proof. Base changing a projective factorization gives

$$\begin{tikzcd} X \times_Y Y' \arrow[r, hook] \arrow[dr] & \mathbb{P}^n_Y \times_Y Y' \arrow[d, "\sim"] \\ & \mathbb{P}^n_{Y'}, \end{tikzcd}$$

and closed, open, and locally closed immersions are stable under base change. The assertion follows from corollary 3.53. □

The composition theorem is the first place where the Segre embedding naturally appears.

Proposition 3.56 (Relative Segre embedding)**.** *For every scheme S there is a closed immersion*

$$\mathbb{P}^m_S \times_S \mathbb{P}^n_S \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}_S$$

which on homogeneous coordinates is given by

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \longmapsto [x_i y_j]_{0 \leq i \leq m,\, 0 \leq j \leq n}.$$

Proof. The assertion is local on S, so take $S = \operatorname{Spec} A$. Write $x_0, \ldots, x_m$ and $y_0, \ldots, y_n$ for the homogeneous coordinates on the two factors and z_{ij} for the $(m+1)(n+1)$ homogeneous coordinates on the target.

The product is covered by affine charts

$$D_+(x_i) \times_A D_+(y_j).$$

On such a chart define a morphism to $D_+(z_{ij})$ by the coordinate homomorphism

$$\frac{z_{i'j'}}{z_{ij}} \longmapsto \frac{x_{i'}}{x_i} \frac{y_{j'}}{y_j}.$$

On overlaps these maps agree, since they are obtained by the same homogeneous rule $z_{i'j'} = x_{i'} y_{j'}$. Hence they glue to the Segre morphism.

Let I be the homogeneous ideal generated by the 2×2 minors

$$z_{ij} z_{i'j'} - z_{ij'} z_{i'j}.$$

The Segre morphism factors through $V_+(I)$. On the chart $D_+(z_{ij})$, the minor relations imply

$$\frac{z_{i'j'}}{z_{ij}} = \frac{z_{i'j}}{z_{ij}}\frac{z_{ij'}}{z_{ij}}.$$

Thus the coordinate ring of $V_+(I) \cap D_+(z_{ij})$ is freely generated over A by the m ratios $z_{i'j}/z_{ij}$ with $i' \neq i$ and the n ratios $z_{ij'}/z_{ij}$ with $j' \neq j$. It is therefore canonically the coordinate ring of

$$D_+(x_i) \times_A D_+(y_j).$$

These inverse affine identifications agree on overlaps. Consequently the Segre morphism identifies the product with the closed subscheme $V_+(I)$ and is a closed immersion. □

Proposition 3.57. *The composition of projective morphisms is projective.*

Proof. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be projective. Choose closed immersions

$$X \hookrightarrow \mathbb{P}^m_Y, \qquad Y \hookrightarrow \mathbb{P}^n_Z.$$

Using $\mathbb{P}^m_Y \cong \mathbb{P}^m_Z \times_Z Y$, the closed immersion $Y \hookrightarrow \mathbb{P}^n_Z$ base changes along the projection

$$\mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z \longrightarrow \mathbb{P}^n_Z$$

to give a closed immersion

$$\begin{tikzcd} \mathbb{P}^m_Y \arrow[r, hook] \arrow[d] & \mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z \arrow[d] \\ Y \arrow[r, hook] & \mathbb{P}^n_Z. \end{tikzcd}$$

Composing $X \hookrightarrow \mathbb{P}^m_Y$ with this map and then with the relative Segre embedding yields a closed immersion

$$X \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}_Z.$$

Thus $g \circ f$ is projective. □

The crucial theorem is that projective geometry is automatically proper.

Proposition 3.58. *The structural morphism*

$$\mathbb{P}^n_Y \longrightarrow Y$$

is proper.

Proof. By proposition 3.42(6), properness is local on the target, so take $Y = \operatorname{Spec} A$. The morphism is of finite type because the finitely many standard opens $D_+(T_i)$ are affine n-spaces of finite type over A. It is separated: the diagonal in

$$\mathbb{P}^n_A \times_A \mathbb{P}^n_A$$

is the closed subscheme defined by

$$X_iY_j - X_jY_i = 0 \qquad (0 \leq i, j \leq n).$$

Indeed, on a product of standard charts these equations say exactly that the affine coordinate ratios on the two factors agree, so they define the diagonal scheme-theoretically.

We verify the valuative criterion. Let R be a valuation ring with fraction field K and consider

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathbb{P}^n_A \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} A. \end{tikzcd}$$

The K-point is represented by homogeneous coordinates

$$[a_0 : \cdots : a_n], \qquad a_i \in K,$$

not all zero. The principal fractional ideals a_iR are totally ordered. Choose a_j such that a_jR contains all the a_iR. After scaling by a_j^{-1}, we obtain

$$[b_0 : \cdots : b_n], \qquad b_i \in R, \quad b_j = 1.$$

These coordinates define an R-point extending the given K-point. Thus the existence part of the valuative criterion holds. Since $\mathbb{P}^n_A \to \operatorname{Spec} A$ is quasi-compact, theorem 3.44 shows that it is universally closed. Together with finite type and separatedness proved above, this gives properness. (Uniqueness of the extension is also automatic from separatedness.) □

Theorem 3.59. *Every projective morphism is proper.*

Proof. A projective morphism factors as a closed immersion followed by $\mathbb{P}^n_Y \to Y$. Closed immersions are proper by proposition 3.42, and the projective-space projection is proper by proposition 3.58. Proper morphisms are stable under composition. □

Example 3.60. The structural morphism

$$\mathbb{A}^1_k \longrightarrow \operatorname{Spec} k$$

is not proper. Let

$$R = k[t]_{(t)}, \qquad K = k(t).$$

The K-point corresponding to $t^{-1} \in K$ does not extend to an R-point of $\mathbb{A}^1_k$, because an R-point is an element of R and $t^{-1} \notin R$. Thus the valuative criterion fails.

Projectivity is stronger than properness, but Chow's lemma shows that for Noetherian geometry the two notions are birationally close. We state the form most useful for the applications in these notes.

Theorem 3.61 (Chow's Lemma, proper integral form)**.** *Let S be a Noetherian scheme and let X be an integral scheme proper over S. Then there exist a projective S-scheme X' and a proper surjective morphism*

$$g : X' \longrightarrow X$$

for which there is a nonempty open subset $U \subseteq X$ such that

$$g^{-1}(U) \xrightarrow{\sim} U.$$

In particular, g is birational: the isomorphism over the nonempty open subset U identifies the generic local rings, so $K(X') \cong K(X)$ in the sense of definition 3.21.

Proof. Write $f : X \to S$ for the structure morphism. Because S is Noetherian, choose a finite affine cover $S = \bigcup_\alpha V_\alpha$. The morphism $X \to S$ is of finite type and hence quasi-compact, so each $f^{-1}(V_\alpha)$ admits a finite affine open cover. Enumerating all these affine opens gives finitely many nonempty affine opens

$$U_i = \operatorname{Spec} B_i \subseteq f^{-1}(V_i), \qquad V_i = \operatorname{Spec} A_i \subseteq S,$$

covering X. Since B_i is a finitely generated A_i-algebra, a choice of generators gives a closed immersion

$$U_i \hookrightarrow \mathbb{A}^{n_i}_{V_i}.$$

The standard open immersion $\mathbb{A}^{n_i}_{V_i} \hookrightarrow \mathbb{P}^{n_i}_{V_i}$ followed by the open immersion $\mathbb{P}^{n_i}_{V_i} \hookrightarrow \mathbb{P}^{n_i}_S$ realizes U_i as a locally closed subscheme of $\mathbb{P}^{n_i}_S$. Let P_i be its reduced closure there. Then P_i is projective over S, and U_i is an open dense subscheme of P_i.

Since X is integral, the intersection

$$U := U_1 \cap \cdots \cap U_r$$

is nonempty and dense. Put

$$P := P_1 \times_S \cdots \times_S P_r,$$

which is projective over S by repeated use of base-change stability and composition of projective morphisms. The open immersions $U \hookrightarrow U_i \hookrightarrow P_i$ define a morphism $U \to P$. Together with the inclusion $U \hookrightarrow X$, this gives the graph-like morphism

$$\varphi : U \longrightarrow X \times_S P.$$

Let X' be the reduced closure of $\varphi(U)$ in $X \times_S P$. Since U is integral, $\varphi(U)$ is irreducible; hence X' is integral. Moreover, $\varphi(U)$ is locally closed in $X \times_S P$ (it is the graph inside the open subscheme $U \times_S P$), so it is an open dense subscheme of its closure X'. We henceforth identify this open subscheme with U. Let

$$\begin{tikzcd} & X' \arrow[dl, "g"'] \arrow[d, hook] \arrow[dr, "h"] & \\ X & X \times_S P \arrow[l, "p_X"] \arrow[r, "p_P"'] & P \end{tikzcd}$$

be the induced projections.

We first show that $h : X' \to P$ is an immersion. Let $p_i : P \to P_i$ be the projections and put

$$W_i := p_i^{-1}(U_i) \subseteq P.$$

The open subsets $g^{-1}(U_i)$ cover X'. On $g^{-1}(U_i)$ there are two morphisms to P_i: the i-th component $p_i \circ h$, and the composite

$$g^{-1}(U_i) \xrightarrow{g} U_i \hookrightarrow P_i.$$

They agree on the dense open U. Since X' is reduced and P_i is separated over S, proposition 3.36 shows that they agree everywhere on $g^{-1}(U_i)$. Hence $g^{-1}(U_i) \subseteq h^{-1}(W_i)$.

Conversely, on $h^{-1}(W_i)$ the i-th projective coordinate gives a morphism to U_i and hence to X. This morphism and g agree on the dense open $U \cap h^{-1}(W_i)$. Since X is separated over S and $h^{-1}(W_i)$ is reduced, proposition 3.36 implies that they agree everywhere. Thus $h^{-1}(W_i) \subseteq g^{-1}(U_i)$, and therefore

$$g^{-1}(U_i) = h^{-1}(W_i).$$

Consider the cartesian scheme

$$\Gamma_i := U_i \times_{P_i} P.$$

The natural morphism

$$\Gamma_i \longrightarrow U_i \times_S P$$

is the base change of the diagonal $\Delta_{P_i/S}$ and is therefore a closed immersion. The projection $\Gamma_i \to P$ identifies Γ_i with the open subscheme W_i. The morphism $U \to U_i \times_S P$ factors through Γ_i. Since Γ_i is closed in $U_i \times_S P$, the closure of U inside $U_i \times_S P$, namely

$$X' \times_X U_i = g^{-1}(U_i),$$

is a closed subscheme of Γ_i. Therefore

$$h^{-1}(W_i) = g^{-1}(U_i) \longrightarrow W_i$$

is a closed immersion. These opens cover X', so h is an immersion.

Next, h is proper. Indeed, X' is closed in $X \times_S P$, and the projection

$$X \times_S P \longrightarrow P$$

is the base change of the proper morphism $X \to S$. Hence h is proper. A proper immersion is a closed immersion: an immersion identifies its source with a locally closed subscheme of the target, while properness makes its image closed; once the image is closed, the immersion is a closed immersion. Therefore

$$X' \hookrightarrow P$$

is closed. Since P is projective over S, X' is projective over S.

Finally, over U the morphism φ is the graph of the morphism $U \to P$. Since $P \to S$ is separated, this graph is closed in $U \times_S P$. Therefore

$$X' \cap (U \times_S P) = \varphi(U),$$

and the first projection induces an isomorphism

$$g^{-1}(U) \xrightarrow{\sim} U.$$

The morphism g is proper because X' is closed in $X \times_S P$ and the projection $X \times_S P \to X$ is projective, hence proper. Its image is therefore closed in X; since it contains the dense open subset U, it is all of X. Thus g is surjective. □

Remark 3.62. The standard Noetherian form of Chow's lemma is more general: for a separated finite-type morphism $X \to S$, one finds a proper surjective morphism $X' \to X$ with X' quasi-projective over S, and when X is integral the morphism may be chosen to be an isomorphism over a dense open subset. If $X \to S$ is proper, the quasi-projective model becomes projective, which is the form proved above.

We finish by translating the relative language back into the classical geometry of varieties. A *variety over k* means an integral separated scheme of finite type over the field k, and a *curve over k* is a variety of dimension one.

Corollary 3.63 (Projective varieties are proper)**.** *Every projective variety over a field is proper over that field.*

Proof. A projective variety is a closed subscheme of some $\mathbb{P}^n_k$, so the result follows from theorem 3.59. $\square$

Remark 3.64. Over an algebraically closed field, this is the scheme-theoretic form of the classical statement that projective varieties are complete. Properness is the relative notion that preserves this "no points escape under specialization" behaviour after every extension of the base.

Proposition 3.65 (Morphisms out of proper schemes)**.** *Let X be proper over S, let Y be separated over S, and let $f : X \to Y$ be an S-morphism. Then f is proper. In particular, the image $f(X)$ is closed in Y.*

Proof. By the graph criterion,

$$\Gamma_f : X \hookrightarrow X \times_S Y$$

is a closed immersion. The projection $X \times_S Y \to Y$ is a base change of $X \to S$ and is proper. Thus f is a composition of proper morphisms and is proper. Proper morphisms are closed. $\square$

Corollary 3.66 (Regular functions on proper varieties)**.** *Let k be algebraically closed and let X be a proper variety over k. Then*

$$\Gamma(X, \mathscr{O}_X) = k.$$

Proof. A global regular function $a \in \Gamma(X, \mathscr{O}_X)$ determines a morphism

$$\varphi_a : X \longrightarrow \mathbb{A}^1_k.$$

By proposition 3.65, this morphism is proper, so its image is closed. The image is irreducible because X is irreducible. Hence it is either a point or all of $\mathbb{A}^1_k$.

If the image were all of $\mathbb{A}^1_k$, then proposition 3.43 would imply that $\mathbb{A}^1_k$ is proper over k, contradicting the valuative example above. Thus the image is a closed point c. Since k is algebraically closed, c is k-rational. The section $a - c$ vanishes at every point of X; equivalently, each of its germs belongs to every prime ideal of the corresponding local ring and is therefore nilpotent. A variety is reduced, so all these germs are zero. Hence $a = c$ in $\Gamma(X, \mathscr{O}_X)$. $\square$

Corollary 3.67 (Projective birational models)**.** *Every proper variety over a field admits a projective variety mapping properly and birationally to it.*

Proof. Apply theorem 3.61 with $S = \operatorname{Spec} k$. In the construction above X' is the reduced closure of an integral dense open subset, hence is itself integral; therefore it is a projective variety and the morphism $X' \to X$ is proper and birational. $\square$

The chapter has therefore produced a sequence of increasingly geometric interpretations of a morphism. Fiber products make base change possible, fibres isolate the geometry over individual points, constructibility controls images, dimension theory describes generic behaviour, valuation rings encode specialization, properness packages existence and uniqueness of specialization, and projective geometry provides the principal source of proper morphisms. Chow's lemma then explains why proper geometry can often be approached through projective models.

Chapter 4

Local Properties of Schemes

The preceding chapters developed schemes globally and relatively: affine schemes were glued into arbitrary schemes, morphisms were studied through fibres and base change, and proper and projective morphisms supplied the first global finiteness conditions. We now turn in the opposite direction and examine what a scheme looks like near a single point.

The basic object is the local ring

$$\mathscr{O}_{X,x}.$$

Its dimension records the codimension of the point, its maximal ideal records first-order local equations, and its factorization properties govern how rational functions vanish along codimension-one subsets. These ideas lead naturally from regularity and singularities to normality and normalization, and then to discrete valuation rings. In dimension one the resulting geometry is especially rigid: normal Noetherian integral schemes are Dedekind schemes, and the basic arithmetic examples are $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathscr{O}_K$ for a number field K.

The chapter deliberately stops short of smoothness. Regularity is an intrinsic condition on a local ring, whereas smoothness is a relative condition that depends on a morphism and is stable under arbitrary base change. Their precise relationship will be treated only after differentials and flatness have been developed.

4.1 Regularity and Singularities

Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. By Nakayama's lemma, the vector space

$$\mathfrak{m}/\mathfrak{m}^2$$

measures the minimal number of generators of the maximal ideal. Thus it records the number of independent first-order local parameters that are visible at the closed point of $\operatorname{Spec} A$.

Definition 4.1. The *embedding dimension* of $(A, \mathfrak{m}, \kappa)$ is

$$\operatorname{edim}(A) := \dim_\kappa \mathfrak{m}/\mathfrak{m}^2.$$

The local ring A is *regular* if

$$\dim A = \operatorname{edim}(A).$$

A locally Noetherian scheme X is *regular at $x \in X$* if $\mathscr{O}_{X,x}$ is a regular local ring. It is *regular* if it is regular at every point. A point at which X is not regular will be called a *singular point*.

The inequality behind this definition is a direct consequence of the height theorem.

Proposition 4.2. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then*

$$\dim A \leq \operatorname{edim}(A).$$

Proof. Let

$$r = \dim_\kappa \mathfrak{m}/\mathfrak{m}^2.$$

Choose elements $x_1, \ldots, x_r \in \mathfrak{m}$ whose classes form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Nakayama's lemma gives

$$\mathfrak{m} = (x_1, \ldots, x_r).$$

Krull's height theorem therefore gives

$$\operatorname{ht}(\mathfrak{m}) \leq r.$$

Since A is local,

$$\dim A = \operatorname{ht}(\mathfrak{m}),$$

and the result follows. □

The equality case says that the maximal ideal can be generated by exactly as many elements as the dimension allows. This is the local algebraic analogue of having a coordinate system with the expected number of parameters.

Example 4.3 (Affine space at the origin)**.** Let

$$A = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}.$$

Its maximal ideal is generated by $x_1, \ldots, x_n$, and their classes form a basis of

$$\mathfrak{m}/\mathfrak{m}^2.$$

Hence $\operatorname{edim}(A) = n$. Since $\dim A = n$, the local ring is regular.

The next examples show that singularities can already be detected directly from the maximal ideal, without using differentials.

Example 4.4 (A node)**.** Let

$$X = \operatorname{Spec} k[x, y]/(xy)$$

and let o be the origin. The local ring $A = \mathscr{O}_{X,o}$ has dimension one, while its maximal ideal is generated by the classes of x and y. The relation $xy = 0$ lies in $\mathfrak{m}^2$, so it produces no linear relation in $\mathfrak{m}/\mathfrak{m}^2$. Therefore

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2 > 1 = \dim A,$$

and the origin is singular.

Example 4.5 (A cusp)**.** Let

$$X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$$

and let o be the origin. Again $\dim \mathscr{O}_{X,o} = 1$. The equation $y^2 - x^3$ lies in $(x, y)^2$, so the classes of x and y remain linearly independent in $\mathfrak{m}_o/\mathfrak{m}_o^2$. Thus

$$\operatorname{edim}(\mathscr{O}_{X,o}) = 2,$$

and the cusp is singular at the origin.

For one-dimensional local domains, regularity is already equivalent to the valuation-theoretic structure that will dominate the rest of this chapter.

Lemma 4.6 (A local characterization of DVRs)**.** *Let $(A, \mathfrak{m})$ be a one-dimensional Noetherian local domain which is not a field. Then the following are equivalent:*

1. *A is a discrete valuation ring;*
2. *$\mathfrak{m}$ is principal.*

Proof. If A is a discrete valuation ring with uniformizer π, then by definition

$$\mathfrak{m} = (\pi).$$

Conversely, suppose $\mathfrak{m} = (\pi)$. We use the Krull intersection theorem in the form

$$\bigcap_{n \geq 0} \mathfrak{m}^n = 0$$

for the Noetherian local domain A. Hence every nonzero $a \in A$ lies in

$$\mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$$

for a unique integer $n \geq 0$. Since $\mathfrak{m}^n = (\pi^n)$, we may write

$$a = \pi^n u$$

with $u \in A$. The condition $a \notin \mathfrak{m}^{n+1}$ implies $u \notin \mathfrak{m}$, hence u is a unit.

Now let $I \subseteq A$ be a nonzero ideal. Choose an element $a \in I$ for which the exponent n in the expression $a = u\pi^n$ is minimal. Every $b \in I$ has the form $v\pi^m$ with $m \geq n$, hence $b \in (\pi^n) = (a)$. Thus $I = (a)$. Therefore A is a local principal ideal domain with a unique nonzero prime ideal, equivalently a discrete valuation ring. □

Corollary 4.7. *Let $(A, \mathfrak{m})$ be a one-dimensional Noetherian local domain. Then A is regular if and only if it is a discrete valuation ring.*

Proof. If A is regular, then

$$\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim A = 1.$$

Nakayama's lemma shows that $\mathfrak{m}$ is generated by one element, and lemma 4.6 gives that A is a DVR.

Conversely, if A is a DVR, its maximal ideal is principal. Hence

$$\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = 1 = \dim A,$$

so A is regular. □

We will also use the following standard theorem from commutative algebra.

Theorem 4.8 (Regular local rings are normal)**.** *Every Noetherian regular local ring is an integrally closed domain.*

Proof. This is a commutative-algebra input. One standard route proves the stronger theorem that every regular local ring is a unique factorization domain. A UFD is an integrally closed domain, which gives the assertion. We will use the theorem only through this consequence; the divisor chapter will later explain geometrically why factoriality is especially useful in codimension one. □

Corollary 4.9. *Every regular locally Noetherian scheme is normal.*

Proof. At every point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is regular and hence integrally closed by theorem 4.8. The theorem also says that it is a domain. Therefore X is normal at every point. □

The word "regular" is intrinsic: it depends only on the local rings of the scheme. This should not be confused with smoothness over a field. Over a perfect field the two notions agree for finite-type schemes at closed points, but over an imperfect field a regular local ring can fail to remain regular after extending the ground field. Smoothness will therefore require both the relative language of Chapter 3 and additional infinitesimal and flatness conditions.

4.2 Normality and Normalization

Regularity is a strong condition. For birational geometry and divisor theory, the weaker condition of normality is often exactly what is needed: it guarantees that rational functions which are locally integral do not create hidden branches inside the function field.

Definition 4.10. A scheme X is *normal at* $x \in X$ if the local ring $\mathscr{O}_{X,x}$ is an integrally closed domain. The scheme is *normal* if it is normal at every point.

For integral schemes, normality can be checked on affine opens.

Lemma 4.11. *Let A be an integral domain with fraction field K. The following are equivalent:*

1. *A is integrally closed in K;*
2. *$A_{\mathfrak{p}}$ is integrally closed for every prime ideal $\mathfrak{p} \subseteq A$;*
3. *$A_{\mathfrak{m}}$ is integrally closed for every maximal ideal $\mathfrak{m} \subseteq A$.*

Proof. If A is integrally closed, then every localization of A is integrally closed; the localization argument is proved explicitly in lemma 4.13. Thus (1) implies (2). The implication (2) implies (3) is immediate.

Assume (3), and let $z \in K$ be integral over A. Suppose that $z \notin A$. Set

$$I := \{a \in A \mid az \in A\}.$$

Then I is a proper ideal, so $I \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Since the same monic polynomial over A is also a monic polynomial over $A_{\mathfrak{m}}$, the element z is integral over $A_{\mathfrak{m}}$. By (3),

$$z \in A_{\mathfrak{m}}.$$

Thus there exists $s \notin \mathfrak{m}$ with $sz \in A$. Then $s \in I \subseteq \mathfrak{m}$, a contradiction. Hence $z \in A$, proving (1). □

Proposition 4.12 (Affine-local criterion for normality)**.** *Let X be an integral scheme. The following are equivalent:*

1. *X is normal;*
2. *every nonempty affine open $U = \operatorname{Spec} A \subseteq X$ has A integrally closed;*
3. *for every nonempty open subset $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is an integrally closed domain.*

Proof. Assume (1), and let $U = \operatorname{Spec} A$ be a nonempty affine open. For every $\mathfrak{p} \in \operatorname{Spec} A$,

$$A_{\mathfrak{p}} \cong \mathscr{O}_{X,\mathfrak{p}}$$

is integrally closed. By lemma 4.11, A is integrally closed. Thus (1) implies (2).

Assume (2). Every point $x \in X$ has a nonempty affine neighbourhood $U = \operatorname{Spec} A$. The local ring $\mathscr{O}_{X,x}$ is a localization of A, hence is integrally closed. Since X is integral, it is also a domain. Therefore (2) implies (1).

Assume (2), and let $U \subseteq X$ be nonempty. By proposition 2.51, all rings $\mathscr{O}_X(V)$ for nonempty open V embed into the common function field $K(X)$, and

$$\mathscr{O}_X(U) = \bigcap_{x \in U} \mathscr{O}_{X,x} \subseteq K(X).$$

Let $z \in K(X)$ be integral over $\mathscr{O}_X(U)$. For any affine open $V \subseteq U$, the same monic equation shows that z is integral over $\mathscr{O}_X(V)$. By (2), $z \in \mathscr{O}_X(V)$. Hence z is regular at every point of U, so proposition 2.51 gives $z \in \mathscr{O}_X(U)$. Thus (2) implies (3). The implication (3) implies (2) is immediate. □

If an integral scheme is not normal, there is a canonical way to enlarge its local rings inside its function field until they become integrally closed. We will use one additional piece of terminology. A morphism $f : Y \to X$ is called *integral* if, for every affine open $U = \operatorname{Spec} A \subseteq X$, the inverse image $f^{-1}(U)$ is affine, say $\operatorname{Spec} B$, and B is integral over A. This condition is affine-local on the target; every finite morphism is integral.

Lemma 4.13 (Integral closure and localization)**.** *Let A be a domain with fraction field K, let $\overline{A}$ be the integral closure of A in K, and let $S \subseteq A$ be multiplicatively closed. Then the integral closure of $S^{-1}A$ in K is $S^{-1}\overline{A}$.*

Proof. Every element of $\overline{A}$ is integral over A, so its image after localization is integral over $S^{-1}A$. Hence

$$S^{-1}\overline{A}$$

is contained in the integral closure of $S^{-1}A$.

Conversely, let $z \in K$ be integral over $S^{-1}A$. Write

$$z^n + \frac{b_1}{s_1} z^{n-1} + \cdots + \frac{b_n}{s_n} = 0, \qquad b_i \in A,\ s_i \in S.$$

Put $s = s_1 \cdots s_n$. Multiplying the equation by s^n gives

$$(sz)^n + d_1 (sz)^{n-1} + \cdots + d_n = 0,$$

where

$$d_i = b_i \frac{s^i}{s_i} \in A.$$

Thus sz is integral over A, so $sz \in \overline{A}$. Since $s \in S$, we conclude

$$z \in S^{-1}\overline{A}.$$

This proves the reverse inclusion. □

Theorem 4.14 (Normalization)**.** *Let X be an integral scheme with function field $K(X)$. There exists a normal integral scheme X^ν and an integral dominant morphism*

$$\nu : X^\nu \longrightarrow X$$

with the following properties.

1. *For every nonempty affine open* $U = \operatorname{Spec} A \subseteq X$*,*

$$\nu^{-1}(U) \cong \operatorname{Spec} \overline{A},$$

where $\overline{A}$ *is the integral closure of* A *in* $K(X)$*.*

2. *The induced map on function fields is an isomorphism*

$$K(X) \xrightarrow{\sim} K(X^\nu).$$

3. *If* Z *is a normal integral scheme and* $f : Z \to X$ *is dominant, then there exists a unique morphism* $\widetilde{f} : Z \to X^\nu$ *such that*

$$\begin{tikzcd} Z \arrow[rr, "\widetilde{f}"] \arrow[dr, "f"'] & & X^\nu \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

commutes.

The scheme X^ν *is called the* normalization *of* X*.*

Proof. Choose an affine open cover $X = \bigcup_i U_i$ with $U_i = \operatorname{Spec} A_i$. Let $\overline{A_i}$ be the integral closure of A_i in the common function field $K(X)$, and set

$$U_i^\nu := \operatorname{Spec} \overline{A_i}.$$

If $D(f) \subseteq U_i$ is distinguished, lemma 4.13 gives

$$\overline{(A_i)_f} = (\overline{A_i})_f.$$

Thus normalization is compatible with passing to distinguished open subsets. By the common distinguished-refinement result proposition 2.25, the overlaps $U_i \cap U_j$ are covered by opens which are distinguished in both affine charts. On each such open the two normalizations are canonically the same spectrum, and these canonical identifications satisfy the cocycle condition. The gluing theorem theorem 2.22 therefore produces a scheme X^ν together with a morphism

$$\nu : X^\nu \to X.$$

On every affine chart the ring map is $A_i \hookrightarrow \overline{A_i}$, which is integral. Hence ν is integral. The generic prime $(0) \subset \overline{A_i}$ contracts to $(0) \subset A_i$, so ν sends the generic point of X^ν to the generic point of X and is therefore dominant. Each $\overline{A_i}$ is a domain integrally closed in $K(X)$, so X^ν is normal and integral.

For (2), every $\overline{A_i}$ is contained in $K(X)$ and has fraction field $K(X)$: it contains A_i, whose fraction field is $K(X)$, while it is itself a subring of that field. Hence the generic local ring of X^ν is $K(X)$.

For the universal property, let $f : Z \to X$ be dominant with Z normal and integral. It is enough to construct the factorization locally over an affine open $U = \operatorname{Spec} A \subseteq X$. Let $V = \operatorname{Spec} B \subseteq f^{-1}(U)$ be a nonempty affine open. Since Z is integral, every nonempty open contains its generic point η_Z. Dominance sends η_Z to the generic point of X, hence the induced ring map

$$A \longrightarrow B$$

has zero kernel and is injective. It therefore extends to an embedding

$$K(X) = \operatorname{Frac}(A) \hookrightarrow K(Z) = \operatorname{Frac}(B).$$

If $a \in \overline{A}$, then a satisfies a monic equation with coefficients in $A \subseteq B$. Its image in $K(Z)$ is therefore integral over B. Since B is integrally closed by proposition 4.12, this image lies in B. We obtain a unique ring homomorphism

$$\overline{A} \longrightarrow B$$

extending $A \to B$, hence a morphism $V \to \operatorname{Spec} \overline{A}$. On overlaps these local morphisms agree because they induce the same embedding of $K(X)$ into $K(Z)$. They glue to a morphism $\widetilde{f} : Z \to X^\nu$. Uniqueness is checked on the same affine charts: any factorization must induce the unique extension $\overline{A} \to B$ determined inside $K(Z)$. □

The normalization is therefore birational in the sense of definition 3.21: it is dominant and induces the identity isomorphism on the common function field. In the finite-type situation it is usually also finite.

Theorem 4.15 (Finiteness of normalization over a field)**.** *Let X be an integral scheme of finite type over a field k. Then the normalization morphism*

$$\nu : X^\nu \to X$$

is finite.

Proof. This is the geometric form of a standard finiteness theorem in commutative algebra: if A is a finitely generated domain over a field, then its integral closure in $\operatorname{Frac}(A)$ is a finite A-module. Applying this theorem on every affine open $U = \operatorname{Spec} A \subseteq X$ gives

$$\nu^{-1}(U) = \operatorname{Spec} \overline{A}$$

with $\overline{A}$ finite over A. By the affine-local definition of a finite morphism from Chapter 3, ν is finite. □

Proposition 4.16 (Finite birational morphisms to a normal scheme)**.** *Let $f : Y \to X$ be a finite birational morphism of integral schemes, and suppose that X is normal. Then f is an isomorphism.*

Proof. The assertion is local on X. Let $U = \operatorname{Spec} A \subseteq X$ be a nonempty affine open. Since f is finite,

$$f^{-1}(U) = \operatorname{Spec} B$$

for a finite A-algebra B. Birationality makes $A \to B$ injective and identifies the two function fields, so we may regard

$$A \subseteq B \subseteq K(X).$$

Every element of B is integral over A because B is finite as an A-module. Since A is integrally closed by normality, $B = A$. Therefore $f^{-1}(U) \to U$ is an isomorphism for every affine open U, and these local isomorphisms glue to give $Y \cong X$. □

The cusp gives the fundamental example of normalization separating a singular local algebra from a regular parameter.

Example 4.17 (Normalization of the cusp)**.** Let

$$A = k[x, y]/(y^2 - x^3).$$

The homomorphism
$$A \longrightarrow k[t], \qquad x \longmapsto t^2, \qquad y \longmapsto t^3,$$
identifies A with the subring $k[t^2, t^3] \subseteq k[t]$. Both rings have fraction field $k(t)$. The element t is integral over A, since it satisfies the monic equation
$$T^2 - x = 0.$$
Thus $k[t]$ is integral over A.

Since $k[t]$ is a PID, it is integrally closed. If $z \in k(t)$ is integral over A, then the same monic equation also shows that z is integral over $k[t]$, because its coefficients lie in $A \subseteq k[t]$. Hence $z \in k[t]$. Therefore $k[t]$ is exactly the integral closure of A in $k(t)$.

The normalization is the finite birational morphism
$$\mathbb{A}^1_k = \operatorname{Spec} k[t] \xrightarrow{\ \nu\ } \operatorname{Spec} k[x,y]/(y^2 - x^3), \qquad t \longmapsto (t^2, t^3).$$
The source is regular, while the target is singular at the origin by example 4.5.

Normalization repairs failure of integral closedness, but it does not in general remove every singularity in higher dimension. What makes dimension one special is that normality and regularity coincide. The reason is the codimension-one valuation theory developed next.

4.3 Codimension One, Valuations, and Dedekind Schemes

Let X be an integral locally Noetherian scheme. We write
$$X^{(1)} := \{x \in X \mid \operatorname{codim}_X \overline{\{x\}} = 1\}$$
for the set of codimension-one points. By the local-dimension discussion of Chapter 2, these are precisely the points at which the local ring has dimension one.

Definition 4.18 (Serre's condition (R_1))**.** A locally Noetherian scheme X satisfies *Serre's condition* (R_1) if $\mathscr{O}_{X,x}$ is regular for every point x of codimension at most one.

For an integral scheme the local ring at the generic point is a field and is therefore regular. Thus (R_1) amounts to regularity at codimension-one points.

Proposition 4.19 (Codimension-one local rings)**.** *Let X be an integral locally Noetherian scheme and let $x \in X^{(1)}$. Then*
$$\dim \mathscr{O}_{X,x} = 1, \qquad \operatorname{Frac}(\mathscr{O}_{X,x}) = K(X).$$
If X satisfies (R_1), then $\mathscr{O}_{X,x}$ is a discrete valuation ring.

Proof. Choose an affine neighbourhood $U = \operatorname{Spec} A$ of x, with $x \leftrightarrow \mathfrak{p}$. By proposition 2.55, codimension one means
$$\operatorname{ht}(\mathfrak{p}) = 1.$$
Hence
$$\dim \mathscr{O}_{X,x} = \dim A_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p}) = 1.$$
Since A is a domain,
$$\operatorname{Frac}(A_{\mathfrak{p}}) = \operatorname{Frac}(A) = K(X),$$
the last equality coming from proposition 2.51. If X satisfies (R_1), then $A_{\mathfrak{p}}$ is a one-dimensional regular local domain, hence a DVR by corollary 4.7. □

Thus every codimension-one point of a normal Noetherian integral scheme will carry a discrete valuation. To explain why normality guarantees the required regularity in codimension one, we recall Serre's second condition.

Definition 4.20 (Depth and Serre's condition (S_2))**.** Let $(A, \mathfrak{m})$ be a Noetherian local ring. The *depth* of A, denoted $\operatorname{depth} A$, is the maximal length of an A-regular sequence contained in $\mathfrak{m}$.

A locally Noetherian scheme X satisfies *Serre's condition* (S_2) if

$$\operatorname{depth} \mathscr{O}_{X,x} \geq \min\{2, \dim \mathscr{O}_{X,x}\}$$

for every $x \in X$.

The following theorem is one of the central local criteria for normality.

Theorem 4.21 (Serre's criterion for normality)**.** *Let X be a locally Noetherian integral scheme. Then X is normal if and only if it satisfies (R_1) and (S_2).*

Proof. We use the standard commutative-algebra form of Serre's criterion: a Noetherian ring A is normal if and only if every localization $A_\mathfrak{p}$ at a prime of height at most one is regular and

$$\operatorname{depth} A_\mathfrak{p} \geq \min\{2, \dim A_\mathfrak{p}\}$$

for every prime $\mathfrak{p}$.

The assertion for schemes is local. On an affine open $U = \operatorname{Spec} A$, condition (R_1) for U is exactly the first condition in the ring-theoretic theorem, because

$$\mathscr{O}_{U,\mathfrak{p}} = A_\mathfrak{p}$$

and codimension is height. Condition (S_2) is exactly the displayed depth inequality for the same localizations. The affine-local criterion proposition 4.12 identifies normality of U with normality of A. Applying the ring-theoretic theorem on each affine open proves the result. □

Corollary 4.22. *Let X be a normal Noetherian integral scheme. For every $x \in X^{(1)}$, the local ring $\mathscr{O}_{X,x}$ is a discrete valuation ring.*

Proof. By theorem 4.21, a normal Noetherian scheme satisfies (R_1). The result now follows from proposition 4.19. □

Definition 4.23 (The valuation at a codimension-one point)**.** Let X be a normal Noetherian integral scheme and let $x \in X^{(1)}$. The DVR $\mathscr{O}_{X,x} \subseteq K(X)$ determines a normalized discrete valuation

$$v_x : K(X)^\times \longrightarrow \mathbb{Z}.$$

If π_x is a uniformizer of $\mathscr{O}_{X,x}$, then every $f \in K(X)^\times$ can be written uniquely as

$$f = u\pi_x^{v_x(f)}$$

with $u \in \mathscr{O}_{X,x}^\times$.

This is the local mechanism behind orders of zero and pole. The divisor theory developed later will package the integers $v_x(f)$ simultaneously over all codimension-one points.

Dimension one is the case in which every non-generic point is already codimension one.

Definition 4.24 (Dedekind scheme)**.** In these notes, a *Dedekind scheme* is an integral Noetherian normal scheme of dimension one.

Some authors allow dimension zero components as well. Our convention is chosen so that the basic examples $\operatorname{Spec}\mathbb{Z}$ and $\operatorname{Spec}\mathscr{O}_K$ fit the definition directly and the generic point is separated from the closed arithmetic points.

Proposition 4.25 (Local structure of a Dedekind scheme)**.** *Let X be an integral Noetherian scheme of dimension one. The following are equivalent:*

1. *X is a Dedekind scheme;*
2. *every local ring at a closed point is a discrete valuation ring;*
3. *X is regular.*

Moreover, every point other than the generic point is closed.

Proof. Let η be the generic point. If $x \neq \eta$ were not closed, there would exist a strict specialization $y \in \overline{\{x\}} \setminus \{x\}$. Then the irreducible closed subsets

$$\overline{\{y\}} \subsetneq \overline{\{x\}} \subsetneq X$$

would form a chain of length two, contradicting $\dim X = 1$. Hence every non-generic point is closed. Equivalently, on an affine chart $\operatorname{Spec} A$, every nonzero prime has height one and is therefore maximal.

Assume (1). At a closed point x, normality makes $\mathscr{O}_{X,x}$ a one-dimensional Noetherian integrally closed local domain. Equivalently, by corollary 4.22, it is a DVR. Thus (1) implies (2).

Assume (2). Every closed-point local ring is regular by corollary 4.7, while

$$\mathscr{O}_{X,\eta} = K(X)$$

is a field and hence a zero-dimensional regular local ring. Thus X is regular, proving (2) implies (3).

Finally, (3) implies that X is normal by corollary 4.9. Together with the standing hypotheses that X is integral, Noetherian, and one-dimensional, this says that X is Dedekind. Hence (3) implies (1). □

Example 4.26 (The arithmetic curve $\operatorname{Spec}\mathbb{Z}$)**.** The ring $\mathbb{Z}$ is Noetherian, integrally closed, and of Krull dimension one. Hence

$$\operatorname{Spec}\mathbb{Z}$$

is a Dedekind scheme. Its generic point is (0), with residue field

$$\kappa((0)) = \mathbb{Q}.$$

Its closed points are the maximal ideals

$$(p),$$

one for each rational prime p. At such a point,

$$\mathscr{O}_{\operatorname{Spec}\mathbb{Z},(p)} = \mathbb{Z}_{(p)}$$

is a DVR with uniformizer p, and the associated valuation on $\mathbb{Q}^\times$ is the usual p-adic order

$$v_p : \mathbb{Q}^\times \to \mathbb{Z}.$$

Thus prime numbers literally occur as codimension-one points of the arithmetic curve $\operatorname{Spec}\mathbb{Z}$.

The same picture persists for number fields. If $K/\mathbb{Q}$ is finite, the ring of integers $\mathscr{O}_K$ is the integral closure of $\mathbb{Z}$ in K, and it is a finite $\mathbb{Z}$-module. Thus the finite morphism

$$\operatorname{Spec} \mathscr{O}_K \longrightarrow \operatorname{Spec} \mathbb{Z}$$

may be viewed as the normalization of the arithmetic curve $\operatorname{Spec} \mathbb{Z}$ in the finite extension $K/\mathbb{Q}$. This is the natural variant of theorem 4.14 in which one normalizes inside an algebraic extension of the function field rather than inside the function field itself.

Example 4.27 (Rings of integers)**.** Let K be a number field and let $\mathscr{O}_K$ be its ring of integers. A standard theorem of algebraic number theory says that $\mathscr{O}_K$ is a Dedekind domain. Therefore

$$X = \operatorname{Spec} \mathscr{O}_K$$

is a Dedekind scheme. Its generic point has function field K, while its closed points are the nonzero prime ideals

$$\mathfrak{p} \subset \mathscr{O}_K.$$

The local ring

$$(\mathscr{O}_K)_{\mathfrak{p}}$$

is a DVR, and its residue field

$$\kappa(\mathfrak{p}) = \mathscr{O}_K/\mathfrak{p}$$

is finite.

The morphism induced by the inclusion $\mathbb{Z} \hookrightarrow \mathscr{O}_K$

$$\pi : \operatorname{Spec} \mathscr{O}_K \longrightarrow \operatorname{Spec} \mathbb{Z}$$

allows the factorization of rational primes to be read geometrically from scheme-theoretic fibres.

Proposition 4.28 (Arithmetic fibres and prime factorization)**.** *Let K be a number field and let p be a rational prime. Suppose*

$$p\mathscr{O}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$$

is the factorization into distinct nonzero prime ideals. Then the fibre of

$$\pi : \operatorname{Spec} \mathscr{O}_K \to \operatorname{Spec} \mathbb{Z}$$

over the point (p) is

$$X_p \cong \operatorname{Spec}(\mathscr{O}_K/p\mathscr{O}_K) \cong \coprod_{i=1}^{g} \operatorname{Spec}(\mathscr{O}_K/\mathfrak{p}_i^{e_i}).$$

Its underlying set consists of the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_g$ above p. The residue field at $\mathfrak{p}_i$ is

$$\kappa(\mathfrak{p}_i) = \mathscr{O}_K/\mathfrak{p}_i,$$

and the fibre is reduced if and only if every $e_i = 1$.

Proof. By the definition of a fibre from Chapter 3 and the affine fiber-product formula theorem 3.3,

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] & \operatorname{Spec} \mathscr{O}_K \arrow[d, "\pi"] \\ \operatorname{Spec} \mathbb{F}_p \arrow[r] & \operatorname{Spec} \mathbb{Z} \end{tikzcd}$$

is cartesian and

$$X_p \cong \operatorname{Spec}(\mathscr{O}_K \otimes_{\mathbb{Z}} \mathbb{F}_p) \cong \operatorname{Spec}(\mathscr{O}_K/p\mathscr{O}_K).$$

The ideals $\mathfrak{p}_i^{e_i}$ are pairwise comaximal because the $\mathfrak{p}_i$ are distinct maximal ideals of the Dedekind domain $\mathscr{O}_K$. The Chinese remainder theorem therefore gives

$$\mathscr{O}_K/p\mathscr{O}_K \cong \prod_{i=1}^{g} \mathscr{O}_K/\mathfrak{p}_i^{e_i}.$$

Taking spectra converts a finite product of rings into a disjoint union, proving the displayed decomposition of X_p.

Each factor $\mathscr{O}_K/\mathfrak{p}_i^{e_i}$ has a unique prime ideal, namely the image of $\mathfrak{p}_i$, so the underlying points of the fibre are exactly the primes above p. The corresponding residue field is

$$(\mathscr{O}_K/\mathfrak{p}_i^{e_i})/(\mathfrak{p}_i/\mathfrak{p}_i^{e_i}) \cong \mathscr{O}_K/\mathfrak{p}_i.$$

If $e_i = 1$, the factor is a field and therefore reduced. Suppose $e_i > 1$. The powers of $\mathfrak{p}_i$ are strictly decreasing after localizing at $\mathfrak{p}_i$, so we may choose

$$a \in \mathfrak{p}_i^{e_i-1} \setminus \mathfrak{p}_i^{e_i}.$$

Its class in $\mathscr{O}_K/\mathfrak{p}_i^{e_i}$ is nonzero, while

$$a^2 \in \mathfrak{p}_i^{2e_i-2} \subseteq \mathfrak{p}_i^{e_i}.$$

Thus this class is nilpotent, and the factor is nonreduced. Hence the whole fibre is reduced exactly when all $e_i = 1$. □

Remark 4.29 (Ramification as nonreduced fibre). The preceding proposition is an elementary scheme-theoretic shadow of ramification. A prime with $e_i > 1$ contributes a nilpotent thickening at the corresponding point of the fibre. Thus the scheme structure remembers information that would disappear if one retained only the set

$$\pi^{-1}((p)) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_g\}.$$

The residue degree

$$f_i = [\kappa(\mathfrak{p}_i) : \mathbb{F}_p]$$

records another part of the arithmetic of the fibre.

Remark 4.30 (Preview of divisors and ideal classes). On a Dedekind scheme every closed point has codimension one. Consequently, the free abelian group on closed points

$$\bigoplus_{x \in X_{\mathrm{cl}}} \mathbb{Z}[x]$$

is exactly the group that will later become the Weil divisor group. For

$$X = \operatorname{Spec} \mathscr{O}_K,$$

the unique factorization of nonzero fractional ideals

$$I = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$$

identifies the group of nonzero fractional ideals with this free abelian group. Principal fractional ideals correspond to principal divisors through the valuations $v_{\mathfrak{p}}$. The resulting quotient is the ideal class group. We postpone the formal divisor language until the divisor chapter, but the local algebra needed for it is already in place.

The three themes of this chapter now meet. Regular local rings provide the intrinsic notion of nonsingularity; normalization repairs integral closedness without changing the function field; and normality in codimension one produces DVRs and valuations. In dimension one these structures become Dedekind schemes, where geometry and arithmetic share the same local language. We are now ready to return to sheaves, this time not merely as sheaves of sets or rings, but as modules over the structure sheaf. Quasi-coherent and coherent sheaves will turn local algebraic modules into global geometric objects.

Chapter 5

Quasi-Coherent and Coherent Sheaves

The preceding chapters treated schemes first as spaces carrying local rings and then as objects related by morphisms, fibres, properness, projective embeddings, and local algebra. We now add the linear objects that live on a scheme. On an affine scheme $X = \operatorname{Spec} A$, an A-module should become a sheaf of $\mathscr{O}_X$-modules, and the basic principle of this chapter is that this passage loses essentially no information when one restricts to quasi-coherent sheaves.

This affine dictionary has several consequences. It makes pullback a geometric form of extension of scalars, turns finite modules into coherent sheaves, and allows graded modules to be placed on projective schemes. Twisting by $\mathscr{O}_X(1)$ then leads to Serre's global-generation theorem and to the useful fact that coherent sheaves on projective schemes are quotients of finite locally free sheaves. Finally, symmetric algebras and relative spectra turn modules back into geometric objects; vector bundles are the resulting geometric incarnation of locally free sheaves.

5.1 Modules on Schemes and the Affine Dictionary

Let $(X, \mathscr{O}_X)$ be a ringed space. The structure sheaf is not merely a sheaf of rings: it is the ring over which all linear constructions on X are performed.

Definition 5.1. A *sheaf of $\mathscr{O}_X$-modules*, or simply an $\mathscr{O}_X$-module, is a sheaf $\mathscr{F}$ of abelian groups such that every $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module and the restriction maps are compatible with scalar multiplication. Morphisms are morphisms of sheaves which are $\mathscr{O}_X$-linear on every open set. We write $\operatorname{Mod}(X)$ for the resulting category.

Kernels and cokernels are computed as for sheaves of abelian groups, with the induced module structures. Exactness may therefore be checked on stalks. The first genuinely new operation is the tensor product.

Definition 5.2. For $\mathscr{F}, \mathscr{G} \in \operatorname{Mod}(X)$, the tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is the sheafification of the presheaf

$$U \longmapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U).$$

Proposition 5.3 (Tensor products and stalks)**.** *For every $x \in X$, there is a natural isomorphism*

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x \cong \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x.$$

Proof. Sheafification does not change stalks. The germ of a local pure tensor $s \otimes t$, with $s \in \mathscr{F}(U)$ and $t \in \mathscr{G}(U)$, therefore defines an element of $(\mathscr{F} \otimes \mathscr{G})_x$, and the rule

$$(s \otimes t)_x \longmapsto s_x \otimes t_x$$

gives a natural $\mathscr{O}_{X,x}$-linear map

$$(\mathscr{F} \otimes \mathscr{G})_x \longrightarrow \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x.$$

Conversely, choose representatives of two germs $u \in \mathscr{F}_x$ and $v \in \mathscr{G}_x$ on neighbourhoods of x. After shrinking to their intersection, they are represented on a common neighbourhood U, and we send $u \otimes v$ to the germ of the corresponding element of

$$\mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U).$$

If the representatives are changed, they agree after further shrinking, so the resulting germ is unchanged. The bilinearity and scalar relations over $\mathscr{O}_{X,x}$ are likewise represented after shrinking to a common neighbourhood. Thus the second construction gives a well-defined inverse to the first. □

Now let

$$f : (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y)$$

be a morphism of ringed spaces. The morphism of structure sheaves

$$f^{-1}\mathscr{O}_Y \longrightarrow \mathscr{O}_X$$

allows us to extend scalars from $f^{-1}\mathscr{O}_Y$ to $\mathscr{O}_X$.

Definition 5.4 (Pullback of modules)**.** For an $\mathscr{O}_Y$-module $\mathscr{G}$, define

$$f^*\mathscr{G} := f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

This is an $\mathscr{O}_X$-module, called the *pullback* or *inverse image as a module* of $\mathscr{G}$.

The distinction between f^{-1} and f^* is important. The first only transports a sheaf to the inverse-image topology; the second also changes the ring of scalars.

Proposition 5.5 (Stalks of pullbacks)**.** *For every $x \in X$, there is a natural isomorphism*

$$(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x}.$$

Proof. By proposition 5.3,

$$(f^*\mathscr{G})_x \cong (f^{-1}\mathscr{G})_x \otimes_{(f^{-1}\mathscr{O}_Y)_x} \mathscr{O}_{X,x}.$$

The inverse-image stalk formula from Chapter 1 gives

$$(f^{-1}\mathscr{G})_x \cong \mathscr{G}_{f(x)}, \qquad (f^{-1}\mathscr{O}_Y)_x \cong \mathscr{O}_{Y,f(x)},$$

and the asserted formula follows. □

The reason for the definition of f^* is its adjunction with direct image.

Theorem 5.6 (Pullback-direct image adjunction)**.** *Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. For every $\mathscr{O}_Y$-module $\mathscr{G}$ and every $\mathscr{O}_X$-module $\mathscr{F}$, there is a natural bijection*

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F}).$$

Thus f^* is left adjoint to f_* on module categories.

Proof. Regard $\mathscr{F}$ as an $f^{-1}\mathscr{O}_Y$-module through the structure homomorphism

$$f^{-1}\mathscr{O}_Y \longrightarrow \mathscr{O}_X.$$

Extension of scalars is left adjoint to restriction of scalars, so an $\mathscr{O}_X$-linear map

$$f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X \longrightarrow \mathscr{F}$$

is equivalent to an $f^{-1}\mathscr{O}_Y$-linear map

$$f^{-1}\mathscr{G} \longrightarrow \mathscr{F}.$$

Explicitly, a map Φ is sent to $g \mapsto \Phi(g \otimes 1)$; conversely, an $f^{-1}\mathscr{O}_Y$-linear map ψ is sent to

$$g \otimes a \longmapsto a\psi(g).$$

These constructions are inverse and compatible with restriction.

The sheaf adjunction $f^{-1} \dashv f_*$ gives a natural bijection

$$\mathrm{Hom}_{f^{-1}\mathscr{O}_Y}(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G}, f_*\mathscr{F}).$$

Because the adjunction unit and counit commute with the action of $\mathscr{O}_Y$, this bijection restricts to module homomorphisms. Composing the two natural bijections proves the theorem. $\square$

The basic formal consequences will be used repeatedly.

Proposition 5.7 (Functoriality of pullback)**.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of ringed spaces. Then there are natural isomorphisms*

$$\mathrm{id}_X^* \mathscr{F} \cong \mathscr{F}, \qquad (g \circ f)^*\mathscr{H} \cong f^*g^*\mathscr{H},$$

and for $\mathscr{O}_Y$-modules $\mathscr{F}, \mathscr{G}$,

$$f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) \cong f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}.$$

The functor f^* is right exact.

Proof. All statements may be checked on stalks. At $x \in X$, they become the usual identities for extension of scalars along the local ring homomorphisms

$$\mathscr{O}_{Z,g(f(x))} \longrightarrow \mathscr{O}_{Y,f(x)} \longrightarrow \mathscr{O}_{X,x}.$$

For example,

$$\big((g \circ f)^*\mathscr{H}\big)_x \cong \mathscr{H}_{g(f(x))} \otimes_{\mathscr{O}_{Z,g(f(x))}} \mathscr{O}_{X,x},$$

while

$$(f^*g^*\mathscr{H})_x \cong \Big(\mathscr{H}_{g(f(x))} \otimes_{\mathscr{O}_{Z,g(f(x))}} \mathscr{O}_{Y,f(x)}\Big) \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x},$$

and associativity of tensor products identifies them. The tensor-product formula is proved in the same way. Finally, extension of scalars is right exact for modules, so the stalk functors of f^* carry right exact sequences to right exact sequences; exactness of sheaves is stalkwise. $\square$

We now specialize to an affine scheme $X = \operatorname{Spec} A$. Ordinary modules over A give the model for all quasi-coherent sheaves.

Definition 5.8 (Associated sheaf on an affine scheme)**.** Let M be an A-module. On the distinguished basis of $X = \operatorname{Spec} A$, set

$$\widetilde{M}(D(f)) := M_f.$$

The localization maps give the restrictions. These data define an $\mathscr{O}_X$-module, called the *sheaf associated to M*.

The required sheaf condition follows from the corresponding localization lemma for modules.

Lemma 5.9 (Localization equalizer for modules)**.** *Let R be a ring, N an R-module, and let $g_1, \dots, g_r \in R$ satisfy*

$$\operatorname{Spec} R = D(g_1) \cup \dots \cup D(g_r).$$

Then

$$0 \longrightarrow N \longrightarrow \prod_i N_{g_i} \rightrightarrows \prod_{i,j} N_{g_i g_j}$$

is an equalizer.

Proof. Since the distinguished opens cover $\operatorname{Spec} R$, the ideal generated by any fixed positive powers $g_1^L, \dots, g_r^L$ has radical equal to R, hence is the unit ideal.

For injectivity, suppose $n \in N$ maps to zero in every N_{g_i}. For each i, some power of g_i kills n. Choose one exponent L which works for every i, and write

$$1 = \sum_i a_i g_i^L.$$

Then

$$n = \sum_i a_i g_i^L n = 0.$$

Now let $n_i \in N_{g_i}$ be a compatible family. Choose a common exponent N and elements $u_i \in N$ such that

$$n_i = \frac{u_i}{g_i^N}.$$

Compatibility in $N_{g_i g_j}$ means that for each pair (i, j) there is an integer $k_{ij} \geq 0$ with

$$(g_i g_j)^{k_{ij}} (g_j^N u_i - g_i^N u_j) = 0.$$

Choose $K \geq k_{ij}$ for every pair, set

$$L := N + K, \qquad m_i := g_i^K u_i,$$

and observe that

$$n_i = \frac{m_i}{g_i^L}, \qquad g_j^L m_i = g_i^L m_j$$

in N for all i, j.

Choose $a_i \in R$ with $\sum_i a_i g_i^L = 1$ and set

$$n := \sum_i a_i m_i \in N.$$

For every j,

$$g_j^L n = \sum_i a_i g_j^L m_i = \sum_i a_i g_i^L m_j = m_j.$$

Thus the image of n in N_{g_j} is $m_j / g_j^L = n_j$. This proves the gluing assertion. $\square$

Apply the lemma with $R = A_f$ and $N = M_f$. Every open cover by distinguished opens admits a finite subcover because $D(f)$ is affine and hence quasi-compact. Therefore the prescription $D(f) \mapsto M_f$ is a sheaf on the distinguished basis, and it extends uniquely to a sheaf on X.

Proposition 5.10 (Basic affine identities)**.** *Let $X = \operatorname{Spec} A$. Then:*

1. *for every $\mathfrak{p} \in \operatorname{Spec} A$,*
$$(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}};$$
2. $\Gamma(X, \widetilde{M}) \cong M$*;*
3. *the functor $M \mapsto \widetilde{M}$ is exact;*
4. *for any family $\{M_i\}_{i \in I}$,*
$$\widetilde{\bigoplus_{i\in I} M_i} \cong \bigoplus_{i\in I} \widetilde{M_i};$$
5. *there is a natural isomorphism*
$$\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}.$$

Proof. The stalk at $\mathfrak{p}$ is the filtered colimit of M_f over $f \notin \mathfrak{p}$, which is precisely $M_{\mathfrak{p}}$. Taking $f = 1$ gives $\Gamma(X, \widetilde{M}) = M$.

For exactness, a sequence of associated sheaves is exact if and only if it is exact at every stalk, and localization $M \mapsto M_{\mathfrak{p}}$ is exact. Direct sums commute with localization, giving (4). For (5), on every distinguished open $D(f)$,

$$(M \otimes_A N)_f \cong M_f \otimes_{A_f} N_f,$$

and these isomorphisms commute with further localization. They therefore glue to the stated sheaf isomorphism. □

There are two useful ways to recognize the sheaves arising from modules. The first is affine and concrete; the second is a local presentation condition.

Definition 5.11 (Quasi-coherent sheaf)**.** An $\mathscr{O}_X$-module $\mathscr{F}$ on a scheme X is *quasi-coherent* if every point has an affine open neighbourhood $U = \operatorname{Spec} A$ for which

$$\mathscr{F}|_U \cong \widetilde{M}$$

for some A-module M.

Proposition 5.12 (Local presentations)**.** *An $\mathscr{O}_X$-module $\mathscr{F}$ is quasi-coherent if and only if the following condition holds: every point admits an open neighbourhood U carrying an exact sequence*
$$\mathscr{O}_U^{(J)} \longrightarrow \mathscr{O}_U^{(I)} \longrightarrow \mathscr{F}|_U \longrightarrow 0$$
for some sets I, J.

Proof. Suppose first that $\mathscr{F}|_U \cong \widetilde{M}$ on an affine open $U = \operatorname{Spec} A$. Choose a presentation of the A-module M,
$$A^{(J)} \longrightarrow A^{(I)} \longrightarrow M \longrightarrow 0.$$
By exactness and compatibility with direct sums in proposition 5.10, sheafification gives

$$\mathscr{O}_U^{(J)} \longrightarrow \mathscr{O}_U^{(I)} \longrightarrow \widetilde{M} \longrightarrow 0.$$

Conversely, shrink the given neighbourhood until it is affine, say $U = \operatorname{Spec} A$. A morphism

$$\mathscr{O}_U^{(J)} \longrightarrow \mathscr{O}_U^{(I)}$$

is determined by the images of the standard basis sections. Since U is quasi-compact, global sections of the direct-sum sheaf $\mathscr{O}_U^{(I)}$ are exactly $A^{(I)}$: a section is locally supported in finitely many summands, and a finite subcover gives one finite set of summands globally. Hence the morphism comes from an A-linear map

$$A^{(J)} \longrightarrow A^{(I)}.$$

Let M be its cokernel. Applying the exact tilde functor shows that its sheaf cokernel is $\widetilde{M}$. Therefore $\mathscr{F}|_U \cong \widetilde{M}$, so $\mathscr{F}$ is quasi-coherent. □

Theorem 5.13 (The affine dictionary)**.** *For $X = \operatorname{Spec} A$, the functors*

$$A\text{-}\mathrm{Mod} \xrightarrow{M \mapsto \widetilde{M}} \mathrm{QCoh}(X) \xrightarrow{\Gamma(X,-)} A\text{-}\mathrm{Mod}$$

are quasi-inverse equivalences. In particular,

$$\operatorname{Hom}_A(M,N) \cong \operatorname{Hom}_{\mathscr{O}_X}(\widetilde{M},\widetilde{N}).$$

Proof. By proposition 5.10, $\Gamma(X,\widetilde{M}) = M$, naturally in M. It remains to show that if $\mathscr{F}$ is quasi-coherent, then the canonical morphism

$$\widetilde{\Gamma(X,\mathscr{F})} \longrightarrow \mathscr{F}$$

is an isomorphism.

Choose a finite distinguished cover

$$X = D(f_1) \cup \cdots \cup D(f_r)$$

such that $\mathscr{F}|_{D(f_i)} \cong \widetilde{M_i}$. Put $M = \Gamma(X,\mathscr{F})$. We claim that the restriction map induces

$$M_{f_i} \xrightarrow{\sim} \Gamma(D(f_i),\mathscr{F}) = M_i.$$

For surjectivity, let $s \in M_i$. On $D(f_i f_j)$, the restriction of s is an element of $(M_j)_{f_i}$. Thus for each j there are $N_j \geq 0$ and $u_j \in M_j$ whose localization equals $f_i^{N_j} s$. Choose $N \geq N_j$ for all j and replace u_j by $f_i^{N-N_j} u_j$; now every u_j restricts to $f_i^N s$ on $D(f_i f_j)$. In particular, on $D(f_j f_k)$ the difference $u_j - u_k$ becomes zero after localizing at f_i. Hence some power of f_i kills this difference. Because there are finitely many pairs, one exponent K works for all of them. The sections

$$f_i^K u_j \in \Gamma(D(f_j),\mathscr{F})$$

therefore agree on every overlap and glue to a global section $t \in M$. On $D(f_i)$ we have

$$t = f_i^{N+K} s,$$

so t/f_i^{N+K} maps to s. Thus $M_{f_i} \to M_i$ is surjective.

For injectivity, let $t \in M$ map to zero in M_i. Write $t_j \in M_j$ for its restriction to $D(f_j)$. Since $(t_j)_{f_i} = 0$, some power of f_i kills t_j. A common exponent N works for the finite cover, so the global section $f_i^N t$ restricts to zero on every $D(f_j)$. By the sheaf axiom, $f_i^N t = 0$ globally, and therefore $t/1 = 0$ in M_{f_i}.

Thus $\widetilde{M}$ and $\mathscr{F}$ have the same sections on each member of the distinguished cover and on all further distinguished subopens. The canonical morphism is therefore an isomorphism locally on a cover, hence globally. This proves the equivalence; the Hom identity follows from full faithfulness. □

Remark 5.14. The argument in the preceding proof is the module-theoretic form of affine communication: localizations on a finite distinguished cover contain enough information to reconstruct the global module. This is the precise sense in which quasi-coherent sheaves are "modules varying algebraically."

We now impose finiteness conditions.

Definition 5.15 (Finite type, finite presentation, and coherence)**.** Let $\mathscr{F}$ be an $\mathscr{O}_X$-module.

1. It is *of finite type* if every point has a neighbourhood U admitting a surjection
$$\mathscr{O}_U^{\oplus n} \twoheadrightarrow \mathscr{F}|_U$$
for some finite n.

2. It is *of finite presentation* if locally there is an exact sequence
$$\mathscr{O}_U^{\oplus m} \longrightarrow \mathscr{O}_U^{\oplus n} \longrightarrow \mathscr{F}|_U \longrightarrow 0$$
with $m, n < \infty$.

3. It is *coherent* if it is of finite type and, for every open $V \subseteq X$ and every morphism
$$\mathscr{O}_V^{\oplus n} \longrightarrow \mathscr{F}|_V,$$
the kernel is of finite type.

Theorem 5.16 (Coherent sheaves on locally Noetherian schemes)**.** *Let X be locally Noetherian and let $\mathscr{F}$ be quasi-coherent. The following are equivalent:*

1. *$\mathscr{F}$ is coherent;*
2. *$\mathscr{F}$ is of finite type;*
3. *for every affine open $U = \operatorname{Spec} A \subseteq X$, the module $\Gamma(U, \mathscr{F})$ is finitely generated over A.*

On an affine open $U = \operatorname{Spec} A$, coherent sheaves therefore correspond exactly to finitely generated A-modules.

Proof. The implication (1)⇒(2) is part of the definition. Suppose $\mathscr{F}$ is of finite type and let $U = \operatorname{Spec} A$ be affine. By the affine dictionary,
$$\mathscr{F}|_U \cong \widetilde{M}, \qquad M = \Gamma(U, \mathscr{F}).$$
Choose a finite distinguished cover $U = \bigcup_i D(f_i)$ on which finitely many sections generate $\mathscr{F}$. By the affine dictionary on each $D(f_i)$, this says that M_{f_i} is finitely generated over A_{f_i}. Choose finitely many elements of M whose images generate all the M_{f_i}, and let $N \subseteq M$ be the submodule they generate. Then
$$(M/N)_{f_i} = 0$$
for every i. Since the $D(f_i)$ cover $\operatorname{Spec} A$, every localization $(M/N)_{\mathfrak{p}}$ is zero. Hence $M/N = 0$, so M is finitely generated. This proves (2)⇒(3).

Assume (3). On $U = \operatorname{Spec} A$, write $M = \Gamma(U, \mathscr{F})$. Since A is Noetherian and M is finitely generated, every kernel of a homomorphism $A^n \to M$ is finitely generated. Under the affine equivalence, morphisms $\mathscr{O}_U^n \to \mathscr{F}|_U$ correspond to homomorphisms $A^n \to M$, and kernels correspond because the tilde functor is exact. Thus $\mathscr{F}|_U$ is coherent. Coherence is local, so $\mathscr{F}$ is coherent. ☐

Corollary 5.17 (Abelian behaviour)**.** *The category* $\mathrm{QCoh}(X)$ *is closed under kernels, images, cokernels, and finite direct sums of morphisms between quasi-coherent sheaves. Hence* $\mathrm{QCoh}(X)$ *is an abelian subcategory of* $\mathrm{Mod}(X)$*. If X is locally Noetherian, the same statements hold for* $\mathrm{Coh}(X)$*.*

Proof. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism between quasi-coherent sheaves. The assertion is local, so take an affine open $U = \operatorname{Spec} A$ and write

$$\mathscr{F}|_U \cong \widetilde{M}, \qquad \mathscr{G}|_U \cong \widetilde{N}.$$

By full faithfulness in theorem 5.13, $\varphi|_U$ comes from a unique homomorphism $M \to N$. Exactness of the tilde functor identifies the kernel, image, and cokernel sheaves with the associated sheaves of the corresponding modules. They are therefore quasi-coherent. Finite direct sums are handled by proposition 5.10.

If X is locally Noetherian and $\mathscr{F}, \mathscr{G}$ are coherent, then M, N are finitely generated over the Noetherian ring A. Their kernels, images, cokernels, and finite direct sums are finitely generated, so the corresponding sheaves are coherent by theorem 5.16. ☐

Definition 5.18 (Locally free and invertible sheaves)**.** An $\mathscr{O}_X$-module $\mathscr{E}$ is *locally free of rank r* if every point has an open neighbourhood U such that

$$\mathscr{E}|_U \cong \mathscr{O}_U^{\oplus r}.$$

It is *finite locally free* if such ranks are finite, possibly varying between connected components. A locally free sheaf of rank one is called *invertible.*

On a locally Noetherian scheme every finite locally free sheaf is coherent, because finite free modules are finitely generated on affine opens.

We can now return to pullback with the affine dictionary available.

Proposition 5.19 (Pullback of quasi-coherent and coherent sheaves)**.** *Let $f : X \to Y$ be a morphism of schemes.*

1. *If $\mathscr{G}$ is quasi-coherent on Y, then $f^*\mathscr{G}$ is quasi-coherent on X.*
2. *If X and Y are locally Noetherian and $\mathscr{G}$ is coherent on Y, then $f^*\mathscr{G}$ is coherent on X.*

If $V = \operatorname{Spec} A \subseteq Y$, $U = \operatorname{Spec} B \subseteq f^{-1}(V)$, *and* $\mathscr{G}|_V \cong \widetilde{M}$, *then in the diagram*

$$\begin{tikzcd} U = \operatorname{Spec} B \arrow[r, hook] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V = \operatorname{Spec} A \arrow[r, hook] & Y \end{tikzcd}$$

we have

$$f^*\mathscr{G}|_U \cong \widetilde{M \otimes_A B}.$$

Proof. The morphism $f|_U : U \to V$ corresponds to a ring homomorphism $A \to B$. There is a canonical morphism

$$\widetilde{M \otimes_A B} \longrightarrow f^*\widetilde{M}|_U$$

obtained from the universal bilinear map $M \times B \to f^*\widetilde{M}(U)$. At a point $\mathfrak{q} \in \operatorname{Spec} B$, with $\mathfrak{p} = f(\mathfrak{q})$, its map on stalks is the standard localization isomorphism

$$(M \otimes_A B)_{\mathfrak{q}} \xrightarrow{\sim} M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}}.$$

Hence the canonical sheaf morphism is an isomorphism by the stalkwise criterion. This proves quasi-coherence locally.

If $\mathscr{G}$ is coherent, then M is finitely generated over the Noetherian ring A. Thus $M \otimes_A B$ is finitely generated over B. Since B is Noetherian by the local Noetherian hypothesis on X, its associated sheaf is coherent by theorem 5.16. □

5.2 Coherent Sheaves on Projective Schemes

The affine dictionary has a graded analogue. Let

$$S = \bigoplus_{d \geq 0} S_d$$

be a graded ring. We will call S *standard graded* if it is generated as an S_0-algebra by finitely many elements of degree one. Let

$$M = \bigoplus_{d \in \mathbb{Z}} M_d$$

be a graded S-module. For a homogeneous element $f \in S_+$, write

$$S_{(f)} := (S_f)_0, \qquad M_{(f)} := (M_f)_0.$$

Recall from Chapter 2 that

$$D_+(f) = \operatorname{Spec} S_{(f)}.$$

Proposition 5.20 (The sheaf associated to a graded module)**.** *Let $X = \operatorname{Proj} S$. There is a unique quasi-coherent $\mathscr{O}_X$-module $\widetilde{M}$ such that*

$$\widetilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$$

for every homogeneous $f \in S_+$. These identifications are compatible with restriction to intersections.

Proof. The opens $D_+(f)$ form a basis of $\operatorname{Proj} S$. On an overlap

$$D_+(f) \cap D_+(g) = D_+(fg),$$

both restrictions are obtained from the same graded localization M_{fg}. More explicitly, if one first passes to M_f and then restricts to the open on which g is invertible, the resulting degree-zero module is canonically the same as $(M_{fg})_0$; the same holds with f and g interchanged. Thus the two associated sheaves on the overlap are canonically identified. These identifications satisfy the cocycle condition because every map is induced by localization. Hence the affine quasi-coherent sheaves $\widetilde{M_{(f)}}$ glue uniquely to $\widetilde{M}$. □

The construction forgets graded information supported at the irrelevant ideal. For example, if every element of a graded module is annihilated by a power of S_+, its associated sheaf is zero. Thus $M \mapsto \widetilde{M}$ on $\operatorname{Proj} S$ is not as faithful as the affine tilde functor.

Definition 5.21 (Shifts and twisting sheaves)**.** For $n \in \mathbb{Z}$, define the shifted graded module $M(n)$ by

$$M(n)_d = M_{n+d}.$$

On $X = \operatorname{Proj} S$, define

$$\mathscr{O}_X(n) := \widetilde{S(n)}, \qquad \mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n).$$

Proposition 5.22 (Twists in the standard graded case)**.** *Assume that S is standard graded. Then every $\mathscr{O}_X(n)$ is invertible and there are natural isomorphisms*

$$\mathscr{O}_X(n) \otimes \mathscr{O}_X(m) \cong \mathscr{O}_X(n+m).$$

Moreover,

$$\widetilde{M(n)} \cong \widetilde{M}(n).$$

Proof. Because S is generated in degree one, the opens $D_+(x)$ with $x \in S_1$ cover X. On each such open, multiplication by x^n identifies the degree-zero localizations of S and $S(n)$, with the evident interpretation for negative n after inverting x. Hence

$$\mathscr{O}_X(n)|_{D_+(x)} \cong \mathscr{O}_{D_+(x)}.$$

Thus $\mathscr{O}_X(n)$ is locally free of rank one. The same localization calculation gives

$$S(n)_{(x)} \otimes_{S_{(x)}} S(m)_{(x)} \cong S(n+m)_{(x)},$$

so the tensor-product isomorphisms glue. Replacing S by M gives $\widetilde{M(n)} \cong \widetilde{M}(n)$. □

For a quasi-coherent sheaf, the analogue of a graded module of sections is obtained by collecting all twists.

Definition 5.23 (The graded module of twisted sections)**.** Let $X = \operatorname{Proj} S$ with S standard graded and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Set

$$\Gamma_*(\mathscr{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n)).$$

Multiplication by homogeneous elements of S makes this a graded S-module.

There is a natural evaluation morphism

$$\widetilde{\Gamma_*(\mathscr{F})} \longrightarrow \mathscr{F}.$$

On a standard open $D_+(f)$, a homogeneous section $t \in \Gamma(X, \mathscr{F}(n))$ and a homogeneous denominator f^m of the same total degree determine, after trivializing the twist by a power of f, an ordinary section of $\mathscr{F}$ on $D_+(f)$. These local evaluation maps commute with further localization and therefore glue to the displayed morphism.

The key technical fact behind projective sheaf theory is that sections defined on a standard affine chart become global after multiplication by a sufficiently high power of the coordinate defining that chart.

Lemma 5.24 (Extension from a standard affine open)**.** *Let X be a separated Noetherian scheme with an invertible sheaf $\mathscr{L}$ generated by global sections*

$$x_0, \dots, x_r \in \Gamma(X, \mathscr{L})$$

such that the opens

$$U_i := \{x \in X \mid (x_i)_x \text{ generates } \mathscr{L}_x\}$$

are affine and cover X. Let $\mathscr{F}$ be quasi-coherent. If $s \in \Gamma(U_i, \mathscr{F})$, then for some $N \geq 0$ the section

$$x_i^N s \in \Gamma(U_i, \mathscr{F} \otimes \mathscr{L}^{\otimes N})$$

extends to a global section of $\mathscr{F} \otimes \mathscr{L}^{\otimes N}$.

Proof. Fix i and trivialize $\mathscr{L}$ on each U_j using x_j. On the affine open U_j, the ratio

$$a_{ij} := x_i/x_j$$

is a regular function, and

$$U_i \cap U_j = D(a_{ij}) \subseteq U_j.$$

After the chosen trivializations, the restriction of s to $U_i \cap U_j$ is an element of the localization of the module corresponding to $\mathscr{F}|_{U_j}$ at a_{ij}. Hence there exists N_j such that $a_{ij}^{N_j} s$ is represented by a section on U_j. Taking one N at least as large as all N_j, we obtain sections

$$t_j \in \Gamma(U_j, \mathscr{F} \otimes \mathscr{L}^{\otimes N})$$

whose restriction to $U_i \cap U_j$ equals $x_i^N s$.

The sections t_j need not yet agree on $U_j \cap U_k$, but their difference restricts to zero after further restriction to

$$U_i \cap U_j \cap U_k.$$

Because X is separated and U_j, U_k are affine, the intersection $U_j \cap U_k$ is affine. In the trivialization by x_j, the open $U_i \cap U_j \cap U_k$ is obtained by inverting the regular function $a_{ij} = x_i/x_j$. The affine dictionary therefore implies that some power of a_{ij}, equivalently some power of the section x_i, kills $t_j - t_k$ on $U_j \cap U_k$. There are only finitely many pairs (j, k), so one common power x_i^M kills every difference. Replacing each t_j by $x_i^M t_j$, the sections agree on all overlaps and therefore glue. Their restriction to U_i is $x_i^{N+M} s$, which proves the lemma after replacing N by $N + M$. □

Theorem 5.25 (Reconstruction from twisted sections). *Let S be a Noetherian standard graded ring, $X = \operatorname{Proj} S$, and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then the natural morphism*

$$\widetilde{\Gamma_*(\mathscr{F})} \longrightarrow \mathscr{F}$$

is an isomorphism.

Proof. It is enough to check on the standard affine cover $U_i = D_+(x_i)$ determined by finitely many degree-one generators. On U_i, the source corresponds to the $S_{(x_i)}$-module

$$(\Gamma_*(\mathscr{F})_{x_i})_0.$$

The evaluation map sends a homogeneous global section divided by a power of x_i to its restriction on U_i, with the twist trivialized by x_i.

Surjectivity follows from lemma 5.24: any local section $s \in \Gamma(U_i, \mathscr{F})$ becomes the restriction of a global section of $\mathscr{F}(N)$ after multiplication by x_i^N, so

$$s = \frac{t}{x_i^N}$$

in the localized degree-zero module.

For injectivity, suppose a homogeneous global section $t \in \Gamma(X, \mathscr{F}(N))$ restricts to zero on U_i. On each affine U_j, its restriction becomes zero after localizing at the function x_i/x_j; hence some power of x_i kills $t|_{U_j}$. A common power works for the finite cover, and therefore $x_i^M t = 0$ globally. Thus t/x_i^N is zero in the localization. The evaluation map is therefore an isomorphism on every U_i, hence globally. □

We now pass from the particular line bundle $\mathscr{O}_X(1)$ to the intrinsic notion that produces projective embeddings.

Definition 5.26 (Very ample invertible sheaf)**.** Let $X \to Y$ be a morphism. An invertible sheaf $\mathscr{L}$ on X is *very ample relative to* Y if there exists a closed immersion

$$i : X \hookrightarrow \mathbb{P}^r_Y$$

for some r such that

$$\mathscr{L} \cong i^* \mathscr{O}_{\mathbb{P}^r_Y}(1).$$

The following theorem is the projective analogue of finite generation over an affine ring.

Theorem 5.27 (Serre's theorem on global generation)**.** *Let A be a Noetherian ring, let X be projective over* $\operatorname{Spec} A$*, and choose a very ample invertible sheaf $\mathscr{O}_X(1)$. If $\mathscr{F}$ is coherent, then there exists n_0 such that $\mathscr{F}(n)$ is generated by finitely many global sections for every $n \geq n_0$.*

Proof. Choose a closed immersion

$$i : X \hookrightarrow \mathbb{P}^r_A$$

realizing $\mathscr{O}_X(1)$. Let $x_0, \dots, x_r$ be the restrictions to X of the standard homogeneous coordinates. The open sets

$$U_i = X \cap D_+(x_i)$$

are affine and cover X, and x_i trivializes $\mathscr{O}_X(1)$ on U_i.

Because $\mathscr{F}$ is coherent, each $\Gamma(U_i, \mathscr{F})$ is a finitely generated module over $\Gamma(U_i, \mathscr{O}_X)$. Choose finitely many local generators s_{ij}. By lemma 5.24, for each s_{ij} there is an integer n_{ij} and a global section

$$t_{ij} \in \Gamma(X, \mathscr{F}(n_{ij}))$$

whose restriction to U_i is $x_i^{n_{ij}} s_{ij}$. Let

$$n_0 = \max_{i,j} n_{ij}.$$

Multiplying t_{ij} by $x_i^{n_0 - n_{ij}}$ gives a global section of $\mathscr{F}(n_0)$ whose restriction to U_i is $x_i^{n_0} s_{ij}$. Since x_i is invertible on U_i, these global sections generate $\mathscr{F}(n_0)|_{U_i}$. As the U_i cover X, they generate $\mathscr{F}(n_0)$ globally.

For $n \geq n_0$, the line bundle $\mathscr{O}_X(n - n_0)$ is generated by the restrictions of degree $n - n_0$ monomials in $x_0, \dots, x_r$. Tensoring the surjection

$$\mathscr{O}_X^{\oplus N} \twoheadrightarrow \mathscr{F}(n_0)$$

with $\mathscr{O}_X(n - n_0)$, and then composing with the finite set of global generators of $\mathscr{O}_X(n - n_0)$, gives a finite set of global sections generating $\mathscr{F}(n)$. $\square$

Corollary 5.28 (Coherent sheaves are quotients of vector bundles)**.** *Let A be Noetherian and let X be projective over* $\operatorname{Spec} A$*. Every coherent $\mathscr{O}_X$-module $\mathscr{F}$ is a quotient of a finite locally free sheaf. More precisely, for some n and N, there is a surjection*

$$\mathscr{O}_X(-n)^{\oplus N} \twoheadrightarrow \mathscr{F}.$$

Proof. Choose n such that $\mathscr{F}(n)$ is generated by finitely many global sections. These sections give a surjection

$$\mathscr{O}_X^{\oplus N} \twoheadrightarrow \mathscr{F}(n).$$

Tensor with the invertible sheaf $\mathscr{O}_X(-n)$. Tensoring by an invertible sheaf is exact because it is locally tensoring with a free module of rank one. Hence

$$\mathscr{O}_X(-n)^{\oplus N} \twoheadrightarrow \mathscr{F}.$$

The source is finite locally free. $\square$

Remark 5.29. The conclusion is stronger than local finite presentation: on a projective scheme the local generators can, after twisting, be replaced by finitely many global generators. This fact is one of the main reasons coherent sheaves interact so effectively with projective geometry.

5.3 Algebra Constructions and Relative Spectrum

Tensor products can be iterated, quotiented, and then converted back into schemes. This section develops precisely the algebra needed for that passage.

Definition 5.30 (Internal Hom and dual)**.** For $\mathscr{O}_X$-modules $\mathscr{F}, \mathscr{G}$, define the internal Hom sheaf by

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})(U) := \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

The dual of $\mathscr{F}$ is

$$\mathscr{F}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X).$$

Definition 5.31 (Tensor, symmetric, and exterior algebras)**.** Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Define the tensor algebra

$$T(\mathscr{F}) := \bigoplus_{n\geq 0} \mathscr{F}^{\otimes n}, \qquad \mathscr{F}^{\otimes 0} := \mathscr{O}_X.$$

The *symmetric algebra* $\mathrm{Sym}(\mathscr{F})$ is the quotient of $T(\mathscr{F})$ by the sheaf of ideals locally generated by

$$s \otimes t - t \otimes s.$$

Its degree-n part is denoted $\mathrm{Sym}^n(\mathscr{F})$.

The *exterior algebra* $\bigwedge(\mathscr{F})$ is the quotient of $T(\mathscr{F})$ by the sheaf of ideals locally generated by

$$s \otimes s.$$

Its degree-n part is denoted $\bigwedge^n \mathscr{F}$.

These constructions are characterized by universal properties. A morphism $\mathscr{F} \to \mathscr{A}$ into a commutative $\mathscr{O}_X$-algebra extends uniquely to an algebra morphism

$$\mathrm{Sym}(\mathscr{F}) \longrightarrow \mathscr{A},$$

while an alternating n-linear map $\mathscr{F}^n \to \mathscr{G}$ factors uniquely through $\bigwedge^n \mathscr{F}$. Since the assertions are local and are the usual universal properties for modules over a ring, the local factorizations agree on overlaps and glue.

Proposition 5.32 (Compatibility with quasi-coherence and pullback)**.** *Let X be a scheme.*

1. *If $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent, then $\mathscr{F} \otimes \mathscr{G}$, $\mathrm{Sym}^n \mathscr{F}$, and $\bigwedge^n \mathscr{F}$ are quasi-coherent. The full algebras $T(\mathscr{F})$, $\mathrm{Sym}(\mathscr{F})$, and $\bigwedge(\mathscr{F})$ are quasi-coherent graded $\mathscr{O}_X$-algebras.*

2. *If X is locally Noetherian and $\mathscr{F}, \mathscr{G}$ are coherent, then $\mathscr{F} \otimes \mathscr{G}$, $\mathrm{Sym}^n \mathscr{F}$, and $\bigwedge^n \mathscr{F}$ are coherent for every fixed n.*

3. *For a morphism $f : X \to Y$, there are natural isomorphisms*

$$f^*T(\mathscr{F}) \cong T(f^*\mathscr{F}), \quad f^* \mathrm{Sym}(\mathscr{F}) \cong \mathrm{Sym}(f^*\mathscr{F}), \quad f^* \bigwedge(\mathscr{F}) \cong \bigwedge(f^*\mathscr{F}).$$

Proof. All assertions are local. On an affine open $\operatorname{Spec} A$, write $\mathscr{F} = \widetilde{M}$ and $\mathscr{G} = \widetilde{N}$. The affine tensor identity gives

$$\mathscr{F} \otimes \mathscr{G} \cong \widetilde{M \otimes_A N}.$$

The symmetric and exterior powers are cokernels of maps between finite tensor powers, so the affine dictionary identifies them with

$$\widetilde{\operatorname{Sym}^n_A M}, \qquad \widetilde{\bigwedge^n_A M}.$$

Direct sums remain quasi-coherent, proving (1). If A is Noetherian and M, N are finitely generated, each fixed tensor, symmetric, and exterior power is finitely generated, proving (2). The full tensor or symmetric algebra is generally an infinite direct sum and need not be coherent.

For (3), pullback is extension of scalars on affine opens. Extension of scalars commutes with tensor products and with quotients by the symmetry or alternating relations. The resulting affine isomorphisms are natural and therefore glue. □

Finite locally free sheaves behave exactly like finite-dimensional vector spaces with coefficients in $\mathscr{O}_X$.

Proposition 5.33 (Duality for finite locally free sheaves)**.** *Let $\mathscr{E}$ be locally free of finite rank. Then the evaluation morphism*

$$\mathscr{E} \longrightarrow \mathscr{E}^{\vee\vee}$$

is an isomorphism. Moreover, for every $\mathscr{F}$,

$$\mathscr{H}om(\mathscr{E}, \mathscr{F}) \cong \mathscr{E}^{\vee} \otimes \mathscr{F}.$$

If $\mathscr{E}$ has constant rank r, its determinant

$$\det(\mathscr{E}) := \bigwedge^r \mathscr{E}$$

is invertible.

Proof. The statements are local on X. Where $\mathscr{E} \cong \mathscr{O}_X^{\oplus r}$, they become the standard identities for a finite free module:

$$A^r \cong (A^r)^{\vee\vee}, \qquad \operatorname{Hom}_A(A^r, M) \cong (A^r)^{\vee} \otimes_A M,$$

and $\bigwedge^r A^r \cong A$. These isomorphisms are canonical and therefore agree on overlaps. □

We now turn algebra into geometry. Let Y be a scheme and let $\mathscr{A}$ be a quasi-coherent sheaf of commutative $\mathscr{O}_Y$-algebras.

Theorem 5.34 (Relative spectrum)**.** *There exists an affine morphism*

$$p : \operatorname{Spec}_Y \mathscr{A} \longrightarrow Y$$

with the following property: for every affine open $U = \operatorname{Spec} R \subseteq Y$, there is a commutative square

$$\begin{tikzcd} \operatorname{Spec} \mathscr{A}(U) \arrow[r, hook] \arrow[d] & \operatorname{Spec}_Y \mathscr{A} \arrow[d, "p"] \\ U \arrow[r, hook] & Y \end{tikzcd}$$

whose top row identifies $\operatorname{Spec} \mathscr{A}(U)$ *with* $p^{-1}(U)$*. The construction is unique up to unique isomorphism and satisfies*

$$p_* \mathscr{O}_{\operatorname{Spec}_Y \mathscr{A}} \cong \mathscr{A}$$

as sheaves of $\mathscr{O}_Y$*-algebras.*

Proof. Choose an affine open cover $Y = \bigcup_i U_i$, with $U_i = \operatorname{Spec} R_i$. Quasi-coherence gives

$$\mathscr{A}|_{U_i} \cong \widetilde{A_i}, \qquad A_i = \mathscr{A}(U_i).$$

Set

$$X_i := \operatorname{Spec} A_i.$$

The structural homomorphism $R_i \to A_i$ gives a morphism $X_i \to U_i$.

On an affine open $W \subseteq U_i \cap U_j$ which is distinguished in both charts, quasi-coherence identifies $\mathscr{A}(W)$ with the appropriate localizations of both A_i and A_j. Hence the inverse images of W in X_i and X_j are both canonically

$$\operatorname{Spec} \mathscr{A}(W).$$

These identifications agree on further restrictions and satisfy the cocycle condition, since all of them come from restriction maps of the single sheaf $\mathscr{A}$. The gluing theorem from Chapter 2 therefore produces a scheme X and a morphism

$$p : X \to Y$$

whose restriction over each U_i is $X_i \to U_i$. By construction, p is affine.

Let $U = \operatorname{Spec} R \subseteq Y$ be an arbitrary affine open. By the common distinguished-refinement result of Chapter 2 and quasi-compactness of U, choose finitely many distinguished opens $W_a \subseteq U$ such that each W_a is contained in one of the original U_i. Quasi-coherence gives

$$\mathscr{A}(W_a) \cong \mathscr{A}(U)_{h_a}$$

when $W_a = D(h_a)$ inside U. Hence both $p^{-1}(U)$ and $\operatorname{Spec} \mathscr{A}(U)$ are obtained by gluing the same affine schemes

$$\operatorname{Spec} \mathscr{A}(W_a)$$

along the same localization maps. The uniqueness part of the gluing theorem therefore gives

$$p^{-1}(U) \cong \operatorname{Spec} \mathscr{A}(U).$$

Consequently,

$$(p_* \mathscr{O}_X)(U) = \Gamma(p^{-1}(U), \mathscr{O}_X) \cong \mathscr{A}(U)$$

for all affine opens U, compatibly with restriction. Hence $p_* \mathscr{O}_X \cong \mathscr{A}$. Finally, these affine descriptions uniquely determine all gluing maps, giving uniqueness up to unique isomorphism. □

The construction characterizes affine morphisms.

Theorem 5.35 (Affine schemes over a base and quasi-coherent algebras)**.** *For a scheme* Y*, relative spectrum gives an anti-equivalence*

$$(\textit{affine } Y\textit{-schemes})^{\mathrm{op}} \simeq \operatorname{QCohAlg}(Y),$$

where $\operatorname{QCohAlg}(Y)$ *denotes quasi-coherent commutative* $\mathscr{O}_Y$*-algebras. Explicitly, if* $f : X \to Y$ *is affine, then* $f_* \mathscr{O}_X$ *is quasi-coherent and the canonical morphism*

$$X \longrightarrow \operatorname{Spec}_Y(f_* \mathscr{O}_X)$$

is an isomorphism.

Proof. Let $f : X \to Y$ be affine and let $U = \operatorname{Spec} R \subseteq Y$ be affine. Then $f^{-1}(U) = \operatorname{Spec} B$ for some R-algebra B, and

$$(f_*\mathscr{O}_X)(U) = B.$$

For a distinguished open $D(g) \subseteq U$,

$$f^{-1}(D(g)) = D(f^\# g) \subseteq \operatorname{Spec} B,$$

so

$$(f_*\mathscr{O}_X)(D(g)) = B_g.$$

Thus $f_*\mathscr{O}_X$ is quasi-coherent on U, hence on Y. The relative spectrum of this algebra restricts over U to $\operatorname{Spec} B = f^{-1}(U)$. These affine isomorphisms are compatible on overlaps and glue to

$$X \xrightarrow{\sim} \operatorname{Spec}_Y(f_*\mathscr{O}_X).$$

Conversely, theorem 5.34 shows that $\operatorname{Spec}_Y \mathscr{A} \to Y$ is affine and recovers $\mathscr{A}$ by direct image. A morphism of Y-schemes between affine objects becomes, on every affine open of Y, the opposite ring homomorphism; these local homomorphisms glue exactly when they define a morphism of quasi-coherent $\mathscr{O}_Y$-algebras. Hence the two constructions are quasi-inverse anti-equivalences. □

There is also a relative form of the affine dictionary for modules. If $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, write $\operatorname{QCoh}(\mathscr{A})$ for the category of $\mathscr{A}$-modules which are quasi-coherent as $\mathscr{O}_Y$-modules.

Theorem 5.36 (Quasi-coherent modules under an affine morphism)**.** *Let $f : X \to Y$ be affine and set*

$$\mathscr{A} := f_*\mathscr{O}_X.$$

Then direct image induces an equivalence

$$\operatorname{QCoh}(X) \simeq \operatorname{QCoh}(\mathscr{A}),$$

where the right-hand side denotes quasi-coherent $\mathscr{A}$-modules on Y.

Proof. By theorem 5.35, identify X with $\operatorname{Spec}_Y \mathscr{A}$. Let $U = \operatorname{Spec} R \subseteq Y$ be affine. Then

$$f^{-1}(U) = \operatorname{Spec} A, \qquad A = \mathscr{A}(U).$$

The affine dictionary gives an equivalence between quasi-coherent sheaves on $f^{-1}(U)$ and A-modules. A quasi-coherent $\mathscr{A}$-module $\mathscr{M}$ restricts on U to the sheaf associated to the A-module $\mathscr{M}(U)$; assigning the associated sheaf on $f^{-1}(U)$ is compatible with restriction to distinguished subopens, so these local sheaves glue to a quasi-coherent sheaf on X.

Conversely, for $\mathscr{F} \in \operatorname{QCoh}(X)$, the module $\Gamma(f^{-1}(U), \mathscr{F})$ is naturally an A-module. The affine dictionary and localization show that these modules form a quasi-coherent $\mathscr{A}$-module on Y, namely $f_*\mathscr{F}$. On every affine U, the two constructions are inverse by theorem 5.13; hence they are inverse globally. □

5.4 Vector Bundles and Locally Free Sheaves

We finish by applying the symmetric algebra construction to finite locally free sheaves. The result is the scheme-theoretic form of a vector bundle.

Definition 5.37 (Geometric vector bundle)**.** A *geometric vector bundle of rank r* over a scheme Y is a morphism

$$p : E \longrightarrow Y$$

together with an open cover $Y = \bigcup_i U_i$ and isomorphisms

$$p^{-1}(U_i) \xrightarrow{\sim} \mathbb{A}^r_{U_i}$$

over U_i, such that on overlaps the transition automorphisms are linear. Equivalently, after trivializing on $U_i \cap U_j$, the transition map is given by a matrix

$$g_{ij} \in \mathrm{GL}_r(\mathscr{O}_Y(U_i \cap U_j))$$

after refining the overlap if necessary.

For any morphism $p : E \to Y$, a section over $U \subseteq Y$ is represented by a commutative triangle

$$\begin{tikzcd} & E \arrow[d, "p"] \\ U \arrow[ur, "s"] \arrow[r, hook] & Y, \end{tikzcd}$$

and the sections form a sheaf of sets

$$\mathscr{S}(E/Y)(U) := \{s : U \to E \mid p \circ s = \mathrm{id}_U\}.$$

For a vector bundle, linearity of the transition functions makes this a sheaf of modules.

Proposition 5.38 (Sections of a vector bundle)**.** *If $p : E \to Y$ is a geometric vector bundle of rank r, then $\mathscr{S}(E/Y)$ has a natural structure of a locally free $\mathscr{O}_Y$-module of rank r.*

Proof. The assertion is local on Y. On a trivializing affine open $U = \operatorname{Spec} A$,

$$E|_U \cong \mathbb{A}^r_U = \operatorname{Spec} A[x_1, \dots, x_r].$$

A section $U \to \mathbb{A}^r_U$ is the same as an A-algebra homomorphism

$$A[x_1, \dots, x_r] \longrightarrow A,$$

which is uniquely determined by the r-tuple of images of the variables. Thus

$$\mathscr{S}(E/Y)(U) \cong A^r.$$

Restriction is coordinatewise. On overlaps, the given linear transition maps act on these r-tuples by the same matrices g_{ij}, so the local free module structures glue. Hence $\mathscr{S}(E/Y)$ is locally free of rank r. □

Conversely, a locally free sheaf produces a vector bundle through relative spectrum. The dual is necessary in order that sections of the geometric bundle recover the original sheaf rather than its dual.

Definition 5.39 (Total space of a locally free sheaf)**.** Let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite rank. Define

$$\mathbf{V}(\mathscr{E}) := \operatorname{Spec}_Y \operatorname{Sym}(\mathscr{E}^\vee).$$

The structural morphism $\mathbf{V}(\mathscr{E}) \to Y$ is called the *total space* of $\mathscr{E}$.

Proposition 5.40 (The total space is a vector bundle). *If $\mathscr{E}$ is locally free of rank r, then*

$$\mathbf{V}(\mathscr{E}) \longrightarrow Y$$

is a geometric vector bundle of rank r.

Proof. On an open set U on which $\mathscr{E}|_U \cong \mathscr{O}_U^{\oplus r}$, we also have

$$\mathscr{E}^\vee|_U \cong \mathscr{O}_U^{\oplus r}.$$

Therefore

$$\operatorname{Sym}(\mathscr{E}^\vee)|_U \cong \mathscr{O}_U[T_1, \dots, T_r],$$

and relative spectrum gives

$$\mathbf{V}(\mathscr{E})|_U \cong \mathbb{A}^r_U.$$

If two trivializations of $\mathscr{E}$ differ by a matrix $g \in \mathrm{GL}_r(\mathscr{O}_U)$, the dual trivializations differ by the contragredient matrix, and the induced algebra map on symmetric algebras produces the corresponding linear change of fibre coordinates. Hence the transition maps of the total space are linear. □

The symmetric algebra universal property identifies sections of this bundle.

Proposition 5.41 (Recovering the sheaf of sections). *For a finite locally free sheaf $\mathscr{E}$, there is a natural isomorphism*

$$\mathscr{S}(\mathbf{V}(\mathscr{E})/Y) \cong \mathscr{E}.$$

Proof. Let $U \subseteq Y$ be open. By the universal property of relative spectrum, a section

$$U \longrightarrow \mathbf{V}(\mathscr{E})|_U = \operatorname{Spec}_U \operatorname{Sym}(\mathscr{E}^\vee|_U)$$

is equivalent to an $\mathscr{O}_U$-algebra homomorphism

$$\operatorname{Sym}(\mathscr{E}^\vee|_U) \longrightarrow \mathscr{O}_U.$$

By the universal property of the symmetric algebra, this is equivalent to an $\mathscr{O}_U$-linear map

$$\mathscr{E}^\vee|_U \longrightarrow \mathscr{O}_U,$$

i.e. to a section of $\mathscr{E}^{\vee\vee}$ over U. By proposition 5.33,

$$\mathscr{E}^{\vee\vee} \cong \mathscr{E}.$$

All identifications are natural in U, so they define an isomorphism of sheaves. □

To formulate the correspondence functorially, a morphism of vector bundles over Y is required to be fibrewise linear in local trivializations.

Theorem 5.42 (Vector bundles and locally free sheaves). *For every scheme Y, the constructions*

$$\mathscr{E} \longmapsto \mathbf{V}(\mathscr{E}) \qquad \text{and} \qquad E \longmapsto \mathscr{S}(E/Y)$$

give an equivalence between the category of finite locally free $\mathscr{O}_Y$-modules and the category of geometric vector bundles over Y with fibrewise linear morphisms. Rank is preserved. In rank one this identifies invertible sheaves with line bundles.

Proof. Starting with a finite locally free sheaf $\mathscr{E}$, proposition 5.41 gives a natural isomorphism

$$\mathscr{S}(\mathbf{V}(\mathscr{E})/Y) \cong \mathscr{E}.$$

Conversely, let $p : E \to Y$ be a vector bundle and put

$$\mathscr{E} := \mathscr{S}(E/Y).$$

By proposition 5.38, $\mathscr{E}$ is finite locally free. On each trivializing open U, both $E|_U$ and $\mathbf{V}(\mathscr{E})|_U$ are canonically the affine space $\mathbb{A}^r_U$ once the same frame is chosen. On overlaps, the transition matrices for $\mathscr{E}$ are exactly the matrices describing the linear transition maps of E; dualization followed by relative spectrum recovers the same change of fibre coordinates. Hence the local identifications glue to an isomorphism

$$\mathbf{V}(\mathscr{S}(E/Y)) \xrightarrow{\sim} E.$$

A morphism $\varphi : \mathscr{E} \to \mathscr{F}$ induces

$$\varphi^\vee : \mathscr{F}^\vee \to \mathscr{E}^\vee$$

and therefore an algebra morphism

$$\mathrm{Sym}(\mathscr{F}^\vee) \longrightarrow \mathrm{Sym}(\mathscr{E}^\vee).$$

Relative spectrum reverses this arrow and produces a fibrewise linear morphism

$$\mathbf{V}(\mathscr{E}) \longrightarrow \mathbf{V}(\mathscr{F}).$$

Locally this is the usual matrix map between affine spaces, so the construction is compatible with composition and identities. The preceding natural isomorphisms show that the two functors are quasi-inverse. □

The chapter began with the affine equivalence

$$A\text{-}\mathrm{Mod} \simeq \mathrm{QCoh}(\mathrm{Spec}\, A)$$

and ends with a geometric realization of finite locally free objects. The next step is to study the global sections functor on these sheaves. It is left exact but not exact, and its derived functors will provide the cohomological machinery needed for projective geometry and divisors.

Chapter 6

Sheaf Cohomology

The global-section functor is left exact but need not be exact. Sheaf cohomology measures this failure and, in algebraic geometry, records genuinely global information that cannot be read from a single affine chart. The aim of this chapter is to develop the theory with as little homological overhead as possible. We use the standard formalism of injective resolutions and long exact sequences, but all comparison arguments will be carried out by explicit resolutions, dimension shifting, and Čech complexes; no spectral sequences will be used.

The chapter has two complementary themes. First, we develop enough general sheaf cohomology to understand vanishing and affineness. Second, we turn cohomology into a computational tool on separated and projective schemes. The affine dictionary of theorem 5.13, the projective twisting sheaves of definition 5.21, and Serre's global-generation theorem theorem 5.27 will all reappear in cohomological form.

6.1 Flasque Sheaves and Derived Cohomology

Definition 6.1 (Flasque sheaf)**.** Let X be a topological space. A sheaf $\mathscr{F}$ of abelian groups on X is *flasque* if for every inclusion of open subsets $V \subseteq U$ the restriction map

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V)$$

is surjective. The same terminology is used for sheaves of modules by forgetting the module structure.

Example 6.2 (Skyscraper sheaves)**.** Let $i_x : \{x\} \hookrightarrow X$ and let M be an abelian group. The skyscraper sheaf $i_{x,*}M$ is flasque: every restriction map is either the identity, the zero map $M \to 0$, or the identity of 0.

Example 6.3 (Discontinuous sections)**.** For a sheaf $\mathscr{F}$ of abelian groups define

$$C^0(\mathscr{F})(U) := \prod_{x \in U} \mathscr{F}_x.$$

Restriction is projection onto the factors indexed by the smaller open subset. Thus $C^0(\mathscr{F})$ is flasque. The germ maps define a canonical monomorphism

$$\mathscr{F} \hookrightarrow C^0(\mathscr{F}), \qquad s \longmapsto (s_x)_{x \in U}.$$

A section is determined by its germs, so the map is injective. Iterating this construction after taking cokernels produces the Godement flasque resolution.

We next record the injective objects needed to define cohomology. No general theory of Grothendieck categories is required here.

Theorem 6.4 (Enough injectives)**.** *Let* $(X, \mathscr{O}_X)$ *be a ringed space. The category* $\mathrm{Mod}(X)$ *of sheaves of* $\mathscr{O}_X$*-modules has enough injective objects.*

Proof. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. For each $x \in X$, choose an embedding of $\mathscr{O}_{X,x}$-modules

$$\mathscr{F}_x \hookrightarrow I_x$$

with I_x injective. Give $i_{x,*}I_x$ the natural $\mathscr{O}_X$-module structure through the homomorphisms $\mathscr{O}_X(U) \to \mathscr{O}_{X,x}$ whenever $x \in U$. There is a natural adjunction

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, i_{x,*}I_x) \cong \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{G}_x, I_x).$$

Indeed a morphism into the skyscraper sheaf is determined by its map on the stalk at x, and the module structures match through $\mathscr{O}_{X,x}$. Since the stalk functor is exact and I_x is injective, the functor on the right is exact in $\mathscr{G}$. Hence $i_{x,*}I_x$ is injective.

Set

$$\mathscr{I} := \prod_{x \in X} i_{x,*}I_x.$$

Products of injective objects are injective here because

$$\mathrm{Hom}_{\mathscr{O}_X}\left(\mathscr{G}, \prod_x i_{x,*}I_x\right) \cong \prod_x \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, i_{x,*}I_x),$$

and products of surjective maps of abelian groups are surjective. The chosen stalk embeddings correspond by adjunction to morphisms $\mathscr{F} \to i_{x,*}I_x$ and therefore to a single morphism

$$\mathscr{F} \longrightarrow \prod_{x \in X} i_{x,*}I_x = \mathscr{I}.$$

At a point y, the projection of the stalk map $\mathscr{F}_y \to \mathscr{I}_y$ onto the y-component is precisely the chosen injection $\mathscr{F}_y \hookrightarrow I_y$. Hence the map is injective on every stalk and therefore is a monomorphism of sheaves. □

Definition 6.5 (Sheaf cohomology)**.** Let X be a topological space and let

$$\Gamma(X, -) : \mathbf{Ab}(X) \longrightarrow \mathbf{Ab}$$

be the global-section functor. Its right derived functors are denoted

$$H^i(X, \mathscr{F}) := R^i\Gamma(X, \mathscr{F}).$$

For an $\mathscr{O}_X$-module we take the cohomology of its underlying sheaf of abelian groups. Thus

$$H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F}).$$

We will use the standard formal facts about derived functors: an injective resolution computes H^i, a short exact sequence gives a natural long exact sequence, and any resolution by Γ-acyclic sheaves computes the same groups. These are the only general homological-algebra inputs needed in this chapter.

Proposition 6.6 (Injective sheaves are flasque)**.** *Every injective* $\mathscr{O}_X$*-module is flasque.*

Proof. Let $V \subseteq U$ be open. Write $j_U : U \hookrightarrow X$ and $j_V : V \hookrightarrow X$. For an open immersion $j : W \hookrightarrow X$, let $j_! \mathscr{O}_W$ denote extension by zero. Its stalks are $\mathscr{O}_{X,x}$ for $x \in W$ and 0 otherwise. We use the same notation $j_! \mathscr{A}$ for extension by zero of a sheaf of abelian groups $\mathscr{A}$ on W. The inclusion $V \subseteq U$ induces a monomorphism

$$j_{V,!}\mathscr{O}_V \hookrightarrow j_{U,!}\mathscr{O}_U.$$

For every $\mathscr{O}_X$-module $\mathscr{G}$ there is a natural identification

$$\mathrm{Hom}_{\mathscr{O}_X}(j_{U,!}\mathscr{O}_U, \mathscr{G}) \cong \Gamma(U, \mathscr{G}),$$

and similarly for V. If $\mathscr{I}$ is injective, applying $\mathrm{Hom}_{\mathscr{O}_X}(-, \mathscr{I})$ to the monomorphism gives a surjection. Under the preceding identifications this is exactly the restriction map:

$$\begin{tikzcd} \mathrm{Hom}(j_{U,!}\mathscr{O}_U, \mathscr{I}) \arrow[r, two heads] \arrow[d, "\sim"'] & \mathrm{Hom}(j_{V,!}\mathscr{O}_V, \mathscr{I}) \arrow[d, "\sim"] \\ \mathscr{I}(U) \arrow[r, two heads] & \mathscr{I}(V). \end{tikzcd}$$

Hence $\mathscr{I}$ is flasque. □

Lemma 6.7 (Sections of a quotient by a flasque subsheaf)**.** *Let*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

be exact and suppose $\mathscr{F}$ is flasque. Then for every open $U \subseteq X$ the sequence

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{G}(U) \longrightarrow \mathscr{H}(U) \longrightarrow 0$$

is exact. If $\mathscr{G}$ is also flasque, then $\mathscr{H}$ is flasque.

Proof. Only surjectivity on the right needs proof. Let $s \in \mathscr{H}(U)$. Consider pairs (V, t) with $V \subseteq U$ open and $t \in \mathscr{G}(V)$ lifting $s|_V$, ordered by extension. Every chain has an upper bound obtained by gluing the compatible sections, so Zorn's lemma gives a maximal pair (V, t). If $V \neq U$, choose $x \in U \setminus V$. Since $\mathscr{G} \to \mathscr{H}$ is surjective as a sheaf map, there is an open neighbourhood $W \subseteq U$ of x and a lift $u \in \mathscr{G}(W)$ of $s|_W$. On $V \cap W$ the difference $t - u$ is the image of some $a \in \mathscr{F}(V \cap W)$. Flasqueness extends a to $\widetilde{a} \in \mathscr{F}(W)$. Replacing u by $u + \widetilde{a}$ makes it agree with t on $V \cap W$, so the two sections glue over $V \cup W$, contradicting maximality. Hence $V = U$ and $\mathscr{G}(U) \to \mathscr{H}(U)$ is surjective.

Now suppose $\mathscr{G}$ is flasque and let $V \subseteq U$. Given $h \in \mathscr{H}(V)$, lift it to $g_V \in \mathscr{G}(V)$ by the first part, extend g_V to $g_U \in \mathscr{G}(U)$, and take the image of g_U in $\mathscr{H}(U)$. This extends h, so $\mathscr{H}$ is flasque. □

Proposition 6.8 (Flasque sheaves are acyclic)**.** *If $\mathscr{F}$ is flasque, then*

$$H^i(X, \mathscr{F}) = 0 \qquad (i > 0).$$

Consequently any flasque resolution computes sheaf cohomology.

Proof. Embed $\mathscr{F}$ in an injective sheaf $\mathscr{I}$ and let $\mathscr{Q} = \mathscr{I}/\mathscr{F}$. The injective sheaf $\mathscr{I}$ is flasque by proposition 6.6. Then lemma 6.7 shows that $\mathscr{Q}$ is flasque and

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{I}) \longrightarrow \Gamma(X, \mathscr{Q}) \longrightarrow 0$$

is exact. The long exact cohomology sequence for

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{Q} \to 0$$

therefore gives $H^1(X, \mathscr{F}) = 0$ and, for $q > 0$,

$$H^{q+1}(X, \mathscr{F}) \cong H^q(X, \mathscr{Q}).$$

Repeating the argument with $\mathscr{Q}$ gives the vanishing in all positive degrees. Since flasque sheaves are Γ-acyclic, the standard acyclic-resolution lemma shows that any flasque resolution computes H^i. □

We now prove the first global vanishing theorem. The proof is purely topological and uses Noetherian induction together with extension by zero. The only general input beyond the formal derived-functor facts above is that cohomology on a Noetherian space commutes with filtered colimits of abelian sheaves. We first make the finite devissage step explicit.

Lemma 6.9 (Finite filtration inside the constant sheaf)**.** *Let X be a Noetherian topological space and let*

$$\mathscr{H} \subseteq \underline{\mathbf{Z}}_X$$

be a subsheaf generated by finitely many local sections. Then there is a finite filtration

$$0 = \mathscr{H}_0 \subseteq \mathscr{H}_1 \subseteq \cdots \subseteq \mathscr{H}_N = \mathscr{H}$$

such that every successive quotient $\mathscr{H}_i/\mathscr{H}_{i-1}$ fits into a short exact sequence

$$0 \longrightarrow j_{V,!}\underline{\mathbf{Z}}_V \longrightarrow j_{U,!}\underline{\mathbf{Z}}_U \longrightarrow \mathscr{H}_i/\mathscr{H}_{i-1} \longrightarrow 0$$

for suitable open subsets $V \subseteq U \subseteq X$.

Proof. Choose finitely many generators $s_i \in \underline{\mathbf{Z}}_X(U_i)$. Every open subset of a Noetherian space is quasi-compact. Since s_i is locally constant, U_i has a finite decomposition into open-and-closed subsets on each of which s_i is constant. Replacing one generator by the finitely many resulting constant pieces, discarding zero pieces, and changing signs if necessary, we may therefore assume that the generators are positive integers

$$n_i \in \mathbf{Z}_{>0}$$

viewed as constant sections over the opens U_i.

Enlarge this finite list as follows. For every nonempty subset I of the index set, add the open

$$U_I := \bigcap_{i \in I} U_i$$

when it is nonempty, together with the constant section

$$\gcd(n_i \mid i \in I)$$

on U_I. The enlarged list is still finite and generates the same subsheaf. It has the useful property that, for every $x \in X$, the subgroup $\mathscr{H}_x \subseteq \mathbf{Z}$ is generated by one of the integers occurring on a neighbourhood of x.

For $m \geq 0$, let $\mathscr{H}_{\leq m}$ be the subsheaf generated by those listed sections whose integer is at most m. Only finitely many different subsheaves occur, and for m large we have $\mathscr{H}_{\leq m} = \mathscr{H}$. It is therefore enough to understand the quotient

$$\mathscr{Q}_m := \mathscr{H}_{\leq m}/\mathscr{H}_{\leq m-1}.$$

Put
$$U := \bigcup_{n_i \le m} U_i, \qquad V := \bigcup_{n_i \le m-1} U_i.$$
The class of the constant section m gives a surjection
$$j_{U,!}\underline{\mathbf{Z}}_U \longrightarrow \mathscr{Q}_m.$$
We check its kernel on stalks. If $x \notin U$, both stalks are zero. If $x \in U \setminus V$, every listed generator through x which occurs in $\mathscr{H}_{\le m}$ has value exactly m, so $(\mathscr{Q}_m)_x \cong m\mathbf{Z} \cong \mathbf{Z}$ and the stalk map is an isomorphism. If $x \in V$, the gcd-closure of the list implies that adding the sections of value m does not enlarge the subgroup already generated by values at most $m-1$; hence $(\mathscr{Q}_m)_x = 0$, and the kernel stalk is $\mathbf{Z}$. Thus the kernel is exactly $j_{V,!}\underline{\mathbf{Z}}_V$. Taking the nontrivial successive values of m gives the required finite filtration. □

Lemma 6.10 (Noetherian devissage)**.** *Let X be a Noetherian topological space and let $d \ge 0$. Suppose that*
$$H^p(X, j_!\underline{\mathbf{Z}}_U) = 0 \qquad (p > d)$$
for every open immersion $j : U \hookrightarrow X$. Then
$$H^p(X, \mathscr{F}) = 0 \qquad (p > d)$$
for every sheaf of abelian groups $\mathscr{F}$ on X.

Proof. Because X is Noetherian, every open subset is quasi-compact. We use two standard facts about cohomology on Noetherian spaces: cohomology commutes with filtered colimits of abelian sheaves, and every sheaf is the filtered union of its subsheaves generated by finitely many local sections. These facts are consequences of quasi-compactness of the open basis and exactness of filtered colimits. Hence it suffices to treat a sheaf generated by finitely many local sections.

Induction on the number of generators reduces to a sheaf generated by one section $s \in \mathscr{F}(U)$. Such a sheaf is a quotient of $j_!\underline{\mathbf{Z}}_U$. Let
$$0 \longrightarrow \mathscr{K} \longrightarrow j_!\underline{\mathbf{Z}}_U \longrightarrow \mathscr{F} \longrightarrow 0.$$
The kernel $\mathscr{K}$ is again a filtered union of subsheaves generated by finitely many local sections. Since
$$j_!\underline{\mathbf{Z}}_U \hookrightarrow \underline{\mathbf{Z}}_X,$$
each such finitely generated subsheaf is covered by lemma 6.9. The assumed vanishing for the two extension-by-zero sheaves in every short exact sequence of that filtration, together with the long exact cohomology sequence, gives the same vanishing for each successive quotient and hence for the finitely generated subsheaf itself. Passing to the filtered colimit gives
$$H^p(X, \mathscr{K}) = 0 \qquad (p > d).$$
The long exact sequence above now gives the desired vanishing for $\mathscr{F}$. □

Theorem 6.11 (Grothendieck vanishing)**.** *Let X be a Noetherian topological space of dimension d. Then for every sheaf of abelian groups $\mathscr{F}$ on X,*
$$H^p(X, \mathscr{F}) = 0 \qquad (p > d).$$

Proof. We argue by induction on d and, inside a fixed dimension, by induction on the number of irreducible components.

Suppose first that X has more than one irreducible component. Choose one component $Z \subseteq X$, put $U = X \setminus Z$, and write $j : U \hookrightarrow X$, $i : Z \hookrightarrow X$. For every sheaf $\mathscr{F}$ there is an exact sequence

$$0 \longrightarrow j_! j^* \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow i_* i^* \mathscr{F} \longrightarrow 0.$$

The sheaf $i_* i^* \mathscr{F}$ is supported on the single component Z, while $j_! j^* \mathscr{F}$ is supported on the closed union Z' of the remaining irreducible components. If $k : W \hookrightarrow X$ is a closed immersion and $\mathscr{G}$ is a sheaf on W, then k_* is exact, sends flasque sheaves to flasque sheaves, and satisfies

$$\Gamma(X, k_* \mathscr{G}) = \Gamma(W, \mathscr{G}).$$

Consequently a flasque resolution of $\mathscr{G}$ gives natural isomorphisms

$$H^q(X, k_* \mathscr{G}) \cong H^q(W, \mathscr{G}) \qquad (q \geq 0).$$

A sheaf whose stalks vanish outside a closed subset is canonically the direct image of its restriction to that subset; hence the same observation applies to both outer terms above. Thus the induction on the number of components and the long exact sequence reduce us to the case where X is irreducible.

If $d = 0$ and X is irreducible, then X has no proper nonempty open subset. Hence every sheaf has exact global sections and all higher cohomology vanishes.

Assume now $d > 0$ and X irreducible. By lemma 6.10 it is enough to prove the required vanishing for $j_! \underline{\mathbf{Z}}_U$, where $U \subseteq X$ is open. If $U = \varnothing$ there is nothing to prove. Otherwise U is dense. Let $i : Z = X \setminus U \hookrightarrow X$. Since X is irreducible and Z is proper closed,

$$\dim Z \leq d - 1.$$

There is an exact sequence

$$0 \longrightarrow j_! \underline{\mathbf{Z}}_U \longrightarrow \underline{\mathbf{Z}}_X \longrightarrow i_* \underline{\mathbf{Z}}_Z \longrightarrow 0.$$

The constant sheaf $\underline{\mathbf{Z}}_X$ is flasque because every nonempty open subset of an irreducible space is connected, so all restriction maps between nonempty opens are identities on $\mathbf{Z}$. Hence its higher cohomology vanishes. By the induction hypothesis on dimension,

$$H^q(Z, \underline{\mathbf{Z}}_Z) = 0 \qquad (q > d - 1).$$

The long exact sequence therefore gives

$$H^p(X, j_! \underline{\mathbf{Z}}_U) = 0 \qquad (p > d),$$

and the devissage lemma finishes the proof. □

6.2 Cohomology on Schemes and Affineness

We now return to schemes. The first basic fact is that quasi-coherent sheaves have no higher cohomology on affine schemes. We state it here because it is the cohomological counterpart of the affine dictionary; a second proof by Čech complexes will appear in section 6.3.

Lemma 6.12 (Sections over a quasi-compact open)**.** *Let A be Noetherian, let M be an A-module, and let*

$$U = D(J) = \operatorname{Spec} A \setminus V(J)$$

for an ideal $J \subseteq A$. Then there is a natural isomorphism

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(J^n, M),$$

where the transition map is restriction along $J^{n+1} \subseteq J^n$.

Proof. Because A is Noetherian, the expression on the right depends only on the open set $D(J)$. Indeed, if $\sqrt{J} = \sqrt{J'}$, then some powers of J and J' are contained in one another, so the two directed systems have cofinal subsystems and hence the same colimit.

As $U = D(J)$ varies, put

$$D_J(M) := \varinjlim_n \operatorname{Hom}_A(J^n, M).$$

If $D(J') \subseteq D(J)$, after replacing both ideals by their radicals we have $J' \subseteq J$; restriction of homomorphisms therefore gives

$$D_J(M) \longrightarrow D_{J'}(M).$$

These maps make $U \mapsto D_J(M)$ into a sheaf on the Noetherian space $\operatorname{Spec} A$. Here is the finite-cover check. Every open cover has a finite refinement by opens $D(J_i)$. After replacing all representatives by sufficiently high powers, compatible homomorphisms on the $J_i^{n_i}$ agree on powers of the products $J_i J_j$. Since

$$D(J) = \bigcup_i D(J_i) \iff \sqrt{J} = \sqrt{J_1 + \cdots + J_r},$$

a sufficiently high power of J is contained in a sufficiently high power of $J_1 + \cdots + J_r$. Expanding that power and using the compatibility on pairwise products gives a unique homomorphism $J^N \to M$ representing the glued section. The same clearing-of-powers argument proves uniqueness.

It remains to identify this sheaf. On a principal open $D(f)$ we have

$$D_{(f)}(M) = \varinjlim_n \operatorname{Hom}_A((f^n), M) \xrightarrow{\sim} M_f.$$

Explicitly, a homomorphism $\varphi : (f^n) \to M$ is sent to

$$\frac{\varphi(f^n)}{f^n},$$

and the usual equality criterion in a localization shows that this is well defined and bijective after passage to the colimit. Since principal opens form a basis and $\widetilde{M}(D(f)) = M_f$, the two sheaves are canonically isomorphic. Taking sections on U gives the formula. $\square$

Lemma 6.13 (Injectives on an affine scheme)**.** *Let A be a Noetherian ring and let I be an injective A-module. Then the associated quasi-coherent sheaf $\widetilde{I}$ on $X = \operatorname{Spec} A$ is flasque.*

Proof. Let $V \subseteq U$ be open. Since X is Noetherian, both opens are quasi-compact, so we may write

$$U = D(J), \qquad V = D(K)$$

with J, K radical ideals. The inclusion $V \subseteq U$ is equivalent to $K \subseteq J$. By lemma 6.12, the restriction map is

$$\varinjlim_n \operatorname{Hom}_A(J^n, I) \longrightarrow \varinjlim_n \operatorname{Hom}_A(K^n, I).$$

For every n the inclusion $K^n \subseteq J^n$ and injectivity of I give a surjection

$$\operatorname{Hom}_A(J^n, I) \twoheadrightarrow \operatorname{Hom}_A(K^n, I).$$

Thus every representative of a section over V lifts at the same stage to a representative over U. The restriction map is surjective, so $\widetilde{I}$ is flasque. □

Theorem 6.14 (Affine acyclicity)**.** *Let A be a Noetherian ring, let $X = \operatorname{Spec} A$, and let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then*

$$H^i(X, \mathscr{F}) = 0 \qquad (i > 0).$$

Proof. By the affine dictionary theorem 5.13, write $\mathscr{F} \cong \widetilde{M}$ with $M = \Gamma(X, \mathscr{F})$. Choose an injective resolution of A-modules

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots.$$

The tilde functor is exact by proposition 5.10, so we obtain an exact sequence of sheaves

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{I^0} \longrightarrow \widetilde{I^1} \longrightarrow \cdots.$$

By lemma 6.13, every $\widetilde{I^q}$ is flasque. This is therefore a flasque resolution of $\mathscr{F}$. Taking global sections gives back the original complex because

$$\Gamma(X, \widetilde{I^q}) \cong I^q.$$

Its positive cohomology vanishes, so proposition 6.8 gives the result. □

The converse phenomenon is striking: on a Noetherian scheme, vanishing of the appropriate first cohomology groups forces affineness.

Theorem 6.15 (Serre's affineness criterion)**.** *Let X be a Noetherian scheme. The following are equivalent:*

(i) X is affine;

(ii) $H^i(X, \mathscr{F}) = 0$ for every quasi-coherent $\mathscr{F}$ and every $i > 0$;

(iii) $H^1(X, \mathscr{I}) = 0$ for every coherent sheaf of ideals $\mathscr{I} \subseteq \mathscr{O}_X$.

Proof. The implication (i)⇒(ii) is theorem 6.14, and (ii)⇒(iii) is immediate. Assume (iii).

Let $x \in X$ be a closed point and choose an affine open neighbourhood $U = \operatorname{Spec} A$ of x. Put $Z = X \setminus U$ and $Z' = Z \cup \{x\}$. Since X is Noetherian, the reduced closed subschemes Z and Z' are cut out by coherent quasi-coherent ideal sheaves $\mathscr{I}$ and $\mathscr{I}'$ with $\mathscr{I}' \subseteq \mathscr{I}$. On U, the quotient $\mathscr{I}/\mathscr{I}'$ is the skyscraper sheaf associated to $A/\mathfrak{m}_x$. Hence

$$\Gamma(X, \mathscr{I}/\mathscr{I}') \cong A/\mathfrak{m}_x.$$

The exact sequence

$$0 \longrightarrow \mathscr{I}' \longrightarrow \mathscr{I} \longrightarrow \mathscr{I}/\mathscr{I}' \longrightarrow 0$$

and $H^1(X, \mathscr{I}') = 0$ show that the restriction map $\Gamma(X, \mathscr{I}) \to A/\mathfrak{m}_x$ is surjective. Choose $f \in \Gamma(X, \mathscr{I})$ mapping to 1. Then f vanishes on Z but is a unit at x, so

$$x \in X_f \subseteq U.$$

Inside the affine scheme U, the open X_f is the principal open $D(f|_U)$ and is therefore affine.

Let W be the union of all affine opens of the form X_f with $f \in \Gamma(X, \mathscr{O}_X)$. The preceding argument shows that every closed point lies in W. If $X \setminus W$ were nonempty, then as a nonempty closed subset of the Noetherian space X it would contain a closed point, a contradiction. Hence $W = X$. Quasi-compactness gives finitely many global functions $f_1, \dots, f_r$ such that

$$X = X_{f_1} \cup \cdots \cup X_{f_r}, \qquad X_{f_i} \text{ affine.}$$

Consider the surjection of sheaves

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{O}_X^{\oplus r} \xrightarrow{(f_1,\dots,f_r)} \mathscr{O}_X \longrightarrow 0$$

It is surjective stalkwise because the X_{f_i} cover X. Define

$$\mathscr{K}_j := \mathscr{K} \cap \mathscr{O}_X^{\oplus j} \qquad (0 \le j \le r),$$

where the first j summands are used. Then

$$0 = \mathscr{K}_0 \subseteq \mathscr{K}_1 \subseteq \cdots \subseteq \mathscr{K}_r = \mathscr{K},$$

Projection to the j-th coordinate induces an injection $\mathscr{K}_j/\mathscr{K}_{j-1} \hookrightarrow \mathscr{O}_X$; its image is therefore a quasi-coherent ideal sheaf. Since X is Noetherian and $\mathscr{K}_j$ is coherent, this quotient ideal is coherent. By hypothesis its H^1 vanishes. Induction on j using the long exact sequences gives $H^1(X, \mathscr{K}) = 0$. Taking global sections in the preceding short exact sequence therefore yields a surjection

$$\Gamma(X, \mathscr{O}_X)^{\oplus r} \twoheadrightarrow \Gamma(X, \mathscr{O}_X),$$

so there exist $a_i \in \Gamma(X, \mathscr{O}_X)$ with

$$\sum_{i=1}^{r} a_i f_i = 1.$$

The affineness criterion theorem 2.31 now applies to the affine opens X_{f_i} and gives that X is affine. □

Remark 6.16. For the implication (iii)⇒(i), it is enough to assume that X is quasi-compact and that $H^1(X, \mathscr{I}) = 0$ for every quasi-coherent ideal sheaf. The Noetherian hypothesis is convenient here because every quasi-coherent ideal is locally of finite type and all closed subsets carry coherent defining ideals.

6.3 Čech Cohomology and Comparison

Let X be a topological space, let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering, and write

$$U_{i_0 \cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.$$

Fix a well-ordering on I.

Definition 6.17 (Čech complex)**.** For a sheaf of abelian groups $\mathscr{F}$, define

$$\check{C}^p(\mathfrak{U}, \mathscr{F}) := \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0 \cdots i_p}).$$

The differential is

$$(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \cdots \widehat{i_j} \cdots i_{p+1}}|_{U_{i_0 \cdots i_{p+1}}}.$$

Then $d^2 = 0$. The cohomology of this complex is denoted

$$\check{H}^p(\mathfrak{U}, \mathscr{F}).$$

Proposition 6.18. *For every sheaf $\mathscr{F}$,*

$$\check{H}^0(\mathfrak{U}, \mathscr{F}) \cong \Gamma(X, \mathscr{F}).$$

Proof. A 0-cocycle is a family $s_i \in \mathscr{F}(U_i)$ satisfying

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

The sheaf gluing axiom identifies such compatible families with unique global sections. □

For comparison with derived cohomology it is useful to sheafify the entire construction. Put

$$\check{\mathscr{C}}^p(\mathfrak{U}, \mathscr{F}) := \prod_{i_0 < \cdots < i_p} j_{i_0 \cdots i_p, *}(\mathscr{F}|_{U_{i_0 \cdots i_p}}),$$

where $j_{i_0 \cdots i_p} : U_{i_0 \cdots i_p} \hookrightarrow X$.

Lemma 6.19 (Augmented Čech resolution)**.** *The augmented complex of sheaves*

$$0 \longrightarrow \mathscr{F} \longrightarrow \check{\mathscr{C}}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{\mathscr{C}}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots$$

is exact.

Proof. Exactness may be checked on stalks. Fix $x \in X$ and choose $i \in I$ with $x \in U_i$. On the stalk complex define a homotopy by inserting the fixed index i. If α is a germ of a p-cochain, set schematically

$$(h\alpha)_{i_0 \cdots i_{p-1}} = \pm \alpha_{i, i_0, \ldots, i_{p-1}},$$

with the sign determined by reordering the indices. After shrinking to a neighbourhood of x contained in U_i, every term is defined. A direct alternating-sum computation gives

$$dh + hd = \mathrm{id}\,.$$

Thus the stalk complex is contractible, hence exact. □

Proposition 6.20 (Flasque sheaves and Čech cohomology)**.** *If $\mathscr{F}$ is flasque, then*

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) = 0 \qquad (p > 0).$$

Proof. Each restriction $\mathscr{F}|_{U_{i_0 \cdots i_p}}$ is flasque, its direct image under an open immersion is flasque, and products of flasque sheaves are flasque. Thus lemma 6.19 is a flasque resolution. Its global-section complex is exactly $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$. Since $\mathscr{F}$ itself is flasque and therefore acyclic by proposition 6.8, the positive cohomology of that complex is zero. □

Proposition 6.21 (The comparison map)**.** *For every open cover $\mathfrak{U}$ and every sheaf $\mathscr{F}$ there are natural maps*

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow H^p(X, \mathscr{F}).$$

Proof. Choose an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$. The augmented Čech resolution and the injective resolution both resolve $\mathscr{F}$. The comparison theorem for resolutions gives a morphism of resolutions, unique up to homotopy. Taking global sections and cohomology gives the required map. Naturality follows from the same comparison construction. □

We now specialize to quasi-coherent sheaves on separated schemes. The next lemma is the technical bridge that allows dimension shifting to remain inside the quasi-coherent category.

Lemma 6.22 (Flasque quasi-coherent enlargement)**.** *Let X be a Noetherian separated scheme and let $\mathscr{F}$ be quasi-coherent. Then there exists a monomorphism*

$$\mathscr{F} \hookrightarrow \mathscr{G}$$

with $\mathscr{G}$ both flasque and quasi-coherent.

Proof. Choose a finite affine cover $X = \bigcup_{i=1}^r U_i$ and write

$$U_i = \operatorname{Spec} A_i, \qquad \mathscr{F}|_{U_i} \cong \widetilde{M_i}.$$

Embed M_i into an injective A_i-module I_i. By lemma 6.13, $\widetilde{I_i}$ is flasque and quasi-coherent on U_i.

Let $j_i : U_i \hookrightarrow X$. The direct image $j_{i,*}\widetilde{I_i}$ is flasque. It is also quasi-coherent: for an affine open $V \subseteq X$, separatedness gives that $V \cap U_i$ is affine, and localization on distinguished opens shows that

$$\Gamma(V, j_{i,*}\widetilde{I_i}) = \Gamma(V \cap U_i, \widetilde{I_i})$$

has the affine localization property of theorem 5.13. Thus

$$\mathscr{G} := \bigoplus_{i=1}^{r} j_{i,*}\widetilde{I_i}$$

is quasi-coherent; the sum is finite, hence it is also flasque.

The local injections $\mathscr{F}|_{U_i} \hookrightarrow \widetilde{I_i}$ correspond by adjunction to maps $\mathscr{F} \to j_{i,*}\widetilde{I_i}$. Together they give

$$\mathscr{F} \longrightarrow \bigoplus_i j_{i,*}\widetilde{I_i} = \mathscr{G}.$$

At every point x, choose i with $x \in U_i$; projection to the i-th summand gives the chosen injection on the stalk at x. Hence the global map is a monomorphism. □

Theorem 6.23 (Čech equals sheaf cohomology on separated schemes)**.** *Let X be a Noetherian separated scheme, let*

$$\mathfrak{U} = \{U_1, \ldots, U_r\}$$

be a finite affine open cover, and let $\mathscr{F}$ be quasi-coherent. Then for every $p \geq 0$ the natural comparison map is an isomorphism:

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^p(X, \mathscr{F}).$$

Proof. Because X is separated, every finite intersection $U_{i_0\cdots i_q}$ is affine. By theorem 6.14, quasi-coherent sheaves have no higher cohomology on these intersections.

Embed $\mathscr{F}$ in a flasque quasi-coherent sheaf $\mathscr{G}$ using lemma 6.22, and put $\mathscr{R} = \mathscr{G}/\mathscr{F}$. By corollary 5.17, $\mathscr{R}$ is quasi-coherent. On every finite affine intersection the section functor is exact for this short exact sequence because the first higher cohomology of $\mathscr{F}$ there vanishes. Consequently we obtain a short exact sequence of Čech complexes

$$0 \to \check{C}^\bullet(\mathfrak{U}, \mathscr{F}) \to \check{C}^\bullet(\mathfrak{U}, \mathscr{G}) \to \check{C}^\bullet(\mathfrak{U}, \mathscr{R}) \to 0.$$

Thus there is a long exact sequence in Čech cohomology. The same short exact sequence gives a long exact sequence in sheaf cohomology. Since $\mathscr{G}$ is flasque, both its positive Čech cohomology and its positive sheaf cohomology vanish. The comparison maps form a commutative ladder:

$$\begin{tikzcd} \check{H}^{p-1}(\mathfrak{U}, \mathscr{R}) \arrow[r, "\delta_{\check{C}}"] \arrow[d] & \check{H}^p(\mathfrak{U}, \mathscr{F}) \arrow[r] \arrow[d] & \check{H}^p(\mathfrak{U}, \mathscr{G}) \arrow[d] \\ H^{p-1}(X, \mathscr{R}) \arrow[r, "\delta_H"] & H^p(X, \mathscr{F}) \arrow[r] & H^p(X, \mathscr{G}). \end{tikzcd}$$

For $p > 1$ the horizontal connecting maps on the left are isomorphisms. The case $p = 0$ is proposition 6.18. For $p = 1$, the two long exact sequences begin with compatible exact rows

$$H^0(\mathscr{G}) \longrightarrow H^0(\mathscr{R}) \longrightarrow H^1(\mathscr{F}) \longrightarrow 0$$

and

$$\check{H}^0(\mathfrak{U}, \mathscr{G}) \longrightarrow \check{H}^0(\mathfrak{U}, \mathscr{R}) \longrightarrow \check{H}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow 0.$$

The first two vertical comparison maps are isomorphisms by degree zero, so their cokernels are isomorphic. This proves the case $p = 1$. Induction on p, applied to $\mathscr{R}$, then proves the result in every degree. □

Corollary 6.24 (Affine morphisms and cohomology)**.** *Let $f : X \to Y$ be an affine morphism of Noetherian separated schemes and let $\mathscr{F}$ be quasi-coherent on X. Then*

$$H^p(X, \mathscr{F}) \cong H^p(Y, f_*\mathscr{F})$$

for every $p \geq 0$.

Proof. Choose a finite affine cover $\mathfrak{V} = \{V_i\}$ of Y. Since f is affine, $f^{-1}(V_i)$ and all their finite intersections are affine. The two Čech complexes are termwise identical:

$$\check{C}^\bullet(f^{-1}\mathfrak{V}, \mathscr{F}) =\!=\!=\!=\!=\!= \check{C}^\bullet(\mathfrak{V}, f_*\mathscr{F}).$$

Apply theorem 6.23 to both sides. □

We now remove the separatedness and affine-cover hypotheses in degree one by refining the cover.

Definition 6.25 (Refinement and global Čech cohomology)**.** A cover $\mathfrak{V} = \{V_j\}_{j\in J}$ is a *refinement* of $\mathfrak{U} = \{U_i\}_{i\in I}$ if for every j one chooses an index $c(j)$ with $V_j \subseteq U_{c(j)}$. Restriction gives a morphism of Čech complexes and hence maps

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \check{H}^p(\mathfrak{V}, \mathscr{F}).$$

Different choices of c induce the same map on cohomology. To see this explicitly, extend ordered cochains alternately to arbitrary tuples of indices. If c and c' are two choices, the standard prism operator

$$(h\alpha)_{j_0\cdots j_{p-1}} = \sum_{\nu=0}^{p-1}(-1)^\nu \alpha_{c(j_0),\dots,c(j_\nu),c'(j_\nu),\dots,c'(j_{p-1})}\big|_{V_{j_0\cdots j_{p-1}}}$$

satisfies $c'^* - c^* = dh + hd$. Thus the two refinement maps are chain homotopic and induce the same map on cohomology. Define

$$\check{H}^p(X,\mathscr{F}) := \varinjlim_{\mathfrak{U}} \check{H}^p(\mathfrak{U},\mathscr{F}),$$

where the colimit runs over all open covers ordered by refinement.

Theorem 6.26 (Degree-one comparison for arbitrary spaces)**.** *For every topological space X and every sheaf of abelian groups $\mathscr{F}$, the canonical map*

$$\check{H}^1(X,\mathscr{F}) \xrightarrow{\sim} H^1(X,\mathscr{F})$$

is an isomorphism.

Proof. Embed $\mathscr{F}$ into an injective sheaf $\mathscr{I}$ and put $\mathscr{Q} = \mathscr{I}/\mathscr{F}$. Since $\mathscr{I}$ is injective,

$$H^1(X,\mathscr{F}) \cong \operatorname{coker}(\Gamma(X,\mathscr{I}) \to \Gamma(X,\mathscr{Q})).$$

We first describe the comparison map for a fixed cover $\mathfrak{U} = \{U_i\}$. A Čech cocycle $c = (c_{ij})$ with values in $\mathscr{F}$ is also a cocycle in the flasque sheaf $\mathscr{I}$. By proposition 6.20, there exist $t_i \in \mathscr{I}(U_i)$ such that

$$c_{ij} = t_j - t_i.$$

The images $\bar{t}_i \in \mathscr{Q}(U_i)$ agree on overlaps and therefore glue to $q \in \Gamma(X,\mathscr{Q})$. The class of q modulo $\Gamma(X,\mathscr{I})$ depends only on the class of c and is exactly the comparison map

$$\check{H}^1(\mathfrak{U},\mathscr{F}) \longrightarrow H^1(X,\mathscr{F}).$$

If this class is zero, choose $t \in \Gamma(X,\mathscr{I})$ lifting q. Then $t_i - t|_{U_i}$ is a section of $\mathscr{F}(U_i)$ and

$$c_{ij} = (t_j - t) - (t_i - t),$$

so c is a coboundary. Thus the map for every fixed cover is injective.

For surjectivity after taking the colimit, let a class in $H^1(X,\mathscr{F})$ be represented by $q \in \Gamma(X,\mathscr{Q})$. Since $\mathscr{I} \to \mathscr{Q}$ is surjective as a sheaf map, there is an open cover $\mathfrak{U} = \{U_i\}$ and local lifts $t_i \in \mathscr{I}(U_i)$ of $q|_{U_i}$. Their differences

$$c_{ij} := t_j - t_i \in \mathscr{F}(U_i \cap U_j)$$

form a Čech 1-cocycle, and the preceding construction sends its class to the chosen element of $H^1(X,\mathscr{F})$. Hence every cohomology class comes from some cover. Passing to the colimit gives the desired isomorphism. □

Remark 6.27. Degree one is special. For a fixed cover, or even after taking the direct limit over ordinary open covers, higher Čech cohomology need not agree with derived sheaf cohomology on an arbitrary space. The acyclic-cover theorem theorem 6.23 is therefore a genuine geometric input, not merely a convenient choice of notation.

6.4 Projective Cohomology and Examples

The standard affine cover of projective space is one of the main reasons Čech cohomology is so effective in algebraic geometry. We begin with a concrete curve calculation and then isolate the general computation it uses.

Example 6.28 (A plane curve)**.** Let k be an algebraically closed field and let

$$C = V_+(F) \subseteq \mathbb{P}^2_k$$

be an integral plane curve of degree $d \geq 1$. Put $S = k[x_0, x_1, x_2]$ and let $U_i = D_+(x_i)$ be the standard affine cover. Every finite intersection of the U_i is affine, so theorem 6.23 identifies the Čech groups of this cover with sheaf cohomology.

Before using the defining equation of C, compute the few twists that will occur. The Čech complex of $\mathscr{O}_{\mathbb{P}^2}(n)$ is

$$\prod_i (S_{x_i})_n \longrightarrow \prod_{i<j} (S_{x_i x_j})_n \longrightarrow (S_{x_0 x_1 x_2})_n.$$

Decompose it by the $\mathbf{Z}^3$-weight of a Laurent monomial $x_0^{e_0} x_1^{e_1} x_2^{e_2}$. If all $e_i \geq 0$, its weight complex contributes only in degree 0; if all $e_i < 0$, it occurs only in the last term and contributes in degree 2; in every mixed-sign case, inserting an index with nonnegative exponent gives a contracting homotopy. Consequently

$$H^1(\mathbb{P}^2_k, \mathscr{O}(n)) = 0 \qquad \text{for all } n,$$

$$H^0(\mathbb{P}^2_k, \mathscr{O}) = k, \qquad H^0(\mathbb{P}^2_k, \mathscr{O}(-d)) = 0, \qquad H^2(\mathbb{P}^2_k, \mathscr{O}) = 0,$$

and $H^2(\mathbb{P}^2_k, \mathscr{O}(-d))$ has the Laurent-monomial basis

$$x_0^{-a_0} x_1^{-a_1} x_2^{-a_2}, \qquad a_0, a_1, a_2 \geq 1, \qquad a_0 + a_1 + a_2 = d.$$

Multiplication by the defining equation gives an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-d) \xrightarrow{cdotF} \mathscr{O}_{\mathbb{P}^2} \longrightarrow \mathscr{O}_C \longrightarrow 0.$$

The associated long exact sequence and the preceding calculation give

$$H^0(C, \mathscr{O}_C) \cong k, \qquad H^1(C, \mathscr{O}_C) \cong H^2(\mathbb{P}^2, \mathscr{O}(-d)).$$

The number of triples of positive integers with sum d is

$$\binom{d-1}{2}.$$

Thus

$$h^0(C, \mathscr{O}_C) = 1, \qquad h^1(C, \mathscr{O}_C) = \frac{(d-1)(d-2)}{2}.$$

No regularity assumption was used; the number on the right will reappear in definition 6.39 as the arithmetic genus of the plane curve.

The same weight decomposition works in every dimension and over an arbitrary Noetherian base ring.

Theorem 6.29 (Cohomology of twisting sheaves on projective space)**.** *Let R be a Noetherian ring, put $S = R[x_0, \ldots, x_r]$, and let $r \geq 0$. For every $n \in \mathbf{Z}$,*

$$H^q(\mathbb{P}^r_R, \mathscr{O}(n)) \cong \begin{cases} S_n, & q = 0,\ n \geq 0, \\ \left(\dfrac{1}{x_0 \cdots x_r} R[x_0^{-1}, \ldots, x_r^{-1}]\right)_n, & q = r,\ n \leq -r-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Here the subscript denotes total degree for the grading $\deg(x_i^{-1}) = -1$.

Proof. Let $U_i = D_+(x_i)$. Every finite intersection is affine, so theorem 6.23 computes cohomology from the ordered Čech complex

$$\check{C}^p(\mathfrak{U}, \mathscr{O}(n)) = \bigoplus_{i_0 < \cdots < i_p} \left(S_{x_{i_0} \cdots x_{i_p}}\right)_n.$$

Give every term the finer $\mathbf{Z}^{r+1}$-grading in which $x_0^{e_0} \cdots x_r^{e_r}$ has weight $e = (e_0, \ldots, e_r)$. The differential preserves weight, so the complex is the direct sum of its weight subcomplexes.

Fix e with $\sum e_i = n$ and put

$$N(e) = \{i \mid e_i < 0\}.$$

The monomial of weight e occurs in the summand indexed by $J = \{i_0, \ldots, i_p\}$ exactly when $N(e) \subseteq J$.

If $N(e) = \varnothing$, the weight subcomplex is the ordinary cochain complex of the full r-simplex; it contributes one copy of R in degree 0 and nothing in higher degrees. These weights are precisely the degree-n ordinary monomials, giving S_n.

If $N(e) = \{0, \ldots, r\}$, the weight occurs only in the top term $\check{C}^r$. It contributes one copy of R to H^r and is represented by a Laurent monomial in which every exponent is negative. Summing these weights gives

$$\left(\frac{1}{x_0 \cdots x_r} R[x_0^{-1}, \ldots, x_r^{-1}]\right)_n.$$

Finally suppose $N(e)$ is nonempty and proper. Choose $j \notin N(e)$. Inserting or deleting the vertex j, with the usual alternating sign, defines a contracting homotopy on the weight subcomplex. Hence it has no cohomology. Summing over all weights proves the formula. □

Corollary 6.30 (Field case)**.** *Let k be a field. Then*

$$H^q(\mathbb{P}^r_k, \mathscr{O}(n)) = 0 \qquad (0 < q < r),$$

and for $n \leq -r-1$ the standard monomial bases determine a perfect pairing

$$H^r(\mathbb{P}^r_k, \mathscr{O}(n)) \times H^0(\mathbb{P}^r_k, \mathscr{O}(-n-r-1)) \longrightarrow k,$$

hence

$$H^r(\mathbb{P}^r_k, \mathscr{O}(n)) \cong H^0(\mathbb{P}^r_k, \mathscr{O}(-n-r-1))^\vee.$$

Proof. The vanishing is immediate from theorem 6.29. Pair

$$x_0^{-a_0-1} \cdots x_r^{-a_r-1}$$

with $x_0^{a_0} \cdots x_r^{a_r}$. The degree condition is $\sum a_i = -n-r-1$, and the two displayed monomial bases are dual. □

The next theorem is the cohomological form of Serre's projective theory. Its proof uses only the quotient by sums of twists obtained in corollary 5.28, the preceding explicit computation, and long exact sequences.

Theorem 6.31 (Serre finiteness and vanishing)**.** *Let A be a Noetherian ring, let X be projective over* $\operatorname{Spec} A$*, choose a very ample sheaf $\mathscr{O}_X(1)$, and let $\mathscr{F}$ be coherent. Then:*

(a) each $H^p(X,\mathscr{F})$ is a finitely generated A-module;

(b) for every $p>0$ there exists n_0 such that

$$H^p(X,\mathscr{F}(n))=0 \qquad (n\geq n_0).$$

Proof. Choose a closed immersion $i : X \hookrightarrow \mathbb{P}^r_A$ inducing the given $\mathscr{O}_X(1)$. The pushforward $i_*\mathscr{F}$ is coherent (affinely this is restriction of scalars along a quotient of Noetherian rings), and

$$H^p(X,\mathscr{F})\cong H^p(\mathbb{P}^r_A, i_*\mathscr{F}),$$

because i is affine and corollary 6.24 applies. Thus we may replace X by $\mathbb{P}^r_A$.

By corollary 5.28, there is an exact sequence

$$0\longrightarrow \mathscr{R}\longrightarrow \mathscr{E}\longrightarrow \mathscr{F}\longrightarrow 0,$$

where $\mathscr{E}$ is a finite direct sum of twists $\mathscr{O}(a_j)$ and $\mathscr{R}$ is coherent. By theorem 6.29, the cohomology of every $\mathscr{E}(n)$ is finitely generated over A, and for $p>0$ it vanishes when $n\gg 0$.

For finite generation, argue by descending induction on p. Grothendieck vanishing gives $H^p(\mathbb{P}^r_A,-)=0$ for $p>r$, so the induction starts above r. The long exact sequence contains

$$H^p(\mathscr{E})\longrightarrow H^p(\mathscr{F})\longrightarrow H^{p+1}(\mathscr{R}).$$

The outer modules are finitely generated by the explicit computation and the induction hypothesis. The kernel of $H^p(\mathscr{F})\to H^{p+1}(\mathscr{R})$ is a quotient of the finite module $H^p(\mathscr{E})$, while its image is a submodule of the finite module $H^{p+1}(\mathscr{R})$. Since A is Noetherian, both are finite, and therefore $H^p(\mathscr{F})$ is finite.

For vanishing, again descend on $p>0$. Twist the exact sequence by $\mathscr{O}(n)$:

$$0\longrightarrow \mathscr{R}(n)\longrightarrow \mathscr{E}(n)\longrightarrow \mathscr{F}(n)\longrightarrow 0.$$

The relevant part of the long exact sequence is

$$H^p(\mathscr{E}(n))\longrightarrow H^p(\mathscr{F}(n))\longrightarrow H^{p+1}(\mathscr{R}(n)).$$

For $n\gg 0$, the left group vanishes by theorem 6.29, while the right group vanishes by the descending induction hypothesis. Therefore the middle group vanishes. There are only finitely many degrees $1\leq p\leq r$, so a single n_0 may be chosen for all positive p if desired. □

6.5 Euler Characteristics and Arithmetic Genus

From now on let k be a field and let X be projective over k unless otherwise stated. By theorems 6.11 and 6.31, the cohomology of a coherent sheaf is finite-dimensional and zero in sufficiently high degree.

Definition 6.32 (Euler characteristic)**.** For a coherent sheaf $\mathscr{F}$ on X, define

$$\chi(X, \mathscr{F}) := \sum_{i \geq 0} (-1)^i \dim_k H^i(X, \mathscr{F}).$$

When X is fixed we write simply $\chi(\mathscr{F})$.

Proposition 6.33 (Additivity of Euler characteristic)**.** *For every short exact sequence of coherent sheaves*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

one has

$$\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'').$$

Proof. The associated long exact sequence consists of finite-dimensional vector spaces and is zero in sufficiently high degree. For every finite exact sequence of finite-dimensional vector spaces, the alternating sum of the dimensions is zero. Applying this to the cohomology sequence gives the formula. □

Definition 6.34 (Numerical polynomial)**.** A polynomial $P(t) \in \mathbf{Q}[t]$ is *numerical* if $P(n) \in \mathbf{Z}$ for every $n \in \mathbf{Z}$.

Lemma 6.35 (Discrete antiderivatives)**.** *If $Q(t)$ is a numerical polynomial, then there exists a numerical polynomial $P(t)$ with*

$$P(n) - P(n-1) = Q(n) \qquad (n \in \mathbf{Z}).$$

Proof. The polynomials

$$\binom{t}{m} = \frac{t(t-1)\cdots(t-m+1)}{m!}$$

form a $\mathbf{Z}$-basis of the ring of numerical polynomials. Since

$$\binom{t+1}{m+1} - \binom{t}{m+1} = \binom{t}{m},$$

each basis element admits a numerical discrete antiderivative, and the same is true for every numerical polynomial. □

We will need one elementary base-change observation in order to avoid placing any restriction on the cardinality of k.

Lemma 6.36 (Extension of the ground field)**.** *Let K/k be a field extension, let X be a projective k-scheme, and let $\mathscr{F}$ be coherent. Then*

$$H^i(X_K, \mathscr{F}_K) \cong H^i(X, \mathscr{F}) \otimes_k K$$

for every i.

Proof. Choose a finite affine open cover $\mathfrak{U}$ of X coming from standard affine opens in a projective embedding. All finite intersections are affine. Its pullback $\mathfrak{U}_K$ is an affine-intersection cover of X_K. By theorem 6.23, both cohomologies are computed by their Čech complexes. On every finite affine intersection, base change of the associated module gives

$$\Gamma(U_K, \mathscr{F}_K) \cong \Gamma(U, \mathscr{F}) \otimes_k K.$$

Hence

$$\check{C}^\bullet(\mathfrak{U}_K, \mathscr{F}_K) \xrightarrow{\quad\sim\quad} \check{C}^\bullet(\mathfrak{U}, \mathscr{F}) \otimes_k K.$$

Because K is flat over k, taking cohomology commutes with tensoring by K. □

Theorem 6.37 (Polynomiality of the Euler characteristic)**.** *Let X be a projective k-scheme, let $\mathscr{O}_X(1)$ be very ample, and let $\mathscr{F}$ be coherent. There exists a numerical polynomial*

$$P_{\mathscr{F}}(t) \in \mathbf{Q}[t]$$

such that

$$\chi(\mathscr{F}(n)) = P_{\mathscr{F}}(n) \qquad \text{for every } n \in \mathbf{Z}.$$

Moreover

$$\deg P_{\mathscr{F}} \leq \dim \operatorname{Supp}(\mathscr{F}).$$

Proof. By lemma 6.36, replacing k by the infinite extension $k(t)$ does not change any of the dimensions entering $\chi(\mathscr{F}(n))$; the dimension of the support is also unchanged under a field extension. We may therefore assume k infinite.

We argue by induction on $m = \dim \operatorname{Supp}(\mathscr{F})$. If $\mathscr{F} = 0$ there is nothing to prove. Suppose $m = 0$. Replace the set-theoretic support by the finite scheme-theoretic support $i : Z \hookrightarrow X$ defined by the annihilator of $\mathscr{F}$. Then Z is a zero-dimensional projective scheme, hence a finite affine Artinian scheme, and $\mathscr{F} = i_*\mathscr{G}$ for a coherent $\mathscr{O}_Z$-module $\mathscr{G}$. Every invertible sheaf on an Artinian scheme is trivial: after decomposing into finitely many Artinian local factors, a rank-one projective module is free on each factor. Hence $\mathscr{O}_X(1)|_Z \cong \mathscr{O}_Z$ and $\mathscr{F}(n) \cong \mathscr{F}$ noncanonically for all n. Grothendieck vanishing gives no higher cohomology on Z, and therefore $\chi(\mathscr{F}(n))$ is constant. This is a numerical polynomial of degree 0.

Assume $m > 0$. Choose a projective embedding $X \hookrightarrow \mathbb{P}^r_k$ realizing $\mathscr{O}_X(1)$. Since k is infinite, there is a hyperplane $H = V(s)$ containing no irreducible component of $\operatorname{Supp}(\mathscr{F})$ of dimension m. Multiplication by s gives

$$\mathscr{F}(-1) \xrightarrow{\cdot s} \mathscr{F}.$$

Let $\mathscr{R}$ and $\mathscr{Q}$ be its kernel and cokernel:

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{F}(-1) \xrightarrow{\cdot s} \mathscr{F} \longrightarrow \mathscr{Q} \longrightarrow 0.$$

Outside H, the section s is invertible, so the middle map is an isomorphism. Hence

$$\operatorname{Supp}(\mathscr{R}), \operatorname{Supp}(\mathscr{Q}) \subseteq H \cap \operatorname{Supp}(\mathscr{F}).$$

Because H contains no maximal-dimensional component of $\operatorname{Supp}(\mathscr{F})$,

$$\dim \operatorname{Supp}(\mathscr{R}),\ \dim \operatorname{Supp}(\mathscr{Q}) < m.$$

Twisting the exact sequence by $\mathscr{O}_X(n)$ and using additivity of the Euler characteristic gives

$$\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathscr{Q}(n)) - \chi(\mathscr{R}(n)).$$

By induction, the right-hand side is a numerical polynomial $Q(n)$ of degree at most $m - 1$. By lemma 6.35, there is a numerical polynomial $P(t)$ of degree at most m with

$$P(n) - P(n-1) = Q(n).$$

Therefore the difference

$$D(n) := \chi(\mathscr{F}(n)) - P(n)$$

is independent of n. Adding this constant to P gives a numerical polynomial equal to $\chi(\mathscr{F}(n))$ for all integers n. The degree bound follows simultaneously from the induction. □

Remark 6.38. For $n \gg 0$, Serre vanishing gives

$$P_{\mathscr{F}}(n) = \chi(\mathscr{F}(n)) = \dim_k \Gamma(X, \mathscr{F}(n)),$$

so the polynomial above is the usual Hilbert polynomial of $\mathscr{F}$ with respect to $\mathscr{O}_X(1)$.

Definition 6.39 (Arithmetic genus)**.** Let X be a projective scheme of pure dimension r over an algebraically closed field k. Its *arithmetic genus* is

$$p_a(X) := (-1)^r\bigl(\chi(X, \mathscr{O}_X) - 1\bigr).$$

For a projective curve this becomes

$$p_a(X) = 1 - \chi(\mathscr{O}_X).$$

The preceding plane-curve calculation gives immediately

$$p_a(C) = \frac{(d-1)(d-2)}{2}$$

for every integral plane curve of degree d, including singular ones.

The final result explains the precise birational invariance that can already be proved with the machinery developed so far.

Theorem 6.40 (Birational invariance for regular projective curves)**.** *Let X and Y be regular projective integral curves over an algebraically closed field k. If X and Y are birational over k, then they are isomorphic over k. In particular,*

$$p_a(X) = p_a(Y).$$

Proof. Choose a birational map over k

$$\varphi : X \dashrightarrow Y.$$

By proposition 3.22, it identifies the function fields

$$K(Y) \xrightarrow{\sim} K(X).$$

We use this isomorphism to regard the two function fields as the same field. Since every nonempty open subset of the integral scheme X contains its generic point, φ is defined there. Every non-generic point of X is closed by proposition 4.25. Let $x \in X$ be such a closed point. Since X is regular and one-dimensional, the same proposition implies that $\mathscr{O}_{X,x}$ is a discrete valuation ring with fraction field $K(X)$. The rational map gives the top arrow in the diagram

$$\begin{tikzcd} \operatorname{Spec} K(X) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ \operatorname{Spec} \mathscr{O}_{X,x} \arrow[r] & \operatorname{Spec} k. \end{tikzcd}$$

The structure morphism $Y \to \operatorname{Spec} k$ is proper by corollary 3.63. The valuative criterion theorem 3.45 therefore gives a unique dotted lift

$$\begin{tikzcd} \operatorname{Spec} K(X) \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ \operatorname{Spec} \mathscr{O}_{X,x} \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k. \end{tikzcd}$$

The lift gives a morphism $\operatorname{Spec}\mathscr{O}_{X,x} \to Y$. Choose an affine neighbourhood $W = \operatorname{Spec} B$ of the image of the closed point. Since every open neighbourhood of the closed point of the local scheme $\operatorname{Spec}\mathscr{O}_{X,x}$ is the whole local scheme, the lift factors through W. The ring B is a finitely generated k-algebra, so the finitely many images of a set of generators are represented by regular functions on some open neighbourhood $V_x \subseteq X$ of x; after shrinking V_x, the defining relations hold there as well. Thus the lift comes from an honest morphism $V_x \to W \subseteq Y$.

Doing this for every closed point extends φ to an open neighbourhood of each point of X. The local extensions agree with the original rational map on a dense open subset, and because X is reduced and Y is separated, proposition 3.36 forces them to agree on overlaps. Hence they glue to a morphism

$$f : X \longrightarrow Y$$

extending the original rational map.

Applying the same argument to the inverse birational map produces $g : Y \to X$. The composites $g \circ f$ and $f \circ g$ restrict to the identity on the common dense generic open. Since the targets are separated and the source curves are reduced, proposition 3.36 implies that the composites are the identity everywhere. Therefore f is an isomorphism. Cohomology and Euler characteristic are invariant under isomorphism, so $p_a(X) = p_a(Y)$. □

Remark 6.41 (Why regularity matters). Arithmetic genus is not a birational invariant for arbitrary singular projective curves. For instance a singular plane cubic has arithmetic genus 1, while its normalization is $\mathbb{P}^1$ and has arithmetic genus 0. The theorem above is therefore deliberately stated for regular projective curves. Higher-dimensional birational invariance of $\chi(\mathscr{O}_X)$ under suitable nonsingularity hypotheses belongs naturally to a later birational-geometric development.

Sheaf cohomology now provides both a structural criterion for affineness and a practical method for projective calculations. In later chapters it will interact with divisors, differentials, and duality; already the examples above show how local algebra, projective embeddings, and global topology are tied together by the groups $H^i(X, \mathscr{F})$.

Chapter 7

Divisors and Linear Systems

Chapter 4 attached a discrete valuation to every codimension-one point of a normal Noetherian integral scheme, and Chapter 6 showed that first cohomology can be recovered from transition data on open covers. Divisor theory brings these two themes together. A Weil divisor records orders of zero and pole along codimension-one subvarieties. A Cartier divisor records the same phenomenon locally by rational equations, and its failure to admit one global rational equation is measured by the same cohomology group that classifies invertible sheaves. Finally, sections of invertible sheaves turn divisor classes into maps to projective space.

Throughout the first and third sections, schemes are assumed integral and Noetherian whenever Weil divisors are discussed. Normality will be imposed whenever the codimension-one intersection theorem is used. For an integral scheme X, its function field is denoted by $K(X)$, as in proposition 2.51. We write

$$X^{(1)} := \{x \in X \mid \operatorname{codim}_X \overline{\{x\}} = 1\}.$$

By corollary 4.22, if X is normal and Noetherian then every $x \in X^{(1)}$ determines a normalized valuation

$$v_x : K(X)^\times \longrightarrow \mathbf{Z}.$$

This is the local input from which the chapter begins.

7.1 Weil Divisors and the Divisor Class Group

Definition 7.1 (Prime and Weil divisors)**.** Let X be an integral Noetherian scheme. A *prime divisor* is an integral closed subscheme $Y \subseteq X$ of codimension one, always taken with its reduced structure. Equivalently, it is the closure of a point $\eta_Y \in X^{(1)}$.

The group of *Weil divisors* is the free abelian group

$$\operatorname{Div}(X) := \bigoplus_{Y \subseteq X \text{ prime divisor}} \mathbf{Z}\, Y.$$

Thus a Weil divisor is a finite sum

$$D = \sum_Y n_Y Y, \qquad n_Y \in \mathbf{Z}.$$

It is *effective*, written $D \geq 0$, if $n_Y \geq 0$ for every Y, and its support is

$$\operatorname{Supp}(D) := \bigcup_{n_Y \neq 0} Y.$$

The valuation $v_Y := v_{\eta_Y}$ makes it possible to assign a divisor to a rational function. We first verify that the resulting sum is finite.

Lemma 7.2 (Finiteness of orders)**.** *Let X be an integral Noetherian scheme satisfying (R_1), and let $f \in K(X)^\times$. Then*

$$v_Y(f) \neq 0$$

for only finitely many prime divisors $Y \subseteq X$.

Proof. Choose a finite affine cover

$$X = U_1 \cup \cdots \cup U_r, \qquad U_i = \operatorname{Spec} A_i.$$

Since every nonempty affine open of an integral scheme has fraction field $K(X)$, on U_i we may write

$$f = \frac{a_i}{b_i}, \qquad a_i, b_i \in A_i \setminus \{0\}.$$

Let Y be a prime divisor whose generic point lies in U_i, and let $\mathfrak{p} \subseteq A_i$ be the corresponding height-one prime. If $v_Y(f) > 0$, then $a_i \in \mathfrak{p}$; if $v_Y(f) < 0$, then $b_i \in \mathfrak{p}$. Hence it suffices to prove that a fixed nonzero element $a \in A_i$ belongs to only finitely many height-one primes.

If $\mathfrak{p}$ is height one and contains a, then $\mathfrak{p}$ is minimal over the principal ideal (a). Indeed, a prime

$$(a) \subseteq \mathfrak{q} \subsetneq \mathfrak{p}$$

would satisfy $0 < \operatorname{ht}(\mathfrak{q}) < 1$, impossible because A_i is a domain and $a \neq 0$. A Noetherian ring has only finitely many minimal primes over any ideal. Thus only finitely many height-one primes contain a_i, and similarly only finitely many contain b_i. Summing over the finite affine cover proves the assertion. □

Definition 7.3 (Principal divisor and class group)**.** Under the hypotheses of lemma 7.2, the *principal Weil divisor* of $f \in K(X)^\times$ is

$$\operatorname{div}(f) := \sum_Y v_Y(f) Y.$$

The subgroup of principal divisors is denoted $\operatorname{Prin}(X)$. Two divisors are *linearly equivalent*, written $D \sim D'$, if $D - D'$ is principal. The *Weil divisor class group* is

$$\operatorname{Cl}(X) := \operatorname{Div}(X) / \operatorname{Prin}(X).$$

The next commutative-algebra lemma is the bridge between divisor classes and unique factorization.

Lemma 7.4 (Height-one criterion for unique factorization)**.** *Let A be a Noetherian domain. Then A is a unique factorization domain if and only if every height-one prime ideal of A is principal.*

Proof. Suppose first that A is a UFD, and let $\mathfrak{p}$ be a height-one prime. Choose $0 \neq a \in \mathfrak{p}$ and factor

$$a = u q_1 \cdots q_m$$

with $u \in A^\times$ and q_i irreducible. Since $\mathfrak{p}$ is prime, one q_i belongs to $\mathfrak{p}$. In a UFD every irreducible element is prime, so (q_i) is a nonzero prime contained in $\mathfrak{p}$. Both have height one, hence

$$\mathfrak{p} = (q_i).$$

Conversely, assume every height-one prime is principal. Since A is Noetherian, every nonzero nonunit factors into irreducibles. It remains to show that every irreducible element q is prime. Let $\mathfrak{p}$ be a prime minimal over (q). Krull's principal ideal theorem gives

$$\operatorname{ht}(\mathfrak{p}) \leq 1.$$

Because A is a domain and $q \neq 0$, the prime $\mathfrak{p}$ is nonzero, hence has height one. By assumption $\mathfrak{p} = (r)$. Since $q = rs$ for some $s \in A$ and q is irreducible, one of r, s is a unit. The generator r cannot be a unit because $\mathfrak{p}$ is proper, so s is a unit and

$$(q) = (r) = \mathfrak{p}.$$

Thus (q) is prime. Hence irreducibles are prime and A is a UFD. □

We will repeatedly use the fact that a normal Noetherian domain is determined inside its fraction field by its codimension-one localizations. This is the algebraic form of the principle that regularity of a rational function can be tested along prime divisors.

Lemma 7.5 (Codimension-one intersection theorem)**.** *Let A be a normal Noetherian domain with fraction field K. Then*

$$A = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$$

inside K.

Proof. The inclusion from left to right is clear. For the reverse inclusion, let $x = a/b \in K$ with $a, b \in A$, $b \neq 0$, and suppose $x \in A_{\mathfrak{p}}$ for every height-one prime $\mathfrak{p}$.

Set $M = A/bA$. Since A is normal, Serre's criterion theorem 4.21 says that A satisfies (S_2). Because b is a non-zero-divisor, M satisfies (S_1): if $\mathfrak{p} \supseteq (b)$ has height at least two, then

$$\operatorname{depth} A_{\mathfrak{p}} \geq 2,$$

and quotienting by the regular element b gives

$$\operatorname{depth}(A_{\mathfrak{p}}/bA_{\mathfrak{p}}) \geq 1.$$

Recall that $\operatorname{Ass}(M)$ denotes the set of prime ideals which occur as annihilators of elements of the Noetherian module M. We use the standard Noetherian-module consequence of (S_1) that A/bA has no embedded associated primes; its associated primes are therefore precisely the primes minimal over (b). By Krull's principal ideal theorem all of these have height one.

Let $\overline{a}$ be the class of a in M. For every $\mathfrak{p} \in \operatorname{Ass}(M)$, the assumption $a/b \in A_{\mathfrak{p}}$ gives an element $s \notin \mathfrak{p}$ such that

$$sa \in bA,$$

so $\overline{a}_{\mathfrak{p}} = 0$ in $M_{\mathfrak{p}}$. If $\overline{a} \neq 0$, then the nonzero cyclic submodule $A\overline{a} \subseteq M$ has an associated prime $\mathfrak{q}$, and

$$\mathfrak{q} \in \operatorname{Ass}(A\overline{a}) \subseteq \operatorname{Ass}(M).$$

But localization at an associated prime of a nonzero cyclic module cannot kill its generator, contradicting $\overline{a}_{\mathfrak{q}} = 0$. Hence $\overline{a} = 0$, so $a \in bA$ and $x \in A$. □

Remark 7.6. The only commutative-algebra input in the preceding proof beyond material already used in Chapter 4 is the elementary associated-prime fact for a Noetherian module. One may equivalently take lemma 7.5 as the standard Krull-domain consequence of normality and Noetherianity.

Theorem 7.7 (Class group and unique factorization)**.** *Let A be a Noetherian domain and put $X = \operatorname{Spec} A$. Then*

$$A \text{ is a UFD} \iff A \text{ is normal and } \mathrm{Cl}(X) = 0.$$

Equivalently, for a normal Noetherian domain,

$$\mathrm{Cl}(\operatorname{Spec} A) = 0 \iff A \text{ is a UFD.}$$

Proof. If A is a UFD, then it is integrally closed and hence normal. Let $Y = V(\mathfrak{p})$ be a prime divisor. By lemma 7.4,

$$\mathfrak{p} = (f)$$

for some irreducible $f \in A$. In the DVR $A_{\mathfrak{p}}$, the maximal ideal is generated by f, so $v_{\mathfrak{p}}(f) = 1$. If $\mathfrak{q} \neq \mathfrak{p}$ is another height-one prime, then $f \notin \mathfrak{q}$ and hence f is a unit in $A_{\mathfrak{q}}$, so $v_{\mathfrak{q}}(f) = 0$. Therefore

$$\operatorname{div}(f) = Y.$$

Every prime divisor is principal, hence $\mathrm{Cl}(X) = 0$.

Conversely, suppose A is normal and $\mathrm{Cl}(X) = 0$. Let $\mathfrak{p}$ be a height-one prime and $Y = V(\mathfrak{p})$. Since $[Y] = 0$, there exists $f \in K^{\times}$ with

$$\operatorname{div}(f) = Y.$$

Thus

$$v_{\mathfrak{p}}(f) = 1, \qquad v_{\mathfrak{q}}(f) = 0 \quad (\mathfrak{q} \neq \mathfrak{p},\ \operatorname{ht} \mathfrak{q} = 1).$$

In particular $f \in A_{\mathfrak{q}}$ for every height-one prime $\mathfrak{q}$, so lemma 7.5 gives $f \in A$. Since $v_{\mathfrak{p}}(f) > 0$, we have $f \in \mathfrak{p}$.

Now let $g \in \mathfrak{p}$. Then

$$v_{\mathfrak{p}}(g/f) \geq 0,$$

and for every other height-one prime $\mathfrak{q}$,

$$v_{\mathfrak{q}}(g/f) = v_{\mathfrak{q}}(g) \geq 0.$$

Thus $g/f \in A_{\mathfrak{q}}$ for every height-one $\mathfrak{q}$, and the intersection theorem again gives $g/f \in A$. Therefore $\mathfrak{p} = (f)$. Every height-one prime is principal, so A is a UFD by lemma 7.4. □

Projective space gives the basic non-affine example. Its class group is not trivial because the hyperplane at infinity cannot be the divisor of a global rational function by itself.

Proposition 7.8 (Prime divisors on projective space)**.** *Let k be a field and $n \geq 1$. Every prime divisor on $\mathbf{P}^n_k$ is of the form*

$$V_+(g)$$

for an irreducible homogeneous polynomial

$$g \in k[x_0, \ldots, x_n].$$

Proof. Put $S = k[x_0, \ldots, x_n]$. Let Y be a prime divisor and let $\mathfrak{p} \subseteq S$ be the homogeneous prime defining its closure in $\operatorname{Proj} S$. Choose i such that the generic point of Y lies in $D_+(x_i)$. On the affine chart

$$D_+(x_i) \cong \operatorname{Spec} k\left[\frac{x_0}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right],$$

the intersection $Y \cap D_+(x_i)$ has codimension one. Localizing the homogeneous prime at x_i therefore gives a height-one prime. Since $x_i \notin \mathfrak{p}$, every prime in a chain below $\mathfrak{p}$ also avoids x_i, so localization preserves the height of $\mathfrak{p}$. Hence $\mathfrak{p}$ itself has height one. Since S is a UFD, lemma 7.4 gives

$$\mathfrak{p} = (g)$$

for an irreducible element g. Because $\mathfrak{p}$ is homogeneous, g may be chosen homogeneous. Thus $Y = V_+(g)$. $\square$

Theorem 7.9 (Class group of projective space)**.** *Let $H = V_+(x_0) \subseteq \mathbf{P}^n_k$. Then the degree map induces an isomorphism*

$$\deg : \mathrm{Cl}(\mathbf{P}^n_k) \xrightarrow{\sim} \mathbf{Z},$$

with $[H]$ mapping to 1*. In particular,*

$$\mathrm{Cl}(\mathbf{P}^n_k) \cong \mathbf{Z}[H].$$

Proof. For a prime divisor $Y = V_+(g)$ with g irreducible homogeneous, define

$$\deg Y := \deg g$$

and extend additively to $\mathrm{Div}(\mathbf{P}^n_k)$.

If $d = \deg g$, then the degree-zero homogeneous fraction

$$\frac{g}{x_0^d}$$

defines a rational function on $\mathbf{P}^n_k$. Its divisor is

$$\mathrm{div}\left(\frac{g}{x_0^d}\right) = Y - dH.$$

Hence $Y \sim dH$, and by additivity every divisor D satisfies

$$D \sim (\deg D)H.$$

It remains to check that principal divisors have degree zero. A rational function on projective space may be written as

$$f = \frac{g}{h}$$

with g, h nonzero homogeneous polynomials of the same degree. Factor them into irreducible homogeneous factors:

$$g = u \prod_i g_i^{a_i}, \qquad h = v \prod_j h_j^{b_j}.$$

Then

$$\mathrm{div}(f) = \sum_i a_i V_+(g_i) - \sum_j b_j V_+(h_j),$$

so

$$\deg \mathrm{div}(f) = \deg g - \deg h = 0.$$

Thus degree descends to $\mathrm{Cl}(\mathbf{P}^n_k)$. Every class is a multiple of $[H]$, so the induced map is surjective. If a class has degree zero, the relation $D \sim (\deg D)H$ shows that it is zero. Hence the map is injective. $\square$

Removing a closed subset gives a useful exact sequence. It is the basic computational tool for class groups in the remainder of the section.

Proposition 7.10 (Localization sequence for divisor class groups)**.** *Let X be an integral Noetherian scheme satisfying (R_1), let $Z \subsetneq X$ be closed, and set $U = X \setminus Z$. Let $Z_1, \dots, Z_r$ be the codimension-one irreducible components of Z. Then*

$$\bigoplus_{i=1}^{r} \mathbf{Z}\, Z_i \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0$$

is exact, where Z_i maps to its divisor class.

Proof. A prime divisor $Y \subseteq X$ either lies in Z, in which case $Y \cap U = \varnothing$, or else $Y \cap U$ is a prime divisor on U. Conversely, the closure in X of a prime divisor on U is a prime divisor on X. Thus restriction gives a surjection

$$\mathrm{Div}(X) \twoheadrightarrow \mathrm{Div}(U)$$

whose kernel is freely generated by $Z_1, \dots, Z_r$.

Since U is a nonempty open subset of the integral scheme X, we have $K(U) = K(X)$. The principal divisor of a rational function restricts to the same principal divisor computed on U. Therefore restriction descends to a surjection

$$\mathrm{Cl}(X) \twoheadrightarrow \mathrm{Cl}(U).$$

If $[D]$ maps to zero, then

$$D|_U = \mathrm{div}(f)|_U$$

for some $f \in K(X)^\times$. Hence $D - \mathrm{div}(f)$ restricts to zero on U and is therefore supported on the codimension-one components Z_i. Its class lies in the subgroup generated by $[Z_1], \dots, [Z_r]$, proving exactness. □

Corollary 7.11. *Under the hypotheses of proposition 7.10, if* $\mathrm{codim}_X Z \geq 2$*, then restriction is an isomorphism*

$$\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(X \setminus Z).$$

We now prove the homotopy invariance requested by the geometry of the affine line.

Theorem 7.12 (Homotopy invariance of the class group)**.** *Let X be a normal Noetherian integral scheme, and let*

$$\pi : X \times \mathbf{A}^1 \longrightarrow X$$

be the projection. Pullback of prime divisors,

$$Y \longmapsto Y \times \mathbf{A}^1,$$

induces an isomorphism

$$\pi^* : \mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Cl}(X \times \mathbf{A}^1).$$

Proof. Put

$$Y := X \times \mathbf{A}^1.$$

The scheme Y is again Noetherian, integral, and normal: on an affine chart $\mathrm{Spec}\, A \subseteq X$ its inverse image is $\mathrm{Spec}\, A[t]$, and polynomial extension preserves these properties for a normal

Noetherian domain. Hence its divisor class group is defined in the same codimension-one sense as that of X.

If $Z \subseteq X$ is a prime divisor, then $Z \times \mathbf{A}^1$ is again a prime divisor: on an affine chart $U = \operatorname{Spec} A$ meeting the generic point of Z, it is defined by the height-one prime $\mathfrak{p}A[t]$, where $\mathfrak{p} \subseteq A$ defines $Z \cap U$. Moreover, for every $f \in K(X)^\times$,

$$v_{Z\times\mathbf{A}^1}(f) = v_Z(f),$$

because $\mathscr{O}_{Y,\eta_{Z\times\mathbf{A}^1}}$ is obtained from the DVR $\mathscr{O}_{X,\eta_Z}$ by adjoining t and localizing at the extension of its maximal ideal. Hence

$$\pi^* \operatorname{div}_X(f) = \operatorname{div}_Y(f),$$

so pullback is well defined on divisor class groups.

We show first that π^* is surjective. Let $W \subseteq Y$ be a prime divisor. Choose a nonempty affine open $U = \operatorname{Spec} A \subseteq X$ such that $W \cap (U \times \mathbf{A}^1)$ is nonempty, and let

$$\mathfrak{q} \subseteq A[t]$$

be the height-one prime defining this intersection. Put $K = \operatorname{Frac}(A) = K(X)$.

If $\mathfrak{p} := \mathfrak{q} \cap A$ is nonzero, then $\mathfrak{p}A[t]$ is a nonzero prime contained in $\mathfrak{q}$. Polynomial extension preserves height, so

$$\operatorname{ht}(\mathfrak{p}A[t]) = \operatorname{ht}(\mathfrak{p}) \geq 1.$$

Since $\operatorname{ht}(\mathfrak{q}) = 1$, we obtain

$$\operatorname{ht}(\mathfrak{p}) = 1, \qquad \mathfrak{q} = \mathfrak{p}A[t].$$

Thus W is the pullback of the prime divisor obtained by closing $V(\mathfrak{p}) \subseteq U$ in X.

Suppose instead that $\mathfrak{q} \cap A = (0)$. Localizing at $A \setminus \{0\}$ gives a nonzero height-one prime

$$\mathfrak{q}K[t] \subseteq K[t].$$

Since $K[t]$ is a PID, there is an irreducible polynomial $P(t) \in K[t]$ with

$$\mathfrak{q}K[t] = (P).$$

Multiply P by a nonzero element of A so that it is represented by a polynomial $G \in A[t]$. In the principal divisor of the rational function G on Y, the only horizontal prime divisor is W, and it occurs with multiplicity one. Indeed every horizontal prime divisor meets $U \times \mathbf{A}^1$ in a height-one prime contracting to (0), and after localization to $K[t]$ the horizontal part of the divisor is exactly the divisor of the irreducible polynomial P. All other components of $\operatorname{div}_Y(G)$ are vertical, hence pullbacks of prime divisors on X by the preceding case. Therefore the class $[W]$ lies in the image of π^*.

Every prime divisor on Y is of one of these two types, so π^* is surjective.

For injectivity, suppose

$$\pi^* D = \operatorname{div}_Y(F)$$

for a Weil divisor D on X and $F \in K(X)(t)^\times$. Factor F in the UFD $K(X)[t]$:

$$F = c \prod_{i=1}^{m} P_i(t)^{n_i}, \qquad c \in K(X)^\times,$$

where the P_i are pairwise nonassociate irreducible polynomials and $n_i \in \mathbf{Z}$. Each P_i determines, after clearing denominators on a nonempty affine open of X, a horizontal prime divisor of Y whose coefficient in $\mathrm{div}_Y(F)$ is n_i. But π^*D has only vertical components. Hence every $n_i = 0$, and therefore

$$F \in K(X)^\times.$$

For each prime divisor $Z \subseteq X$ we already observed that

$$v_{Z\times\mathbf{A}^1}(F) = v_Z(F).$$

Comparing coefficients in

$$\pi^*D = \mathrm{div}_Y(F)$$

gives

$$D = \mathrm{div}_X(F).$$

Thus $[D] = 0$, proving injectivity. □

Example 7.13 (Affine and projective space)**.** Since polynomial rings over a field are UFDs,

$$\mathrm{Cl}(\mathbf{A}^n_k) = 0.$$

Repeatedly applying theorem 7.12 gives the same conclusion from $\mathrm{Cl}(\operatorname{Spec} k) = 0$. By contrast,

$$\mathrm{Cl}(\mathbf{P}^n_k) \cong \mathbf{Z},$$

generated by a hyperplane. Passing from affine to projective space therefore creates one new divisor class, represented by the hyperplane at infinity.

Example 7.14 (The quadric cone)**.** Let

$$A = k[x, y, z]/(xy - z^2), \qquad X = \operatorname{Spec} A,$$

and assume k is a field. The ring A is normal. Indeed it is a two-dimensional domain. Every height-one localization is regular: if a height-one prime contained both x and y, then it would also contain z and hence the maximal ideal (x, y, z) of height two, impossible; therefore either x or y is invertible, and after inverting one of them the ring is a localization of a polynomial ring. Thus (R_1) holds. Moreover x is a non-zero-divisor and y remains a non-zero-divisor in

$$A/(x) \cong k[y, z]/(z^2),$$

so the depth at the vertex is at least two; away from the vertex the ring is regular. Hence (S_2) holds, and normality follows from theorem 4.21.

Let

$$D = V(x, z).$$

The open set $D(x)$ is affine with

$$A_x \cong k[x, x^{-1}, z],$$

a UFD. The complement of $D(x)$ has the single codimension-one component D, so proposition 7.10 shows that $\mathrm{Cl}(X)$ is generated by $[D]$.

At the generic point of D, the element y is a unit and the relation gives

$$x = \frac{z^2}{y}.$$

Thus z is a uniformizer and

$$v_D(x) = 2.$$

Hence

$$\operatorname{div}(x) = 2D,$$

so $2[D] = 0$.

The class $[D]$ is not zero. If it were, the argument in the proof of theorem 7.7 would make the height-one prime (x, z) principal. Localize at the vertex $\mathfrak{m} = (x, y, z)$. The ideal

$$(x, z)A_{\mathfrak{m}}$$

requires two generators: the images of x and z are linearly independent in

$$(x, z)A_{\mathfrak{m}}/\mathfrak{m}(x, z)A_{\mathfrak{m}}.$$

By Nakayama's lemma it is not principal. Therefore

$$\operatorname{Cl}(X) \cong \mathbf{Z}/2\mathbf{Z}.$$

This is a concrete normal affine scheme which is not factorial.

Example 7.15 (Dedekind schemes and ideal classes)**.** Let A be a Dedekind domain and $X = \operatorname{Spec} A$. By proposition 4.25, the prime divisors of X are exactly the nonzero prime ideals $\mathfrak{p} \subset A$. Therefore

$$\operatorname{Div}(X) = \bigoplus_{\mathfrak{p}\neq 0} \mathbf{Z}\,\mathfrak{p}.$$

Unique factorization of nonzero fractional ideals gives an isomorphism from this free abelian group to the group of nonzero fractional ideals,

$$\sum_{\mathfrak{p}} n_{\mathfrak{p}}\mathfrak{p} \longmapsto \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}.$$

For $f \in \operatorname{Frac}(A)^\times$, the principal divisor $\operatorname{div}(f)$ maps to the principal fractional ideal (f). Hence

$$\operatorname{Cl}(\operatorname{Spec} A)$$

is canonically the usual ideal class group of the Dedekind domain A. In particular, for the ring of integers $\mathscr{O}_K$ of a number field, divisor classes on the arithmetic curve $\operatorname{Spec} \mathscr{O}_K$ recover ideal classes.

7.2 Cartier Divisors and Cohomological Classification

We now change perspective. A Weil divisor records integers at codimension-one points. A Cartier divisor is instead a global section of a quotient sheaf: locally it is represented by one rational equation, and two local equations are identified when they differ by a unit. This sheaf-theoretic formulation makes the relation with first cohomology immediate.

Definition 7.16 (The sheaf of rational functions)**.** Let X be an integral scheme. Define $\mathscr{K}_X$ by

$$\mathscr{K}_X(U) = K(X)$$

for every nonempty open subset $U \subseteq X$, with identity restriction maps; on the empty set take the zero ring. Its sheaf of units is denoted $\mathscr{K}_X^\times$. Thus

$$\mathscr{K}_X^\times(U) = K(X)^\times$$

for every nonempty open U.

Remark 7.17. For a general scheme one replaces $K(X)$ by the total quotient ring obtained by inverting local non-zero-divisors. The integral case is sufficient for the divisor theory developed here and keeps the relation with the function field transparent.

Lemma 7.18 (The rational function sheaf is flasque)**.** *Let X be integral. Then $\mathscr{K}_X^\times$ is flasque. Consequently*

$$H^i(X, \mathscr{K}_X^\times) = 0 \qquad (i > 0).$$

Proof. If $\varnothing \neq V \subseteq U$, both groups of sections are canonically $K(X)^\times$, and the restriction map is the identity. Restrictions to the empty set are automatically surjective. Hence $\mathscr{K}_X^\times$ is flasque. The cohomological vanishing follows from proposition 6.8. □

Definition 7.19 (Cartier divisor)**.** Let X be an integral scheme. A *Cartier divisor* is a global section of

$$\mathscr{K}_X^\times / \mathscr{O}_X^\times.$$

The group of Cartier divisors is

$$\mathrm{CaDiv}(X) := \Gamma\Big(X, \mathscr{K}_X^\times / \mathscr{O}_X^\times\Big).$$

A Cartier divisor is *principal* if it lies in the image of

$$K(X)^\times = \Gamma(X, \mathscr{K}_X^\times) \longrightarrow \mathrm{CaDiv}(X).$$

The principal Cartier divisor defined by f is denoted $\mathrm{div}_C(f)$. The *Cartier divisor class group* is

$$\mathrm{CaCl}(X) := \mathrm{CaDiv}(X)/\{\text{principal Cartier divisors}\}.$$

The quotient definition immediately produces the cohomological classification.

Theorem 7.20 (Cartier classes as first cohomology)**.** *For every integral scheme X, the connecting homomorphism in the cohomology sequence of*

$$1 \longrightarrow \mathscr{O}_X^\times \longrightarrow \mathscr{K}_X^\times \longrightarrow \mathscr{K}_X^\times / \mathscr{O}_X^\times \longrightarrow 1$$

induces a natural isomorphism

$$\mathrm{CaCl}(X) \xrightarrow{\sim} H^1(X, \mathscr{O}_X^\times).$$

Proof. The relevant part of the long exact sequence is

$$K(X)^\times \longrightarrow \mathrm{CaDiv}(X) \xrightarrow{\ \delta\ } H^1(X, \mathscr{O}_X^\times) \longrightarrow H^1(X, \mathscr{K}_X^\times).$$

By lemma 7.18, the last group is zero. Exactness therefore makes δ surjective, and its kernel is exactly the image of $K(X)^\times$, namely the subgroup of principal Cartier divisors. Passing to the quotient gives

$$\mathrm{CaCl}(X) \cong H^1(X, \mathscr{O}_X^\times).$$

□

We next prove independently that the same cohomology group classifies invertible sheaves. This is the point at which the degree-one Čech comparison from Chapter 6 becomes a geometric theorem rather than merely a computational device.

Definition 7.21 (Picard group)**.** For a scheme X, the *Picard group* $\mathrm{Pic}(X)$ is the set of isomorphism classes of invertible $\mathscr{O}_X$-modules, with group law induced by tensor product:

$$[\mathscr{L}] + [\mathscr{M}] := [\mathscr{L} \otimes \mathscr{M}].$$

The inverse of $[\mathscr{L}]$ is $[\mathscr{L}^\vee]$.

Theorem 7.22 (Picard group and first cohomology)**.** *For every scheme X, there is a natural isomorphism*

$$\mathrm{Pic}(X) \xrightarrow{\sim} H^1(X, \mathscr{O}_X^\times).$$

More explicitly,

$$\mathrm{Pic}(X) \cong \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{O}_X^\times) \cong H^1(X, \mathscr{O}_X^\times),$$

where the colimit runs over open covers ordered by refinement.

Proof. Let $\mathscr{L}$ be invertible. Choose an open cover $\mathfrak{U} = \{U_i\}$ and local frames

$$e_i \in \Gamma(U_i, \mathscr{L})$$

so that $\mathscr{L}|_{U_i} = \mathscr{O}_{U_i} e_i$. On $U_i \cap U_j$ there is a unique unit

$$g_{ij} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^\times)$$

satisfying

$$e_i = g_{ij} e_j.$$

On triple overlaps,

$$e_i = g_{ij} g_{jk} e_k = g_{ik} e_k,$$

so

$$g_{ij} g_{jk} = g_{ik}.$$

Thus (g_{ij}) is a multiplicative Čech 1-cocycle.

If the frame is changed to $e_i' = u_i e_i$ with $u_i \in \mathscr{O}_X^\times(U_i)$, then

$$g_{ij}' = u_i g_{ij} u_j^{-1},$$

which differs from g_{ij} by a multiplicative coboundary. If the trivializing cover is refined, the cocycle simply restricts. Hence the isomorphism class of $\mathscr{L}$ determines a well-defined element of

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{O}_X^\times).$$

Conversely, let (g_{ij}) be a multiplicative 1-cocycle on a cover $\mathfrak{U}$. Take a copy $\mathscr{O}_{U_i} e_i$ of the trivial line bundle on each U_i and identify the restrictions on overlaps by

$$e_i = g_{ij} e_j.$$

The cocycle condition is exactly the compatibility condition on triple overlaps:

$$\begin{tikzcd}
\mathscr{O}_{U_i \cap U_j \cap U_k} e_k \arrow[r, "g_{jk}"] \arrow[dr, "g_{ik}"'] & \mathscr{O}_{U_i \cap U_j \cap U_k} e_j \arrow[d, "g_{ij}"] \\
& \mathscr{O}_{U_i \cap U_j \cap U_k} e_i.
\end{tikzcd}$$

The gluing theorem for sheaves therefore produces an invertible sheaf $\mathscr{L}$. Replacing (g_{ij}) by a cohomologous cocycle amounts to rescaling the local frames and produces an isomorphic invertible sheaf. Passing to a refinement does not change the glued sheaf. Thus we obtain a map in the opposite direction.

The two constructions are inverse: starting from an invertible sheaf and gluing the trivializations by its transition functions recovers the original sheaf, while starting from a cocycle and then reading off the transition functions returns the same cocycle class. Tensor products multiply transition functions, so the bijection respects the group structures. Hence

$$\operatorname{Pic}(X) \cong \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{O}_X^\times).$$

Finally, theorem 6.26 identifies the colimit of degree-one Čech cohomology with derived sheaf cohomology:

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{O}_X^\times) \xrightarrow{\sim} H^1(X, \mathscr{O}_X^\times).$$

This completes the proof. □

Combining the two cohomological classifications gives the central correspondence of the section.

Corollary 7.23 (Cartier classes and invertible sheaves)**.** *For every integral scheme X, there is a natural isomorphism*

$$\operatorname{CaCl}(X) \xrightarrow{\sim} \operatorname{Pic}(X).$$

The quotient-sheaf definition is conceptual, but calculations require local equations. We now recover them without changing the underlying definition.

Proposition 7.24 (Local equations for Cartier divisors)**.** *A Cartier divisor D can be represented by data*

$$\{(U_i, f_i)\}_{i\in I}$$

where $\{U_i\}$ is an open cover,

$$f_i \in K(X)^\times,$$

and

$$\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathscr{O}_X^\times)$$

for every i, j. Two such collections represent the same Cartier divisor if their local equations differ by units on a common refinement.

Proof. A global section of the quotient sheaf $\mathscr{K}_X^\times/\mathscr{O}_X^\times$ is locally represented by sections of $\mathscr{K}_X^\times$. Thus after choosing an open cover, D has lifts $f_i \in K(X)^\times$. Two local lifts define the same section of the quotient on an overlap precisely when their quotient is a unit. Conversely, compatible local classes glue uniquely to a global section of the quotient sheaf. □

Definition 7.25 (The invertible sheaf associated to a Cartier divisor)**.** Let D be represented by local equations (U_i, f_i). Define a subsheaf of $\mathscr{K}_X$ by

$$\mathscr{O}_X(D)|_{U_i} := f_i^{-1}\mathscr{O}_{U_i}.$$

Proposition 7.26 (Basic identities for $\mathscr{O}_X(D)$)**.** *The sheaf $\mathscr{O}_X(D)$ is well defined and invertible. Moreover,*

$$\mathscr{O}_X(D+E) \cong \mathscr{O}_X(D) \otimes \mathscr{O}_X(E),$$
$$\mathscr{O}_X(-D) \cong \mathscr{O}_X(D)^\vee,$$

and if D is principal, then

$$\mathscr{O}_X(D) \cong \mathscr{O}_X.$$

Under the isomorphisms

$$\mathrm{CaCl}(X) \cong H^1(X, \mathscr{O}_X^\times) \cong \mathrm{Pic}(X),$$

the class of D maps to the class of $\mathscr{O}_X(D)$.

Proof. On $U_i \cap U_j$, the quotient f_i/f_j is a unit, so

$$f_i^{-1}\mathscr{O}_X = f_j^{-1}\mathscr{O}_X$$

as subsheaves of $\mathscr{K}_X$. Hence the local definitions glue. Multiplication by f_i^{-1} identifies

$$\mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{O}_X(D)|_{U_i},$$

so $\mathscr{O}_X(D)$ is invertible.

If D and E are locally represented by f_i and g_i, then $D+E$ is represented by f_ig_i, and multiplication gives the local isomorphisms

$$f_i^{-1}\mathscr{O}_{U_i} \otimes g_i^{-1}\mathscr{O}_{U_i} \xrightarrow{\sim} (f_ig_i)^{-1}\mathscr{O}_{U_i}.$$

They agree on overlaps and glue. Taking $E = -D$ gives the duality statement. If $D = \mathrm{div}_C(f)$, then

$$\mathscr{O}_X(D) = f^{-1}\mathscr{O}_X,$$

and multiplication by f trivializes it.

For the compatibility with cohomology, the connecting homomorphism in theorem 7.20 is computed by choosing the local lifts f_i. With the multiplicative Čech convention inherited from Chapter 6, it is represented by

$$g_{ij} = \frac{f_j}{f_i}.$$

On U_i take the local frame

$$e_i = f_i^{-1}$$

of $\mathscr{O}_X(D)$. Then on overlaps

$$e_i = f_i^{-1} = \frac{f_j}{f_i} f_j^{-1} = g_{ij}e_j.$$

Thus the connecting cocycle is exactly the transition cocycle of $\mathscr{O}_X(D)$. The two routes in the diagram

$$\begin{tikzcd}
\mathrm{CaCl}(X) \arrow[r, "\sim"] \arrow[dr, "D \mapsto \mathscr{O}_X(D)"'] & H^1(X, \mathscr{O}_X^\times) \arrow[d, "\sim"] \\
& \mathrm{Pic}(X)
\end{tikzcd}$$

therefore agree. ∎

Definition 7.27 (Effective Cartier divisor)**.** A Cartier divisor D on an integral scheme is *effective* if it admits local equations (U_i, f_i) with

$$f_i \in \Gamma(U_i, \mathscr{O}_X) \setminus \{0\}.$$

Its support is the closed subset on which its local equations fail to be units.

Proposition 7.28 (Effective Cartier divisors and invertible ideals)**.** *Let D be effective. The ideals*

$$(f_i) \subseteq \mathscr{O}_X(U_i)$$

glue to an invertible ideal sheaf $\mathscr{I}_D \subseteq \mathscr{O}_X$, and

$$\mathscr{I}_D \cong \mathscr{O}_X(-D).$$

Conversely, a closed subscheme whose ideal sheaf is locally generated by one non-zero-divisor determines an effective Cartier divisor.

If X is locally Noetherian, every irreducible component of $\mathrm{Supp}(D)$ *has codimension one.*

Proof. On overlaps, f_i/f_j is a unit, so $(f_i) = (f_j)$. The ideals therefore glue to $\mathscr{I}_D$. Since X is integral, every nonzero f_i is a non-zero-divisor, and multiplication by f_i identifies

$$\mathscr{O}_{U_i} \xrightarrow{\sim} (f_i).$$

Thus $\mathscr{I}_D$ is invertible. By definition,

$$\mathscr{O}_X(-D)|_{U_i} = f_i \mathscr{O}_{U_i},$$

so $\mathscr{I}_D \cong \mathscr{O}_X(-D)$.

Conversely, if $\mathscr{I}_Z$ is locally generated by non-zero-divisors f_i, then two generators differ by a unit on overlaps. The pairs (U_i, f_i) therefore define an effective Cartier divisor.

For the codimension statement, let Y be an irreducible component of $\mathrm{Supp}(D)$ with generic point η_Y. Choose an affine neighbourhood $U = \operatorname{Spec} A$ on which D is cut out by a nonzero $f \in A$. The point η_Y corresponds to a prime $\mathfrak{p}$ minimal over (f). Krull's principal ideal theorem gives $\mathrm{ht}(\mathfrak{p}) \leq 1$, while $\mathfrak{p} \neq (0)$ because A is a domain and $f \neq 0$. Hence $\mathrm{ht}(\mathfrak{p}) = 1$. □

There is a useful refinement of the correspondence between divisors and line bundles. A Cartier divisor is not merely a line bundle: it is a line bundle together with a rational trivialization.

Proposition 7.29 (Rational sections and Cartier divisors)**.** *Let X be integral and let $\mathscr{L}$ be invertible. A nonzero rational section s of $\mathscr{L}$ determines a Cartier divisor* $\mathrm{div}(s)$ *and a canonical isomorphism*

$$\mathscr{L} \cong \mathscr{O}_X(\mathrm{div}(s)).$$

Conversely, every Cartier divisor arises in this way from the rational section 1 *of $\mathscr{O}_X(D) \subseteq \mathscr{K}_X$.*

Proof. Choose a trivializing cover $\{U_i\}$ with local frames e_i and write the rational section as

$$s = f_i e_i, \qquad f_i \in K(X)^\times.$$

If $e_i = g_{ij} e_j$, then equality of the rational section on overlaps gives

$$f_j = f_i g_{ij}.$$

Hence $f_i/f_j = g_{ij}^{-1}$ is a unit, so the f_i define a Cartier divisor $D = \operatorname{div}(s)$. The local morphisms

$$\mathscr{L}|_{U_i} \longrightarrow \mathscr{O}_X(D)|_{U_i}, \qquad e_i \longmapsto f_i^{-1},$$

are compatible on overlaps and glue to an isomorphism.

Conversely, if D has local equations f_i, the rational section 1 of the subsheaf

$$\mathscr{O}_X(D) \subseteq \mathscr{K}_X$$

is written in the local frame f_i^{-1} as

$$1 = f_i(f_i^{-1}).$$

Its local coefficients are precisely the equations f_i, so its divisor is D. $\square$

7.3 Cartier and Weil Divisors

We now compare the two languages. Assume throughout this section that X is a normal Noetherian integral scheme. By corollary 4.22, every prime divisor Y carries a valuation v_Y.

Definition 7.30 (Order of a Cartier divisor)**.** Let $D \in \operatorname{CaDiv}(X)$ and let Y be a prime divisor. Choose a local equation

$$f \in K(X)^\times$$

for D on a neighbourhood of the generic point η_Y. Define

$$\operatorname{ord}_Y(D) := v_Y(f).$$

Lemma 7.31. *The integer* $\operatorname{ord}_Y(D)$ *is independent of the choice of local equation. For fixed* D*, it is nonzero for only finitely many prime divisors* Y*.*

Proof. If f and g are two local equations near η_Y, then f/g is a unit in the DVR $\mathscr{O}_{X,\eta_Y}$, so

$$v_Y(f) - v_Y(g) = v_Y(f/g) = 0.$$

Thus the order is well defined.

Choose a finite affine open cover on which D is represented by rational functions $f_1, \dots, f_r$. If $\eta_Y \in U_i$, then

$$\operatorname{ord}_Y(D) = v_Y(f_i).$$

By lemma 7.2, each fixed f_i has nonzero valuation at only finitely many prime divisors. The cover is finite, so only finitely many Y occur for D. $\square$

Definition 7.32 (Associated Weil divisor)**.** The Weil divisor associated to D is

$$[D]_W := \sum_Y \operatorname{ord}_Y(D) Y.$$

Proposition 7.33. *The map*

$$\operatorname{CaDiv}(X) \longrightarrow \operatorname{Div}(X), \qquad D \longmapsto [D]_W$$

is a group homomorphism. It sends principal Cartier divisors to the corresponding principal Weil divisors and effective Cartier divisors to effective Weil divisors.

Proof. Near the generic point of Y, if D and E are represented by f and g, then $D+E$ is represented by fg. Hence

$$\mathrm{ord}_Y(D+E) = v_Y(fg) = v_Y(f) + v_Y(g).$$

This proves additivity. If $D = \mathrm{div}_C(f)$, then the same rational function is a local equation everywhere, so

$$[D]_W = \sum_Y v_Y(f)Y = \mathrm{div}(f).$$

If D is effective, its local equation at η_Y lies in the DVR $\mathscr{O}_{X,\eta_Y}$, and therefore has nonnegative valuation. □

The normality assumption now becomes decisive. A rational function whose orders vanish at every prime divisor is already a unit locally.

Theorem 7.34 (Cartier divisors are locally principal Weil divisors)**.** *Let X be normal, Noetherian, and integral. The natural homomorphism*

$$\mathrm{CaDiv}(X) \longrightarrow \mathrm{Div}(X)$$

is injective. Its image is precisely the subgroup of Weil divisors which are locally principal: for every $x \in X$, there is a neighbourhood U and an $f \in K(X)^\times$ such that

$$D|_U = \mathrm{div}(f)|_U.$$

Proof. Suppose first that a Cartier divisor D maps to zero. Choose an affine open $U = \operatorname{Spec} A$ on which D is represented by $f \in K(X)^\times$. For every height-one prime $\mathfrak{p} \subseteq A$,

$$v_{\mathfrak{p}}(f) = 0.$$

Thus both f and f^{-1} lie in every $A_{\mathfrak{p}}$. By lemma 7.5,

$$f, f^{-1} \in A,$$

so $f \in A^\times$. Hence $D|_U = 0$. Such affine opens cover X, so $D = 0$.

Every Cartier divisor is locally represented by some $f \in K(X)^\times$, and on that neighbourhood its associated Weil divisor is exactly $\mathrm{div}(f)$. Hence the image consists of locally principal Weil divisors.

Conversely, let D be a locally principal Weil divisor. Choose an open cover $\{U_i\}$ and rational functions f_i with

$$D|_{U_i} = \mathrm{div}(f_i)|_{U_i}.$$

On $U_i \cap U_j$,

$$\mathrm{div}(f_i/f_j) = 0.$$

Apply the injectivity argument above to the principal Cartier divisor defined by f_i/f_j. It shows that

$$f_i/f_j \in \Gamma(U_i \cap U_j, \mathscr{O}_X^\times).$$

Thus the f_i are local equations of a Cartier divisor, whose associated Weil divisor is D. □

Definition 7.35 (Locally factorial scheme)**.** An integral scheme X is *locally factorial* if every local ring $\mathscr{O}_{X,x}$ is a unique factorization domain.

Theorem 7.36 (Local factoriality and Cartier divisors)**.** *Let X be a normal Noetherian integral scheme. The following conditions are equivalent:*

1. *X is locally factorial;*
2. *every prime Weil divisor is Cartier;*
3. *every Weil divisor is Cartier;*
4. *the natural map*
$$\mathrm{CaCl}(X) \longrightarrow \mathrm{Cl}(X)$$
is an isomorphism;
5. *the natural map*
$$\mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$$
is an isomorphism.

Proof. Assume (1), and let Y be a prime divisor. Fix $x \in X$. If $x \notin Y$, then Y is zero on $X \setminus Y$. Suppose $x \in Y$ and put
$$A = \mathscr{O}_{X,x}.$$
The germ of the ideal of Y is a height-one prime $\mathfrak{p} \subseteq A$. Since A is a UFD, lemma 7.4 gives
$$\mathfrak{p} = (t).$$
Represent t by a section on a neighbourhood U of x. The inclusion
$$t\mathscr{O}_U \subseteq \mathscr{I}_Y|_U$$
is an equality at the stalk x. Its cokernel is coherent, so after shrinking U it vanishes. Thus Y is locally cut out by one nonzero equation and is Cartier. This proves (1)⇒(2).

The implication (2)⇒(3) follows because every Weil divisor is a finite integral combination of prime divisors.

Assume (3). Fix $x \in X$ and put $A = \mathscr{O}_{X,x}$. Let $\mathfrak{p} \subseteq A$ be a height-one prime. It corresponds, after choosing an affine neighbourhood of x, to a codimension-one point whose closure is a prime Weil divisor Y through x. By assumption Y is Cartier. On $\operatorname{Spec} A$ choose a local equation $f \in \mathrm{Frac}(A)^{\times}$ for the Cartier divisor Y. Then
$$v_{\mathfrak{p}}(f) = 1, \qquad v_{\mathfrak{q}}(f) = 0$$
for every other height-one prime $\mathfrak{q}$ of A. The intersection theorem gives $f \in A$. Since $v_{\mathfrak{p}}(f) > 0$, we have $f \in \mathfrak{p}$.

If $g \in \mathfrak{p}$, then
$$v_{\mathfrak{p}}(g/f) \geq 0$$
and for every $\mathfrak{q} \neq \mathfrak{p}$,
$$v_{\mathfrak{q}}(g/f) = v_{\mathfrak{q}}(g) \geq 0.$$
Hence g/f belongs to every height-one localization of A, and lemma 7.5 gives $g/f \in A$. Therefore $\mathfrak{p} = (f)$. By lemma 7.4, A is a UFD. Since x was arbitrary, X is locally factorial. Thus (3)⇒(1).

By theorem 7.34, the natural map $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ is injective. Hence (3) is equivalent to this map being surjective. If every Weil divisor is Cartier, then the induced map on class groups is clearly an isomorphism, so (3)⇒(4).

Conversely, suppose (4). Let D be a Weil divisor. Surjectivity on class groups gives a Cartier divisor C and an $f \in K(X)^{\times}$ such that
$$D - [C]_W = \mathrm{div}(f).$$

Both $[C]_W$ and $\mathrm{div}(f)$ are Cartier as Weil divisors, so D is Cartier. Thus (4)⇒(3).

Finally, corollary 7.23 identifies $\mathrm{CaCl}(X)$ with $\mathrm{Pic}(X)$ compatibly with the map to $\mathrm{Cl}(X)$, so (4) and (5) are equivalent. □

Corollary 7.37 (Regular schemes)**.** *Let X be a regular Noetherian integral scheme. Then*

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Cl}(X).$$

Proof. A regular Noetherian local ring is a UFD, a standard theorem of commutative algebra. Thus X is locally factorial. The result follows from theorem 7.36. □

Corollary 7.38 (Picard group of projective space)**.** *For $n \geq 1$,*

$$\mathrm{Pic}(\mathbf{P}^n_k) \cong \mathbf{Z},$$

generated by $\mathscr{O}_{\mathbf{P}^n_k}(1)$.

Proof. Projective space is regular, so corollary 7.37 and theorem 7.9 give

$$\mathrm{Pic}(\mathbf{P}^n_k) \cong \mathrm{Cl}(\mathbf{P}^n_k) \cong \mathbf{Z}.$$

The hyperplane divisor $H = V_+(x_0)$ corresponds to $\mathscr{O}(H)$. On the standard chart $U_i = D_+(x_i)$, a local equation is

$$f_i = \frac{x_0}{x_i};$$

for $i = 0$ this is the unit 1, reflecting the fact that $H \cap U_0 = \varnothing$. The local frame f_i^{-1} of $\mathscr{O}(H)$ satisfies on $U_i \cap U_j$

$$f_i^{-1} = \frac{x_i}{x_j} f_j^{-1}.$$

The twisting sheaf $\mathscr{O}(1)$ has local frame x_i on U_i with the same transition rule

$$x_i = \frac{x_i}{x_j} x_j.$$

Thus the local identifications $f_i^{-1} \mapsto x_i$ glue to an isomorphism

$$\mathscr{O}(H) \cong \mathscr{O}_{\mathbf{P}^n_k}(1).$$

Hence $\mathscr{O}(1)$ generates the Picard group. □

Remark 7.39. For a normal scheme which is not locally factorial, the natural injection

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$$

need not be surjective. The quotient measures precisely the obstruction for a codimension-one Weil divisor to be locally cut out by one equation. The quadric cone in example 7.14 is the simplest place where this distinction is visible.

7.4 Linear Systems and Projective Geometry

Chapter 3 treated projective morphisms extrinsically, through closed immersions into relative projective space. Chapter 5 introduced very ample invertible sheaves from the same closed immersions. We now reverse the direction: an invertible sheaf together with sufficiently many sections produces a morphism to projective space. Linear systems are the divisor-theoretic form of this construction.

Definition 7.40 (Generated by global sections)**.** An invertible sheaf $\mathscr{L}$ on a scheme X is *generated by global sections* if there exist sections

$$s_0, \dots, s_n \in \Gamma(X, \mathscr{L})$$

such that the evaluation morphism

$$\mathscr{O}_X^{\oplus(n+1)} \xrightarrow{\text{ev}} \mathscr{L}$$

is surjective. Equivalently, for every $x \in X$ at least one germ $(s_i)_x$ generates the free rank-one module $\mathscr{L}_x$.

The following theorem is the intrinsic form of the homogeneous-coordinate construction.

Theorem 7.41 (Morphisms to projective space)**.** *Let $X \to Y$ be a morphism of schemes. There is a natural bijection between:*

1. *Y-morphisms*
$$\varphi : X \longrightarrow \mathbf{P}^n_Y;$$
2. *isomorphism classes of data $(\mathscr{L}; s_0, \dots, s_n)$ consisting of an invertible sheaf $\mathscr{L}$ on X and global sections s_i generating $\mathscr{L}$.*

Under this correspondence,

$$\mathscr{L} \cong \varphi^* \mathscr{O}_{\mathbf{P}^n_Y}(1),$$

and the s_i are the pullbacks of the standard homogeneous coordinates.

Proof. Suppose first that $\varphi : X \to \mathbf{P}^n_Y$ is a Y-morphism. Put

$$\mathscr{L} := \varphi^* \mathscr{O}_{\mathbf{P}^n_Y}(1).$$

The standard coordinates $T_0, \dots, T_n$ generate $\mathscr{O}(1)$, so their pullbacks

$$s_i := \varphi^* T_i$$

generate $\mathscr{L}$. This gives the map from (1) to (2).

Conversely, let $(\mathscr{L}; s_0, \dots, s_n)$ be given. For each i, let

$$X_i := \{x \in X \mid (s_i)_x \text{ generates } \mathscr{L}_x\}.$$

These are open and cover X. On X_i, the section s_i trivializes $\mathscr{L}$, so for $j \neq i$ there is a unique regular function

$$a_{ji} := \frac{s_j}{s_i} \in \Gamma(X_i, \mathscr{O}_X).$$

The standard chart

$$D_+(T_i) \subseteq \mathbf{P}^n_Y$$

is relative affine n-space over Y with coordinates T_j/T_i. To construct the map rigorously, take an affine open $V = \operatorname{Spec} R \subseteq Y$. On $X_i \cap f^{-1}(V)$, the structural map gives $R \to \Gamma(X_i \cap f^{-1}(V), \mathscr{O}_X)$, and adjoining the functions a_{ji} gives

$$R[T_j/T_i]_{j\neq i} \longrightarrow \Gamma(X_i \cap f^{-1}(V), \mathscr{O}_X).$$

By theorem 2.14, this is a unique morphism to the affine chart $D_+(T_i)|_V$. These morphisms agree when V is restricted, so they glue over Y to a unique Y-morphism

$$\varphi_i : X_i \longrightarrow D_+(T_i)$$

characterized by

$$\varphi_i^*(T_j/T_i) = s_j/s_i.$$

On $X_i \cap X_k$, the coordinate transformations satisfy

$$\frac{s_j}{s_k} = \frac{s_j/s_i}{s_k/s_i},$$

which is exactly the transition rule between the standard projective charts. Hence the local morphisms agree on overlaps and glue:

$$\begin{tikzcd} X_i \cap X_k \arrow[r, hook] \arrow[d, hook] & X_i \arrow[d, "\varphi_i"] \\ X_k \arrow[r, "\varphi_k"'] & \mathbf{P}^n_Y. \end{tikzcd}$$

We obtain a morphism

$$\varphi : X \to \mathbf{P}^n_Y.$$

On $D_+(T_i)$, the section T_i trivializes $\mathscr{O}(1)$. Sending $\varphi^* T_i$ to s_i gives a local isomorphism

$$\varphi^*\mathscr{O}(1)|_{X_i} \xrightarrow{\sim} \mathscr{L}|_{X_i},$$

and the ratio identities show that these local isomorphisms agree on overlaps. Thus

$$\varphi^*\mathscr{O}(1) \cong \mathscr{L}.$$

The two constructions are visibly inverse on the standard charts, proving the bijection. □

Corollary 7.42 (Intrinsic characterization of very ampleness)**.** *Let $X \to Y$ be a morphism. An invertible sheaf $\mathscr{L}$ is very ample relative to Y in the sense of definition 5.26 if and only if there exists a finite generating set of global sections whose associated morphism*

$$X \longrightarrow \mathbf{P}^n_Y$$

is a closed immersion.

Proof. If $\mathscr{L}$ is relatively very ample, by definition there is a closed immersion

$$i : X \hookrightarrow \mathbf{P}^n_Y$$

with $\mathscr{L} \cong i^*\mathscr{O}(1)$. Pulling back the standard coordinates gives a generating set, and theorem 7.41 recovers the same immersion.

Conversely, if a generating set produces a closed immersion φ, then

$$\mathscr{L} \cong \varphi^*\mathscr{O}(1),$$

so $\mathscr{L}$ is very ample by definition. □

Corollary 7.43 (Projective morphisms and very ample sheaves)**.** *A morphism $f : X \to Y$ is projective if and only if X carries an invertible sheaf which is very ample relative to Y.*

Proof. If f is projective, section 3.4 gives a closed immersion

$$i : X \hookrightarrow \mathbf{P}^n_Y$$

for some n, and $i^*\mathscr{O}(1)$ is relatively very ample. Conversely, a relatively very ample invertible sheaf is, by definition 5.26, obtained from such a closed immersion; composing with the projective-space projection recovers f, so f is projective. □

Very ampleness depends on one chosen embedding. Ampleness is weaker: it says that sufficiently positive tensor powers eventually generate every coherent sheaf. The definition below is tailored to the projective Noetherian setting already established in Chapter 5.

Definition 7.44 (Ample invertible sheaf)**.** Let X be a Noetherian scheme. An invertible sheaf $\mathscr{L}$ is *ample* if for every coherent sheaf $\mathscr{F}$ there exists n_0 such that

$$\mathscr{F} \otimes \mathscr{L}^{\otimes n}$$

is generated by finitely many global sections for every $n \geq n_0$.

The following lemma explains why tensoring with a globally generated line bundle does not destroy an embedding.

Lemma 7.45 (Very ample times globally generated is very ample)**.** *Let X be a projective scheme over a ring A. If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by finitely many global sections, then*

$$\mathscr{L} \otimes \mathscr{M}$$

is very ample.

Proof. Choose a closed immersion

$$i : X \hookrightarrow \mathbf{P}^r_A$$

with $\mathscr{L} \cong i^*\mathscr{O}(1)$. Choose global generators of $\mathscr{M}$. By theorem 7.41, they define a morphism

$$j : X \longrightarrow \mathbf{P}^s_A$$

with $\mathscr{M} \cong j^*\mathscr{O}(1)$.

The product morphism

$$(i, j) : X \longrightarrow \mathbf{P}^r_A \times_A \mathbf{P}^s_A$$

is a closed immersion. Indeed it factors as the graph of j followed by the base change of i:

$$X \xhookrightarrow{\Gamma_j} X \times_A \mathbf{P}^s_A \xhookrightarrow{i \times \mathrm{id}} \mathbf{P}^r_A \times_A \mathbf{P}^s_A.$$

The graph is closed because projective space is separated, and the second map is a closed immersion by base-change stability.

Compose with the relative Segre embedding from proposition 3.56:

$$X \xhookrightarrow{\Phi} \mathbf{P}^r_A \times_A \mathbf{P}^s_A \xhookrightarrow{\sigma} \mathbf{P}^{(r+1)(s+1)-1}_A.$$

The Segre coordinates are the products of the coordinates from the two factors, so

$$\sigma^*\mathscr{O}(1) \cong \mathrm{pr}_1^*\,\mathscr{O}(1) \otimes \mathrm{pr}_2^*\,\mathscr{O}(1).$$

Pulling back to X gives

$$\mathscr{L} \otimes \mathscr{M}.$$

Thus $\mathscr{L} \otimes \mathscr{M}$ is the pullback of $\mathscr{O}(1)$ along a closed immersion and is therefore very ample. □

Theorem 7.46 (Ampleness on a projective scheme)**.** *Let A be Noetherian, let X be projective over* $\operatorname{Spec} A$*, and let $\mathscr{L}$ be invertible. Then the following are equivalent:*

1. *$\mathscr{L}$ is ample;*
2. *some positive tensor power $\mathscr{L}^{\otimes m}$ is very ample over A.*

Proof. Assume first that $\mathscr{L}^{\otimes m}$ is very ample. Let $\mathscr{F}$ be coherent. For each residue class $0 \le r < m$, apply Serre's global-generation theorem theorem 5.27 to the coherent sheaf

$$\mathscr{F} \otimes \mathscr{L}^{\otimes r}$$

using the very ample invertible sheaf $\mathscr{L}^{\otimes m}$. There exists q_r such that

$$\mathscr{F} \otimes \mathscr{L}^{\otimes r} \otimes (\mathscr{L}^{\otimes m})^{\otimes q}$$

is globally generated for every $q \ge q_r$. Since there are only finitely many residue classes, one bound works for all sufficiently large

$$n = mq + r.$$

Hence $\mathscr{L}$ is ample.

Conversely, assume $\mathscr{L}$ is ample. Choose once and for all a very ample sheaf $\mathscr{A}$ on X, available because X is projective. Apply ampleness to the coherent invertible sheaf $\mathscr{A}^{-1}$. For $n \gg 0$,

$$\mathscr{A}^{-1} \otimes \mathscr{L}^{\otimes n}$$

is generated by finitely many global sections. By lemma 7.45,

$$\mathscr{L}^{\otimes n} \cong \mathscr{A} \otimes (\mathscr{A}^{-1} \otimes \mathscr{L}^{\otimes n})$$

is very ample. Thus some positive power of $\mathscr{L}$ is very ample. □

Remark 7.47 (Relative ampleness). For a projective morphism $f : X \to Y$ with Y Noetherian, one defines $\mathscr{L}$ to be *f-ample* if its restriction over every affine open $V \subseteq Y$ is ample on $f^{-1}(V)$. Relative very ampleness implies relative ampleness by theorem 5.27. Conversely, on an affine piece of the base, theorem 7.46 shows that a sufficiently high power of an ample invertible sheaf is very ample. Thus the intrinsic language of ample line bundles is locally on the base equivalent to the projective embeddings used in Chapter 3.

We finish with linear systems. From now on let X be an integral projective scheme over a field k, and let $\mathscr{L}$ be invertible.

Definition 7.48 (Linear system and base locus)**.** A *linear system* on $\mathscr{L}$ is a finite dimensional k-subspace

$$V \subseteq H^0(X, \mathscr{L}).$$

Its *base locus* is

$$\operatorname{Bs}(V) := \{x \in X \mid s_x \in \mathfrak{m}_x \mathscr{L}_x \text{ for every } s \in V\}.$$

Equivalently, x is a base point if no section in V generates $\mathscr{L}_x$. The system is *base-point free* if $\operatorname{Bs}(V) = \varnothing$.

The *complete linear system* of $\mathscr{L}$ is the full space $H^0(X, \mathscr{L})$, viewed projectively when scalar multiples are identified.

Proposition 7.49 (The rational map of a linear system)**.** *Let $V \subseteq H^0(X, \mathscr{L})$ be nonzero and finite dimensional. On the open*

$$U_V := X \setminus \mathrm{Bs}(V)$$

there is a canonical morphism

$$\varphi_V : U_V \longrightarrow \mathbf{P}(V^\vee),$$

and hence a canonical rational map

$$X \dashrightarrow \mathbf{P}(V^\vee).$$

If V is base-point free, then $U_V = X$, so φ_V is a morphism and

$$\varphi_V^* \mathscr{O}_{\mathbf{P}(V^\vee)}(1) \cong \mathscr{L}.$$

Proof. Choose a basis $s_0, \dots, s_n$ of V. On

$$U := X \setminus \mathrm{Bs}(V)$$

these sections generate $\mathscr{L}|_U$, so theorem 7.41 gives a morphism

$$U \longrightarrow \mathbf{P}^n_k.$$

Changing the basis of V acts on $\mathbf{P}^n_k$ by the corresponding projective linear automorphism. Therefore the construction is intrinsic and has target $\mathbf{P}(V^\vee)$. Since $V \neq 0$ and X is integral, at least one section is nonzero at the generic point, so U is nonempty. The morphism on U therefore represents a rational map on X in the sense of definition 3.21. We do not claim that U is the maximal open to which this rational map can extend: a chosen linear system may have removable base points. If V has no base points, then $U = X$ and the map is everywhere defined. The pullback identity follows from theorem 7.41. □

For a Cartier divisor, a linear system can be described directly in terms of effective divisors. The key observation is that a section of $\mathscr{O}_X(D)$ is a rational function whose allowed poles are bounded by D.

Proposition 7.50 (Sections of $\mathscr{O}_X(D)$)**.** *Let D be a Cartier divisor on an integral scheme X. Then*

$$H^0(X, \mathscr{O}_X(D)) = \{0\} \cup \{f \in K(X)^\times \mid D + \mathrm{div}_C(f) \geq 0\}.$$

Consequently, D is linearly equivalent to an effective Cartier divisor if and only if

$$H^0(X, \mathscr{O}_X(D)) \neq 0.$$

Proof. Let D have local equations f_i on U_i. By definition,

$$\mathscr{O}_X(D)|_{U_i} = f_i^{-1} \mathscr{O}_{U_i} \subseteq \mathscr{K}_X.$$

A nonzero rational function $g \in K(X)^\times$ is a section of this sheaf over U_i precisely when

$$g f_i \in \mathscr{O}_X(U_i).$$

Equivalently, the Cartier divisor with local equations $g f_i$ is effective. But these local equations represent

$$D + \mathrm{div}_C(g).$$

The local conditions glue, yielding the displayed equality.

If a nonzero section g exists, then

$$E := D + \mathrm{div}_C(g)$$

is effective and linearly equivalent to D. Conversely, if $E \geq 0$ and $E \sim D$, then

$$E = D + \mathrm{div}_C(g)$$

for some $g \in K(X)^\times$, and the first part shows that g is a nonzero global section of $\mathscr{O}_X(D)$. □

Definition 7.51 (Complete linear system of a divisor)**.** For a Cartier divisor D, its *complete linear system* is

$$|D| := \{E \geq 0 \mid E \sim D\}.$$

Theorem 7.52 (Sections and effective divisors)**.** *Assume X is projective and integral over an algebraically closed field k. For every Cartier divisor D there is a natural bijection*

$$\mathbf{P}(H^0(X, \mathscr{O}_X(D))) \xrightarrow{\sim} |D|,$$

where a nonzero section represented by $f \in K(X)^\times$ maps to

$$D + \mathrm{div}_C(f).$$

Proof. By proposition 7.50, every nonzero section f determines an effective divisor linearly equivalent to D, so the map is defined. Multiplying f by a nonzero scalar in k does not change its divisor.

Conversely, every $E \in |D|$ is of the form

$$E = D + \mathrm{div}_C(f)$$

for some $f \in K(X)^\times$, and then f is a nonzero section of $\mathscr{O}_X(D)$.

It remains to identify when two sections determine the same effective divisor. If

$$\mathrm{div}_C(f) = \mathrm{div}_C(g),$$

then f/g has trivial Cartier divisor and therefore lies in $\Gamma(X, \mathscr{O}_X^\times)$. Since X is projective integral over the algebraically closed field k, corollary 3.66 gives

$$\Gamma(X, \mathscr{O}_X) = k, \qquad \Gamma(X, \mathscr{O}_X^\times) = k^\times.$$

Thus f and g differ by a scalar. Hence the fibres of the map are exactly the one-dimensional k-subspaces of $H^0(X, \mathscr{O}_X(D))$. □

Remark 7.53. The preceding theorem explains why complete linear systems are projective spaces. It also shows that the passage

$$D \longmapsto \mathscr{O}_X(D) \longmapsto H^0(X, \mathscr{O}_X(D))$$

turns a divisor class into a family of effective representatives. A choice of a finite dimensional subspace of sections then turns that family into a rational map to projective space.

Example 7.54 (Hyperplanes and the identity map)**.** Let H be a hyperplane in $\mathbf{P}^n_k$. By corollary 7.38,

$$\mathscr{O}(H) \cong \mathscr{O}_{\mathbf{P}^n}(1).$$

The space of global sections is generated by the homogeneous coordinates

$$x_0, \ldots, x_n.$$

They have no common zero in projective space, so the complete linear system is base-point free. The associated morphism

$$\mathbf{P}^n_k \longrightarrow \mathbf{P}\big(H^0(\mathbf{P}^n, \mathscr{O}(1))^\vee\big)$$

is the standard identification of projective space with its projective coordinate space, up to the chosen basis.

Example 7.55 (The Veronese linear system)**.** For $d \geq 1$, the global sections of $\mathscr{O}_{\mathbf{P}^n}(d)$ are the homogeneous degree-d polynomials. The degree-d monomials generate $\mathscr{O}(d)$ everywhere, so they define a morphism

$$\nu_d : \mathbf{P}^n_k \longrightarrow \mathbf{P}^N_k, \qquad N = \binom{n+d}{d} - 1,$$

whose homogeneous coordinates are all monomials of degree d. This is the d-uple Veronese morphism. It is a closed immersion, and therefore

$$\mathscr{O}_{\mathbf{P}^n}(d)$$

is very ample.

One way to see the closed-immersion statement is to pass to the d-th Veronese subring

$$S^{(d)} := \bigoplus_{m \geq 0} S_{md}$$

of $S = k[x_0, \ldots, x_n]$. The natural map from a polynomial ring on the degree-d monomials onto $S^{(d)}$ is surjective, so it gives a closed immersion

$$\operatorname{Proj} S^{(d)} \hookrightarrow \mathbf{P}^N_k.$$

The standard affine charts show

$$\operatorname{Proj} S^{(d)} \cong \operatorname{Proj} S = \mathbf{P}^n_k,$$

and the resulting immersion is exactly ν_d.

The chapter can now be summarized by the two linked chains

$$K(X)^\times \longrightarrow \operatorname{Div}(X) \longrightarrow \operatorname{Cl}(X),$$

$$\operatorname{CaDiv}(X) \longrightarrow H^1(X, \mathscr{O}_X^\times) \xrightarrow{\sim} \operatorname{Pic}(X).$$

and

$$\operatorname{Pic}(X) \longrightarrow \{\text{linear systems}\} \longrightarrow \{X \dashrightarrow \mathbf{P}^n\}.$$

For normal locally factorial schemes the two divisor class groups coincide, and then a codimension-one cycle, an invertible sheaf, and a linear system are three views of the same underlying geometry. The next chapter will introduce differentials and infinitesimal geometry; once canonical sheaves appear, divisor theory will allow them to be represented by canonical divisors.

Chapter 8

Differentials, Smoothness, and Duality

The divisor theory of the preceding chapter described codimension-one geometry by measuring zeros and poles of rational functions. We now turn to a different kind of local information: first-order variation. A derivation remembers the linear term of a function under an infinitesimal displacement, and the module of Kähler differentials is the universal object which records all such first-order changes at once. After sheafification this leads to relative differentials, tangent spaces, smooth morphisms, and infinitesimal lifting. Exterior powers of the sheaf of differentials then produce canonical sheaves, and the same relative viewpoint culminates in duality.

Several constructions in this chapter deliberately return to objects introduced earlier. The diagonal from Chapter 3 reappears through its conormal sheaf, the twisting sheaves and linear systems of Chapters 5 and 7 produce the Euler sequence and Bertini's theorem, and the cohomology of Chapter 6 becomes the absolute case of higher direct images. Thus the purpose of the chapter is not to introduce an isolated calculus on schemes, but to connect the local, projective, and cohomological parts of the theory already developed.

8.1 Kähler Differentials and Relative Differentials

Let $A \to B$ be a homomorphism of commutative rings and let M be a B-module.

Definition 8.1. An A-*derivation* from B to M is an additive map

$$D : B \longrightarrow M$$

such that

$$D(bb') = bD(b') + b'D(b)$$

for all $b, b' \in B$, and $D(a) = 0$ for every a in the image of A. The B-module of A-derivations is denoted

$$\mathrm{Der}_A(B, M).$$

The Leibniz rule immediately gives

$$D(1) = 0, \qquad D(b^m) = mb^{m-1}D(b),$$

and, when b is a unit,

$$D(b^{-1}) = -b^{-2}D(b).$$

The last identity is the reason derivations extend uniquely across localizations.

Definition 8.2. A *module of Kähler differentials of B over A* is a pair $(\Omega_{B/A}, d_{B/A})$ consisting of a B-module and an A-derivation

$$d_{B/A} : B \longrightarrow \Omega_{B/A}$$

with the following universal property: for every B-module M, composition with $d_{B/A}$ induces a natural isomorphism

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M).$$

We usually write db for $d_{B/A}(b)$.

Theorem 8.3 (Existence of Kähler differentials)**.** *For every ring homomorphism $A \to B$, the module $\Omega_{B/A}$ exists and is unique up to a unique isomorphism compatible with the universal derivation.*

Proof. Let F be the free B-module on symbols $[b]$ indexed by $b \in B$. Let $R \subseteq F$ be the submodule generated by

$$[b + b'] - [b] - [b'],$$
$$[bb'] - b[b'] - b'[b],$$

and $[a]$ for a in the image of A. Set

$$\Omega_{B/A} = F/R, \qquad db = [b] \bmod R.$$

The defining relations say exactly that $b \mapsto db$ is an A-derivation.

If $D : B \to M$ is another A-derivation, freeness gives a unique B-linear map $F \to M$ sending $[b]$ to $D(b)$. The derivation identities imply that this map kills R, hence factors through a unique map

$$\Omega_{B/A} \longrightarrow M.$$

This proves the universal property. Uniqueness follows in the usual way from the universal property: if (Ω, d) and (Ω', d') are both universal, d' and d induce mutually inverse maps $\Omega \rightleftarrows \Omega'$. □

The universal property makes computations almost automatic.

Proposition 8.4 (Polynomial algebras)**.** *Let*

$$B = A[x_1, \ldots, x_n].$$

Then $\Omega_{B/A}$ is free with basis $dx_1, \ldots, dx_n$. For every polynomial $f \in B$,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i.$$

Proof. Let $F = \bigoplus_i Be_i$ and define

$$D(f) = \sum_i \frac{\partial f}{\partial x_i} e_i.$$

This is an A-derivation, so the universal property gives a map $\Omega_{B/A} \to F$ sending dx_i to e_i. Conversely the assignment $e_i \mapsto dx_i$ gives a B-linear map $F \to \Omega_{B/A}$. Repeated use of the Leibniz rule proves the displayed formula for df, so the two maps are inverse on the generators of both modules. □

There is a second construction which exhibits the geometric meaning of differentials. Let

$$\mu : B \otimes_A B \longrightarrow B, \qquad b \otimes b' \longmapsto bb',$$

and let $I = \ker \mu$.

Theorem 8.5 (The diagonal ideal). *There is a canonical isomorphism*

$$\Omega_{B/A} \xrightarrow{\sim} I/I^2$$

sending

$$db \longmapsto 1 \otimes b - b \otimes 1 \pmod{I^2}.$$

Proof. The map

$$\delta : B \longrightarrow I/I^2, \qquad b \longmapsto 1 \otimes b - b \otimes 1$$

is an A-derivation. Indeed, the defect in the Leibniz rule is

$$(1 \otimes b - b \otimes 1)(1 \otimes c - c \otimes 1),$$

which lies in I^2. Hence the universal property gives a map $\Omega_{B/A} \to I/I^2$.

Conversely, let $D : B \to M$ be an A-derivation. Define

$$\Phi : B \otimes_A B \longrightarrow M, \qquad \Phi(b \otimes c) = bD(c).$$

On I this is B-linear for the first tensor factor. For generators of I one computes

$$\Phi\big((1 \otimes b - b \otimes 1)(1 \otimes c - c \otimes 1)\big) = D(bc) - bD(c) - cD(b) = 0.$$

Thus Φ kills I^2 and gives a B-linear map $I/I^2 \to M$ carrying $\delta(b)$ to $D(b)$. Finally the elements $\delta(b)$ generate I/I^2: if $\sum_i b_i \otimes c_i \in I$, then $\sum_i b_i c_i = 0$ and

$$\sum_i b_i \otimes c_i = \sum_i (b_i \otimes 1)(1 \otimes c_i - c_i \otimes 1).$$

Hence $(I/I^2, \delta)$ satisfies the same universal property as $\Omega_{B/A}$. □

Geometrically, $B \otimes_A B$ is the coordinate ring of

$$\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B,$$

and I is the ideal of the diagonal. The quotient I/I^2 therefore remembers the diagonal only to first order.

The basic functorial properties will be used repeatedly.

Theorem 8.6 (Transitivity). *For ring homomorphisms*

$$A \longrightarrow B \longrightarrow C$$

there is a natural exact sequence of C-modules

$$C \otimes_B \Omega_{B/A} \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0.$$

Proof. The composite $B \to C \xrightarrow{d_{C/A}} \Omega_{C/A}$ is an A-derivation, hence induces the first arrow. Every B-derivation is in particular an A-derivation, giving the second arrow; it is surjective because $\Omega_{C/B}$ is generated by the elements dc.

Let Q be the cokernel of the first arrow. The universal derivation $C \to \Omega_{C/A}$ followed by $\Omega_{C/A} \to Q$ vanishes on B, so it is a B-derivation. Conversely every B-derivation $C \to M$ factors through $\Omega_{C/A}$ and vanishes on the image of $C \otimes_B \Omega_{B/A}$. Hence Q represents B-derivations from C, and therefore $Q \cong \Omega_{C/B}$. □

The sequence need not be exact on the left. The failure of left exactness is genuine first-order information which an ordinary module of differentials does not retain.

Theorem 8.7 (Conormal sequence)**.** *Let $A \to B$ be a ring homomorphism, let $J \subseteq B$ be an ideal, and put $C = B/J$. There is a natural exact sequence*

$$J/J^2 \xrightarrow{d} \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0,$$

where the first map sends the class of f to $df \otimes 1$.

Proof. Apply theorem 8.6 to $A \to B \to C$. Since every B-derivation from C is zero, $\Omega_{C/B} = 0$. The kernel of

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A}$$

is generated by the $df \otimes 1$ for $f \in J$. The map $J \to \Omega_{B/A} \otimes_B C$ kills J^2 because

$$d(fg) \otimes 1 = f\,dg \otimes 1 + g\,df \otimes 1 = 0$$

for $f, g \in J$. Hence it factors through J/J^2 with precisely the required image. □

Remark 8.8. The first arrow in the conormal sequence is not injective in general. Later we will see that for a closed immersion of nonsingular varieties it becomes an injection and the conormal sequence is short exact. This distinction is important: the left exactness is a geometric consequence, not part of the formal construction.

Proposition 8.9 (Jacobian presentation)**.** *If*

$$B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_r),$$

then there is a right exact sequence

$$B^{\oplus r} \xrightarrow{J} B^{\oplus n} \longrightarrow \Omega_{B/A} \longrightarrow 0,$$

where J sends the i-th basis vector to

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j.$$

Thus J is represented by the Jacobian matrix $(\partial f_i/\partial x_j)$.

Proof. Apply theorem 8.7 to the polynomial algebra $P = A[x_1, \ldots, x_n]$ and the ideal $(f_1, \ldots, f_r)$. By proposition 8.4, $\Omega_{P/A} \otimes_P B$ is free on the dx_j, and the classes of the f_i generate the conormal module. The resulting presentation is exactly the displayed one. □

Proposition 8.10 (Localization and base change)**.** *Let $A \to B$ be a ring homomorphism.*

1. *If $S \subseteq B$ is multiplicatively closed, then*

$$S^{-1}\Omega_{B/A} \cong \Omega_{S^{-1}B/A}.$$

2. *If $A \to A'$ is another ring homomorphism and $B' = B \otimes_A A'$, then*

$$\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_B B'.$$

Proof. For localization, an A-derivation $B \to M$ into an $S^{-1}B$-module extends uniquely to $S^{-1}B$ because
$$d(b/s) = \frac{s\,db - b\,ds}{s^2}.$$
Hence both modules represent the same derivation functor on $S^{-1}B$-modules.

For base change, an A'-derivation $B \otimes_A A' \to M$ is uniquely determined by its restriction to $B \otimes 1$, and that restriction is an A-derivation $B \to M$. Conversely an A-derivation $\delta : B \to M$ extends by
$$\delta'(b \otimes a') = a'\delta(b).$$
Thus the two sides again represent the same functor. □

One simple calculation already shows that differentials remember arithmetic phenomena invisible to dimension alone.

Example 8.11 (Differentials and inseparability)**.** Let L/K be a finite field extension. If L/K is separable, then
$$\Omega_{L/K} = 0.$$
Indeed, after the primitive element theorem write $L = K(\alpha)$ with separable minimal polynomial f. Then
$$0 = d(f(\alpha)) = f'(\alpha)d\alpha,$$
and $f'(\alpha) \neq 0$ is a unit in L, so $d\alpha = 0$.

In characteristic $p > 0$, if
$$L = K[t]/(t^p - a)$$
is a field with $a \notin K^p$, then
$$\Omega_{L/K} \cong L\,dt,$$
because $d(t^p - a) = 0$. Thus Kähler differentials detect the inseparable part of an algebraic extension.

We now pass from rings to schemes.

Theorem 8.12 (The sheaf of relative differentials)**.** *Let $f : X \to S$ be a morphism of schemes. There exists a quasi-coherent $\mathscr{O}_X$-module $\Omega^1_{X/S}$ together with an $f^{-1}\mathscr{O}_S$-linear derivation*
$$d_{X/S} : \mathscr{O}_X \longrightarrow \Omega^1_{X/S}$$
such that whenever $U = \operatorname{Spec} B \subseteq X$ maps into $V = \operatorname{Spec} A \subseteq S$, there is a canonical isomorphism
$$\Omega^1_{X/S}|_U \cong \widetilde{\Omega_{B/A}}.$$
Moreover, for every $\mathscr{O}_X$-module $\mathscr{F}$,
$$\operatorname{Hom}_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{F}) \cong \operatorname{Der}_S(\mathscr{O}_X, \mathscr{F}).$$

Proof. On an affine chart $U = \operatorname{Spec} B$ lying over $V = \operatorname{Spec} A$, take the associated quasi-coherent sheaf $\widetilde{\Omega_{B/A}}$. By proposition 8.10, passage to a distinguished affine open agrees with localization of the module of differentials. Therefore these sheaves are canonically identified on common distinguished refinements of overlaps. The compatibility of the localization maps gives the cocycle condition, so the gluing theorem of Chapter 2 produces $\Omega^1_{X/S}$. The affine universal derivations glue for the same reason.

For the universal property, restrict an S-derivation to principal affine opens. The affine universal property gives compatible local maps from $\widetilde{\Omega_{B/A}}$, and uniqueness forces agreement on overlaps. Conversely any sheaf morphism out of $\Omega^1_{X/S}$ composed with $d_{X/S}$ is an S-derivation. These operations are inverse. □

Corollary 8.13. *If $f : X \to S$ is locally of finite type, then $\Omega^1_{X/S}$ is locally of finite type. If f is locally of finite presentation, then $\Omega^1_{X/S}$ is locally of finite presentation.*

Proof. The assertion is affine-local and follows from the Jacobian presentation and finite generation of differentials for finitely generated algebras. □

Theorem 8.14 (Transitivity for schemes)**.** *For morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} S$$

there is a natural right exact sequence

$$f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0.$$

Proof. On affine charts $\operatorname{Spec} C \to \operatorname{Spec} B \to \operatorname{Spec} A$ this is the sheaf associated to the exact sequence of theorem 8.6. Compatibility with localization makes the local sequences glue. □

Theorem 8.15 (Conormal sequence for a closed immersion)**.** *Let $i : X \hookrightarrow Y$ be a closed immersion over S, with ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_Y$. There is a natural right exact sequence*

$$\mathscr{I}/\mathscr{I}^2 \longrightarrow i^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S} \longrightarrow 0.$$

Proof. The statement is local on Y. On an affine chart it is theorem 8.7; compatibility with localization gives the global sequence. □

The diagonal description now becomes intrinsic.

Proposition 8.16 (Differentials as the conormal sheaf of the diagonal)**.** *Let $f : X \to S$ be separated and let*

$$\Delta_{X/S} : X \hookrightarrow X \times_S X$$

be the diagonal, with ideal sheaf $\mathscr{I}_\Delta$. Then

$$\Omega^1_{X/S} \cong \mathscr{I}_\Delta/\mathscr{I}_\Delta^2.$$

Proof. Separatedness makes the diagonal a closed immersion by Chapter 3. On an affine chart $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$, its ideal is the kernel of

$$B \otimes_A B \longrightarrow B.$$

The result is therefore theorem 8.5, and the affine identifications glue because they are canonical. □

We finish the section with the fundamental projective calculation. It will later determine the canonical sheaf of projective space.

Theorem 8.17 (Euler sequence)**.** *Let S be a scheme and let $\pi : \mathbf{P}^n_S \to S$ be projective n-space. There is a natural short exact sequence*

$$0 \longrightarrow \Omega^1_{\mathbf{P}^n_S/S} \longrightarrow \mathscr{O}_{\mathbf{P}^n_S}(-1)^{\oplus(n+1)} \xrightarrow{\varepsilon} \mathscr{O}_{\mathbf{P}^n_S} \longrightarrow 0,$$

where

$$\varepsilon(s_0, \ldots, s_n) = \sum_{i=0}^{n} X_i s_i.$$

Proof. The assertion is local on S, so take $S = \operatorname{Spec} A$. Put $U_i = D_+(X_i)$ and write

$$t_j^{(i)} = \frac{X_j}{X_i} \qquad (j \neq i).$$

Then $U_i \cong \mathbf{A}_A^n$ and

$$\Omega^1_{U_i/S} \cong \bigoplus_{j \neq i} \mathscr{O}_{U_i}\, dt_j^{(i)}.$$

Let u_i denote the local generator of $\mathscr{O}(-1)|_{U_i}$ characterized by $X_i u_i = 1$. The map ε is surjective on U_i, because the vector with u_i in the i-th component and zero elsewhere maps to 1.

For $j \neq i$, define

$$v_j^{(i)} = e_j u_i - t_j^{(i)} e_i u_i.$$

These n sections form a basis of $\ker \varepsilon|_{U_i}$: the equation $\sum X_j s_j = 0$ uniquely determines the i-th coefficient from the other n coefficients. Hence there is an isomorphism

$$\psi_i : \ker \varepsilon|_{U_i} \xrightarrow{\sim} \Omega^1_{U_i/S}, \qquad v_j^{(i)} \longmapsto dt_j^{(i)}.$$

It remains to check that these maps glue. On $U_i \cap U_\ell$,

$$t_j^{(\ell)} = \frac{t_j^{(i)}}{t_\ell^{(i)}}$$

and therefore

$$dt_j^{(\ell)} = \frac{t_\ell^{(i)} dt_j^{(i)} - t_j^{(i)} dt_\ell^{(i)}}{(t_\ell^{(i)})^2}.$$

The local generators satisfy

$$u_\ell = (t_\ell^{(i)})^{-1} u_i,$$

and a direct substitution gives the identical transition formula

$$v_j^{(\ell)} = \frac{t_\ell^{(i)} v_j^{(i)} - t_j^{(i)} v_\ell^{(i)}}{(t_\ell^{(i)})^2}.$$

Thus the ψ_i agree on overlaps. They glue to

$$\ker \varepsilon \cong \Omega^1_{\mathbf{P}^n_S/S},$$

which proves exactness. $\square$

8.2 Tangent Spaces, Nonsingularity, and Smoothness

The sheaf $\Omega^1_{X/S}$ becomes geometrically visible after passing to a residue field. We begin with the intrinsic local definition already implicit in Chapter 4.

Definition 8.18. Let $(A, \mathfrak{m}, \kappa)$ be a local ring. Its *cotangent space* is

$$\mathfrak{m}/\mathfrak{m}^2,$$

and its *Zariski tangent space* is the dual vector space

$$\operatorname{Hom}_\kappa(\mathfrak{m}/\mathfrak{m}^2, \kappa).$$

For a morphism $f : X \to S$ and $x \in X$, the *relative cotangent space* and *relative tangent space* are

$$T^*_{X/S,x} := \Omega^1_{X/S} \otimes_{\mathscr{O}_{X,x}} \kappa(x),$$
$$T_{X/S,x} := \mathrm{Hom}_{\kappa(x)}(T^*_{X/S,x}, \kappa(x)).$$

Proposition 8.19 (Cotangent space at a rational point)**.** *Let X be a k-scheme and let $x \in X(k)$. Then there is a natural isomorphism*

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \xrightarrow{\sim} \Omega^1_{X/k} \otimes \kappa(x).$$

For an arbitrary point x, there is instead a natural right exact sequence

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \Omega^1_{X/k} \otimes \kappa(x) \longrightarrow \Omega^1_{\kappa(x)/k} \longrightarrow 0.$$

Proof. Apply the conormal sequence to the residue homomorphism

$$\mathscr{O}_{X,x} \longrightarrow \kappa(x).$$

It gives

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \Omega_{\mathscr{O}_{X,x}/k} \otimes \kappa(x) \longrightarrow \Omega_{\kappa(x)/k} \longrightarrow 0.$$

Localization identifies the middle term with $\Omega^1_{X/k} \otimes \kappa(x)$. If x is k-rational, then $\kappa(x) = k$ and the final term is zero.

It remains to prove injectivity in the rational case, since the formal conormal sequence alone does not provide it. Let V be a k-vector space, regarded as an $\mathscr{O}_{X,x}$-module through the residue map. Every k-derivation $D : \mathscr{O}_{X,x} \to V$ kills $\mathfrak{m}_x^2$, because $\mathfrak{m}_x$ acts trivially on V. Thus restriction gives

$$\mathrm{Der}_k(\mathscr{O}_{X,x}, V) \longrightarrow \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, V).$$

Conversely, since the residue map splits the inclusion $k \subseteq \mathscr{O}_{X,x}$, every element has a unique form $c + m$ with $c \in k$ and $m \in \mathfrak{m}_x$. A linear map $\lambda : \mathfrak{m}_x/\mathfrak{m}_x^2 \to V$ defines a derivation by $D(c + m) = \lambda(m)$. These constructions are inverse. Hence $\mathfrak{m}_x/\mathfrak{m}_x^2$ and $\Omega^1_{X/k} \otimes k$ represent the same functor on k-vector spaces, proving the isomorphism. □

Proposition 8.20 (Tangent vectors as dual-number points)**.** *Let $f : X \to S$ and let $x \in X$. Put*

$$\kappa(x)[\varepsilon] := \kappa(x)[\varepsilon]/(\varepsilon^2).$$

Then $T_{X/S,x}$ is naturally in bijection with morphisms

$$v : \mathrm{Spec}\,\kappa(x)[\varepsilon] \longrightarrow X$$

over S whose restriction along $\mathrm{Spec}\,\kappa(x) \hookrightarrow \mathrm{Spec}\,\kappa(x)[\varepsilon]$ *is the point x.*

Proof. Work affinely near x and $f(x)$, say $X = \mathrm{Spec}\,B$, $S = \mathrm{Spec}\,A$, and let $\rho : B \to \kappa(x)$ be the residue map. A lifting of x to the dual numbers is a ring homomorphism

$$\phi : B \longrightarrow \kappa(x)[\varepsilon]$$

whose reduction modulo ε is ρ. Hence uniquely

$$\phi(b) = \rho(b) + \varepsilon D(b).$$

Multiplicativity of ϕ is equivalent, because $\varepsilon^2 = 0$, to

$$D(bb') = \rho(b)D(b') + \rho(b')D(b),$$

and the condition that the map is over S says that D vanishes on A. Thus D is an A-derivation from B to $\kappa(x)$. By the universal property,

$$\mathrm{Der}_A(B, \kappa(x)) \cong \mathrm{Hom}_{\kappa(x)}(T^*_{X/S,x}, \kappa(x)),$$

which is the tangent space. □

A morphism $h : X \to Y$ over S therefore has an actual differential on tangent vectors.

Proposition 8.21. *Let $X \xrightarrow{h} Y \to S$ be morphisms and let $x \in X$, $y = h(x)$. Pullback of differentials induces a natural map*

$$dh_x : T_{X/S,x} \longrightarrow T_{Y/S,y} \otimes_{\kappa(y)} \kappa(x)$$

whenever the cotangent space of Y/S at y is finite dimensional.

Proof. The natural morphism $h^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ gives, after tensoring with $\kappa(x)$,

$$T^*_{Y/S,y} \otimes_{\kappa(y)} \kappa(x) \longrightarrow T^*_{X/S,x}.$$

Dualizing over $\kappa(x)$ gives the asserted map. □

Now let

$$X = \operatorname{Spec} B, \qquad B = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r),$$

and let $x \in X(k)$. The conormal presentation turns the tangent space into ordinary linear algebra.

Theorem 8.22 (Affine Jacobian description)**.** *With the notation above,*

$$T_{X/k,x} = \ker\left(k^n \xrightarrow{J_x} k^r\right),$$

where

$$J_x = \left(\frac{\partial f_i}{\partial x_j}(x)\right).$$

Consequently,

$$\dim_k T_{X/k,x} = n - \mathrm{rk}\, J_x.$$

Proof. Tensor the right exact Jacobian presentation of proposition 8.9 with the residue field k. We obtain

$$k^r \xrightarrow{J_x^{\mathsf{T}}} k^n \longrightarrow T^*_{X/k,x} \longrightarrow 0.$$

Dualizing over the field k is exact and identifies the dual map with $J_x : k^n \to k^r$. Hence the tangent space is its kernel, and rank-nullity gives the dimension formula. □

The node and cusp of Chapter 4 now acquire their familiar tangent-space interpretation. For $xy = 0$ at the origin, and for $y^2 - x^3 = 0$ at the origin when $\operatorname{char} k \neq 2, 3$, every first derivative vanishes, so the tangent space has dimension two while the local dimension is one. The tangent space is therefore too large precisely where the local ring fails to be regular.

We next formulate smoothness using the local model which exposes this Jacobian calculation directly.

Definition 8.23 (Standard smooth algebra)**.** An A-algebra is *standard smooth of relative dimension* $n-r$ if it has the form

$$B = (A[x_1, \dots, x_n]/(f_1, \dots, f_r))_g$$

for which some $r \times r$ minor of the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right)$$

is a unit in B.

A morphism $f : X \to S$ is *smooth of relative dimension* d if every point of X has an affine neighborhood which, over an affine neighborhood in S, is induced by a standard smooth algebra of relative dimension d.

Theorem 8.24 (Differentials of a standard smooth algebra)**.** *If B is standard smooth over A of relative dimension d, then $\Omega_{B/A}$ is finite free of rank d. Consequently, if $f : X \to S$ is smooth of relative dimension d, then $\Omega^1_{X/S}$ is locally free of rank d.*

Proof. Write $d = n-r$ and, after reordering the variables, suppose that the first $r \times r$ block J_0 of the Jacobian matrix is invertible. The relations in the Jacobian presentation are

$$J_0 \begin{pmatrix} dx_1 \\ \vdots \\ dx_r \end{pmatrix} = -J_1 \begin{pmatrix} dx_{r+1} \\ \vdots \\ dx_n \end{pmatrix}.$$

Thus $dx_{r+1}, \dots, dx_n$ generate $\Omega_{B/A}$.

Suppose $\sum_{j>r} a_j dx_j = 0$. In the presentation $B^r \to B^n \to \Omega_{B/A} \to 0$, the vector

$$(0, \dots, 0, a_{r+1}, \dots, a_n)^{\mathsf{T}}$$

therefore lies in the image of the transpose of the relation matrix. Looking at its first r coordinates gives $J_0^{\mathsf{T}} c = 0$ for some $c \in B^r$. Since J_0 is invertible, $c = 0$, and then all $a_j = 0$. Hence the displayed generators form a basis. The scheme statement follows on the standard smooth affine neighborhoods. $\square$

Definition 8.25 (Relative tangent sheaf)**.** If $f : X \to S$ is smooth, its *relative tangent sheaf* is

$$\mathscr{T}_{X/S} := \mathscr{H}om_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{O}_X).$$

It is the dual vector bundle of $\Omega^1_{X/S}$, and its fiber at x is naturally $T_{X/S,x}$.

For varieties over an algebraically closed field, this relative notion recovers the intrinsic regularity of Chapter 4.

Definition 8.26. Let k be algebraically closed and let X be a variety over k. We call X *nonsingular* if it is regular as a scheme, equivalently if every local ring $\mathscr{O}_{X,x}$ is regular.

The following local lemma will also control the conormal sequence. We use here the standard commutative-algebra facts that a regular system of parameters in a regular local ring is a regular sequence and that regular local rings satisfy the expected dimension formula for quotients by prime ideals. These are the same level of commutative-algebra input as the factoriality theorem for regular local rings used in Chapter 4.

Lemma 8.27 (Independent parameters in a regular local ring). *Let $(A, \mathfrak{m}, \kappa)$ be a regular local ring of dimension n. If $f_1, \ldots, f_c \in \mathfrak{m}$ have linearly independent classes in $\mathfrak{m}/\mathfrak{m}^2$, then*

$$A/(f_1, \ldots, f_c)$$

is a regular local ring of dimension $n - c$.

Proof. Extend the classes of $f_1, \ldots, f_c$ to a basis of $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama, representatives of this basis form a minimal generating set of $\mathfrak{m}$, hence a regular system of parameters. In a regular local ring a regular system of parameters is a regular sequence. Therefore $f_1, \ldots, f_c$ is a regular sequence, and successive quotients reduce the dimension by one while preserving regularity. It follows that

$$A/(f_1, \ldots, f_c)$$

is regular local of dimension $n - c$. □

Theorem 8.28 (Nonsingularity and the sheaf of differentials). *Let k be algebraically closed and let X be a variety of dimension d over k. The following conditions are equivalent:*

1. *X is nonsingular;*
2. *$X \to \operatorname{Spec} k$ is smooth of relative dimension d;*
3. *$\Omega^1_{X/k}$ is locally free of rank d.*

Proof. We first show that regularity at a closed point produces a standard smooth neighborhood. Let $x \in X(k)$ and choose an affine neighborhood

$$U = \operatorname{Spec} B, \qquad B = k[x_1, \ldots, x_n]/I.$$

Let $\mathfrak{q} \subset k[x_1, \ldots, x_n]$ be the maximal ideal corresponding to x. If $\mathscr{O}_{X,x}$ is regular, then

$$\dim_k T_{X,x} = d$$

by the definition of regularity and proposition 8.19. Hence the image of the conormal map in k^n has dimension $n - d$. Choose $f_1, \ldots, f_c \in I$, with $c = n - d$, whose differentials give a basis of that image. Some $c \times c$ Jacobian minor is nonzero at x.

Put $J = (f_1, \ldots, f_c)$. Localize the polynomial ring at $\mathfrak{q}$. By lemma 8.27,

$$R := k[x_1, \ldots, x_n]_{\mathfrak{q}}/J_{\mathfrak{q}}$$

is a regular local ring of dimension d, hence a domain by theorem 4.8. The ideal $I_{\mathfrak{q}}/J_{\mathfrak{q}}$ has quotient $\mathscr{O}_{X,x}$, also of dimension d. If this ideal were nonzero, any prime minimal over it would be a nonzero prime of the regular local domain R, hence would have positive height. The dimension formula for regular local rings would then make the quotient have dimension at most $d - 1$. Thus

$$I_{\mathfrak{q}} = J_{\mathfrak{q}}.$$

Because I is finitely generated, after shrinking around x we have $I = J$, and the chosen Jacobian minor remains invertible. Hence X is standard smooth near x.

If X is nonsingular, the preceding argument gives a smooth neighborhood of every closed point. Their union is all of X: otherwise its closed complement, being a nonempty finite-type k-scheme, would contain a closed point. Thus (1) implies (2).

The implication (2)⇒(3) is theorem 8.24.

Finally assume (3). At every closed point x, proposition 8.19 gives

$$\dim_k \mathfrak{m}_x/\mathfrak{m}_x^2 = d.$$

A closed point of the d-dimensional variety has local dimension d, so $\mathscr{O}_{X,x}$ is regular. Repeating the first paragraph gives a standard smooth neighborhood of each closed point. These neighborhoods cover X by the same closed-point argument. On each standard smooth chart, the invertible Jacobian minor and lemma 8.27 show that every closed-point local ring is regular. Regularity of a regular local ring is preserved under localization, so every local ring on the chart is regular. Hence X is regular everywhere, proving (1). □

Remark 8.29. Over a perfect field the same equivalence holds after the appropriate residue-field formulation. Over an imperfect field, regularity need not survive extension of the ground field, and the final term $\Omega_{\kappa(x)/k}$ in proposition 8.19 is exactly where inseparability enters. For a general morphism $X \to S$, local freeness of $\Omega^1_{X/S}$ of the expected rank is necessary for smoothness but is not by itself sufficient; the missing condition is related to flatness and geometric regularity of the fibers.

We can now return to the left end of the conormal sequence.

Theorem 8.30 (Conormal sequence for nonsingular subvarieties)**.** *Let k be algebraically closed and let*

$$i : X \hookrightarrow Y$$

be a closed immersion of nonsingular varieties, of dimensions d and n respectively. Then $\mathscr{I}/\mathscr{I}^2$ is locally free of rank $n-d$, and the conormal sequence is short exact:

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow i^*\Omega^1_{Y/k} \longrightarrow \Omega^1_{X/k} \longrightarrow 0.$$

Proof. Fix a closed point $x \in X$ and put

$$A = \mathscr{O}_{Y,x}, \qquad B = \mathscr{O}_{X,x} = A/I.$$

The rings A and B are regular local of dimensions n and d. The map on cotangent spaces

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$$

is surjective, with kernel of dimension $n-d$. Choose $f_1, \ldots, f_{n-d} \in I$ whose classes form a basis of this kernel. Their classes in $\mathfrak{m}_A/\mathfrak{m}_A^2$ are independent, so lemma 8.27 shows that

$$A/(f_1, \ldots, f_{n-d})$$

is a regular local domain of dimension d. It surjects onto B, which has the same dimension. As in the proof of theorem 8.28, the dimension formula for the regular local source shows that a nonzero kernel would force the quotient dimension to drop. Hence

$$I = (f_1, \ldots, f_{n-d}).$$

The right-hand map in the conormal sequence is already surjective. Both $i^*\Omega^1_{Y/k}$ and $\Omega^1_{X/k}$ are locally free, of ranks n and d by theorem 8.28; the surjection therefore splits locally, and its kernel K is locally free of rank $n-d$. Since the f_i generate I, there is a surjection

$$B^{\oplus(n-d)} \longrightarrow I/I^2,$$

and composition with the conormal map gives

$$B^{\oplus(n-d)} \longrightarrow K, \qquad e_i \longmapsto df_i.$$

On the fiber at x, the vectors df_i form a basis of $K \otimes_B k$ by the choice of the f_i. After shrinking, the displayed map between free modules of the same rank is therefore an isomorphism: in local trivializations, one of its maximal determinants is a unit. Because it factors through the surjection $B^{\oplus(n-d)} \to I/I^2$, that surjection must also be injective, and the conormal map $I/I^2 \to K$ is an isomorphism. Hence I/I^2 is locally free of rank $n-d$ and the conormal sequence is short exact near x. Since the same argument applies at every closed point, these neighborhoods cover X and the sequence is short exact globally. □

Corollary 8.31 (Embedded differential criterion for nonsingularity). *Let k be algebraically closed, let Y be a nonsingular variety of dimension n, and let $X \hookrightarrow Y$ be a closed subvariety of dimension d. For a closed point $x \in X$, the following are equivalent after shrinking around x:*

1. *X is nonsingular at x;*
2. *$\Omega^1_{X/k}$ is locally free of rank d, $\mathscr{I}/\mathscr{I}^2$ is locally free of rank $n-d$, and the conormal sequence is left exact, hence short exact;*
3. *locally the ideal of X is generated by $n-d$ functions $f_1, \ldots, f_{n-d}$ whose differentials are linearly independent along X near x.*

Proof. The implication (1)⇒(2) is theorems 8.28 and 8.30. In the proof of theorem 8.30, the ideal is locally generated by $n-d$ elements whose classes in the cotangent space are independent; this gives (3).

Assume (3). The Jacobian presentation shows that the corresponding conormal map has rank $n-d$. After choosing local coordinates on the smooth variety Y, some $(n-d)\times(n-d)$ Jacobian minor is invertible. Thus X is locally standard smooth of dimension d, and theorem 8.28 gives (1).

Finally, (2) itself implies (1) already from the local freeness of $\Omega^1_{X/k}$ of rank d by theorem 8.28; the extra left exactness records the fact that the embedding is locally a regular one. □

The corollary explains the precise role of the left end of the conormal sequence. Left exactness by itself is not a smoothness criterion: singular complete intersections may still have an injective conormal map. What characterizes nonsingularity here is the expected locally free differential geometry together with the regular embedding behavior.

8.3 Infinitesimal Lifting and Bertini's Theorem

The dual-number description of tangent vectors suggests a broader principle: smooth morphisms should allow maps to be lifted through nilpotent first-order thickenings. The algebraic formulation uses square-zero ideals.

Definition 8.32. Let $A \to B$ be a ring homomorphism.

1. It is *formally smooth* if, for every A-algebra C, every ideal $J \subseteq C$ with $J^2 = 0$, and every A-algebra map $B \to C/J$, there exists an A-algebra lift $B \to C$.
2. It is *formally unramified* if such a lift, whenever it exists, is unique.
3. It is *formally ètale* if every such lifting problem has a unique solution.

For schemes the corresponding conditions are imposed locally on the source for every square-zero closed immersion.

Thus formal smoothness means existence of infinitesimal extensions, while formal unramifiedness means uniqueness. Their difference is measured exactly by derivations.

Lemma 8.33 (Difference of two lifts)**.** *Let $J \subseteq C$ satisfy $J^2 = 0$, and let*

$$\phi_1, \phi_2 : B \longrightarrow C$$

be two A-algebra maps inducing the same map $B \to C/J$. Then

$$D(b) = \phi_1(b) - \phi_2(b)$$

defines an A-derivation $B \to J$, where J is a B-module through the common map $B \to C/J$.

Proof. The difference takes values in J. Writing $\phi_1(b) = \phi_2(b) + D(b)$ and using $J^2 = 0$, we find

$$D(bb') = \phi_2(b)D(b') + \phi_2(b')D(b).$$

The action of B on J depends only on the common reduction modulo J, so this is the Leibniz rule. Additivity and vanishing on A are immediate. ☐

Proposition 8.34 (Differential criterion for formal unramifiedness)**.** *A ring homomorphism $A \to B$ is formally unramified if and only if*

$$\Omega_{B/A} = 0.$$

Proof. If $\Omega_{B/A} = 0$, then lemma 8.33 identifies the difference of two lifts with an element of

$$\mathrm{Der}_A(B, J) \cong \mathrm{Hom}_B(\Omega_{B/A}, J) = 0,$$

so the lift is unique.

Conversely, consider the square-zero extension

$$B \oplus \Omega_{B/A}$$

with multiplication

$$(b, \omega)(b', \omega') = (bb', b\omega' + b'\omega).$$

The projection to B has two A-algebra sections

$$s_0(b) = (b, 0), \qquad s_d(b) = (b, db).$$

The second is multiplicative precisely because d satisfies the Leibniz rule. Formal unramifiedness forces $s_0 = s_d$, so $db = 0$ for every b and therefore $\Omega_{B/A} = 0$. ☐

The existence half is encoded by the Jacobian inverse function calculation.

Lemma 8.35 (First-order Taylor formula)**.** *Let $J \subseteq C$ satisfy $J^2 = 0$. If $y_i \in C$, $z_i \in J$, and $f \in C[x_1, \ldots, x_n]$, then*

$$f(y_1 + z_1, \ldots, y_n + z_n) = f(y_1, \ldots, y_n) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(y_1, \ldots, y_n) z_j.$$

Proof. For a monomial, expand the product after substituting $y_i + z_i$. Every term containing two of the z_i lies in J^2 and vanishes. The remaining terms are the original monomial and the terms containing exactly one z_i, which are the displayed derivatives. Additivity gives the result for every polynomial. ☐

Theorem 8.36 (Standard smooth algebras lift infinitesimally)**.** *Every standard smooth A-algebra is formally smooth over A.*

Proof. Write
$$B = (A[x_1, \dots, x_n]/(f_1, \dots, f_r))_g$$
and suppose, after reordering variables, that the $r \times r$ Jacobian block
$$M = \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i,j \le r}$$
has invertible determinant in B.

Let $C \to C/J$ be square-zero and let $\bar{\phi} : B \to C/J$ be an A-algebra map. Choose arbitrary lifts $y_j \in C$ of $\bar{\phi}(x_j)$. Since the equations vanish modulo J,
$$c_i := f_i(y_1, \dots, y_n) \in J.$$
The matrix $M(y)$ reduces modulo J to an invertible matrix. An element of C which is a unit modulo a square-zero ideal is a unit: if $uv = 1 + j$, then $(1 + j)^{-1} = 1 - j$. Hence $M(y)$ is invertible in $M_r(C)$.

Choose $z_1, \dots, z_r \in J$ satisfying
$$M(y) \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} = - \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix},$$
and set $z_{r+1} = \cdots = z_n = 0$. By lemma 8.35,
$$f_i(y + z) = c_i + \sum_{j=1}^{r} \frac{\partial f_i}{\partial x_j}(y) z_j = 0.$$
Thus $x_j \mapsto y_j + z_j$ defines a lift from the quotient polynomial algebra. The image of g is a unit modulo J, hence a unit in C, so the lift extends to the localization B. □

Corollary 8.37. *Let $f : X \to S$ be smooth. Given a square-zero closed immersion $T_0 \hookrightarrow T$ and a commutative square*
$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ T \arrow[r] & S, \end{tikzcd}$$
every point of T_0 has a neighborhood in T on which a dotted lift $T \dashrightarrow X$ exists.

Proof. The problem is local on T and X. After choosing a standard smooth affine neighborhood in X it becomes exactly the lifting problem solved in theorem 8.36. □

By iterating square-zero extensions one obtains the same local lifting statement for a nilpotent thickening whose nilpotent ideal has finite exponent. One concrete consequence is that tangent vectors on a smooth variety have no finite-order obstruction.

Corollary 8.38 (Prolonging infinitesimal arcs)**.** *Let X be a smooth variety over a field k, let $x \in X(k)$, and let*
$$v : \operatorname{Spec} k[t]/(t^2) \longrightarrow X$$
represent a tangent vector at x. For every $m \ge 2$, after choosing an affine smooth neighborhood of x, the map v extends to a morphism
$$v_m : \operatorname{Spec} k[t]/(t^m) \longrightarrow X.$$

Proof. Assume inductively that v_m is defined. The quotient map

$$k[t]/(t^{m+1}) \longrightarrow k[t]/(t^m)$$

has kernel generated by t^m, whose square is zero because $2m \geq m+1$. Formal smoothness of the chosen affine smooth neighborhood, proved in theorem 8.36, lifts v_m to v_{m+1}. Starting with $v_2 = v$ gives the result by induction. □

Remark 8.39 (Comparison with formal smoothness). The standard comparison theorem for smooth morphisms says that a morphism is smooth if and only if it is locally of finite presentation and formally smooth. The forward implication is already contained in the standard smooth presentation and theorem 8.36. The converse is a deeper commutative-algebra comparison result: from finite presentation and the infinitesimal lifting property one extracts, locally, an invertible Jacobian minor. Equivalently, after developing flatness one may use the criterion that a locally finitely presented morphism is smooth exactly when it is flat with geometrically regular fibers. We record this theorem as context rather than use it as an unproved step below; every lifting statement needed in the remainder of this chapter follows directly from the standard smooth calculation above.

We now combine smoothness with the linear systems of Chapter 7. If $i : X \hookrightarrow \mathbf{P}^N_k$ is a closed immersion of nonsingular varieties, dualizing the short exact conormal sequence gives

$$0 \longrightarrow \mathscr{T}_{X/k} \longrightarrow i^*\mathscr{T}_{\mathbf{P}^N/k} \longrightarrow (\mathscr{I}/\mathscr{I}^2)^\vee \longrightarrow 0.$$

Thus the embedded tangent spaces form a vector subbundle of the ambient tangent bundle; this makes the family of tangent hyperplanes algebraic. The simplest form of Bertini's theorem concerns hyperplane sections of a fixed projective embedding and avoids the inseparability phenomena which can occur for arbitrary linear systems in positive characteristic.

Theorem 8.40 (Bertini for hyperplane sections)**.** *Let k be algebraically closed and let*

$$X \hookrightarrow \mathbf{P}^N_k$$

be a nonsingular projective variety of positive dimension d. Then there is a nonempty open subset

$$U \subseteq (\mathbf{P}^N_k)^\vee$$

such that for every hyperplane $H \in U$, the scheme $X \cap H$ is a nonsingular scheme of dimension $d-1$.

Proof. If $d = N$, then the closed integral subvariety $X \subseteq \mathbf{P}^N$ is all of $\mathbf{P}^N$, and every hyperplane section is $\mathbf{P}^{N-1}$; so assume $d < N$.

For $x \in X$, let $\mathbf{T}_x X \subseteq \mathbf{P}^N_k$ denote the embedded projective tangent space. Since X is nonsingular of dimension d, this is a projective d-plane. Consider the incidence set

$$\mathcal{T} := \{(x, H) \in X \times (\mathbf{P}^N)^\vee : \mathbf{T}_x X \subseteq H\}.$$

The tangent bundle of X is locally free and its map to the restriction of the tangent bundle of $\mathbf{P}^N$ varies algebraically. Consequently the condition that the linear form defining H vanish on $\mathbf{T}_x X$ is given locally by linear equations, so $\mathcal{T}$ is closed. More concretely, $\mathcal{T}$ is the projective bundle of hyperplanes annihilating the embedded tangent bundle.

For a fixed x, the hyperplanes containing a fixed projective d-plane form

$$\mathbf{P}^{N-d-1}.$$

Therefore

$$\dim \mathcal{T} = d + (N - d - 1) = N - 1.$$

The projection

$$\mathcal{T} \longrightarrow (\mathbf{P}^N)^\vee$$

is proper because X is projective. Hence its image B is closed. Since

$$\dim B \leq N - 1 < N = \dim(\mathbf{P}^N)^\vee,$$

it is a proper closed subset. Put

$$U = (\mathbf{P}^N)^\vee \setminus B.$$

Take $H \in U$. The intersection $X \cap H$ is nonempty: otherwise X would be a closed subscheme of the affine open $\mathbf{P}^N \setminus H \cong \mathbf{A}^N$, hence both affine and proper over k; by proposition 3.48 it would be finite over k, contradicting $d > 0$. Let $x \in X \cap H$. By construction H does not contain $\mathbf{T}_x X$. If $\ell \in \mathfrak{m}_x$ is a local equation for $H|_X$, then the class of ℓ in $\mathfrak{m}_x/\mathfrak{m}_x^2$ is nonzero: its dual linear form on $T_x X$ is precisely $d\ell|_{T_x X}$. Since $\mathscr{O}_{X,x}$ is a regular local ring of dimension d, lemma 8.27 applied to the single parameter ℓ shows that

$$\mathscr{O}_{X \cap H, x} \cong \mathscr{O}_{X,x}/(\ell)$$

is a regular local ring of dimension $d - 1$. Thus every closed point of $X \cap H$ is nonsingular. The closed-point argument used in theorem 8.28 then shows that the whole hyperplane section is nonsingular of pure dimension $d - 1$. □

Corollary 8.41 (Bertini for a very ample linear system)**.** *Let X be a nonsingular projective variety over an algebraically closed field and let $\mathscr{L}$ be very ample. A general divisor in the complete linear system $|\mathscr{L}|$ is nonsingular.*

Proof. By corollary 7.42, a finite-dimensional space of sections embeds X into projective space and identifies $\mathscr{L}$ with the pullback of $\mathscr{O}(1)$. Divisors in the corresponding complete hyperplane system are hyperplane sections. Apply theorem 8.40. □

8.4 Canonical Sheaves and Birational Differential Forms

First-order differentials have higher exterior powers. For a smooth morphism the top exterior power is a line bundle, and Chapter 7 allows us to reinterpret that line bundle divisor-theoretically.

Definition 8.42. Let $f : X \to S$ be a morphism. For $p \geq 0$, define

$$\Omega^p_{X/S} := \bigwedge^p_{\mathscr{O}_X} \Omega^1_{X/S}, \qquad \Omega^0_{X/S} = \mathscr{O}_X.$$

If f is smooth of pure relative dimension n, its *relative canonical sheaf* is

$$\omega_{X/S} := \Omega^n_{X/S} = \bigwedge^n \Omega^1_{X/S}.$$

When $S = \operatorname{Spec} k$, we write simply ω_X.

Proposition 8.43. *If $f : X \to S$ is smooth of relative dimension n, then $\Omega^p_{X/S}$ is locally free of rank $\binom{n}{p}$. In particular, $\omega_{X/S}$ is invertible.*

Proof. Locally $\Omega^1_{X/S}$ is free with basis $e_1, \ldots, e_n$ by theorem 8.24. The wedge products

$$e_{i_1} \wedge \cdots \wedge e_{i_p}, \qquad i_1 < \cdots < i_p,$$

form a basis of the p-th exterior power. $\square$

We will use the determinant identity from multilinear algebra.

Proposition 8.44 (Determinant of a short exact sequence)**.** *Let*

$$0 \longrightarrow \mathscr{E}' \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}'' \longrightarrow 0$$

be a short exact sequence of finite locally free sheaves. Then there is a canonical isomorphism

$$\det \mathscr{E} \cong \det \mathscr{E}' \otimes \det \mathscr{E}''.$$

Proof. The statement is local. Since $\mathscr{E}''$ is locally free, the sequence splits locally. Choose a basis of $\mathscr{E}'$ and lift a basis of $\mathscr{E}''$ to $\mathscr{E}$. Sending the top wedge of the resulting basis of $\mathscr{E}$ to the tensor product of the two top wedges gives the desired isomorphism. Changing the lifts adds terms containing too many vectors from $\mathscr{E}'$, hence does not change the top wedge; changing either basis multiplies both sides by the same determinant. Thus the local isomorphisms are canonical and glue. $\square$

The Euler sequence now gives its promised consequence.

Corollary 8.45 (Canonical sheaf of projective space)**.** *For every scheme S,*

$$\omega_{\mathbf{P}^n_S/S} \cong \mathscr{O}_{\mathbf{P}^n_S}(-n-1).$$

Over a field, if H denotes the hyperplane class, then the canonical divisor class satisfies

$$K_{\mathbf{P}^n} \sim -(n+1)H.$$

Proof. Taking determinants in the Euler sequence of theorem 8.17 gives

$$\det\bigl(\mathscr{O}(-1)^{\oplus(n+1)}\bigr) \cong \det \Omega^1_{\mathbf{P}^n_S/S} \otimes \det \mathscr{O}.$$

The left side is $\mathscr{O}(-n-1)$ and $\det \mathscr{O} \cong \mathscr{O}$, so

$$\det \Omega^1_{\mathbf{P}^n_S/S} = \omega_{\mathbf{P}^n_S/S} \cong \mathscr{O}(-n-1).$$

By corollary 7.38, $\mathscr{O}(1)$ corresponds to the hyperplane class, giving the divisor statement. $\square$

For a smooth integral variety, a rational top form is a rational section of the canonical line bundle, so Chapter 7 immediately produces a Cartier divisor.

Definition 8.46 (Canonical divisor and geometric genus)**.** Let X be a nonsingular variety of dimension n over an algebraically closed field k. A nonzero element

$$\eta \in \omega_X \otimes_{\mathscr{O}_X} K(X) \cong \bigwedge^n \Omega_{K(X)/k}$$

is a *rational top differential form.* By proposition 7.29, it determines a Cartier divisor

$$\operatorname{div}(\eta).$$

Its divisor class is called the *canonical class*; any representative is denoted K_X.

If X is projective, its *geometric genus* is

$$p_g(X) := h^0(X, \omega_X) = \dim_k H^0(X, \omega_X).$$

If η' is another nonzero rational top form, then $\eta' = f\eta$ for some $f \in K(X)^\times$, and

$$\operatorname{div}(\eta') = \operatorname{div}(\eta) + \operatorname{div}(f).$$

Thus the canonical class is independent of the chosen rational form.

The conormal sequence also gives adjunction formulas.

Theorem 8.47 (Adjunction for a smooth hypersurface)**.** *Let*

$$X = V_+(F) \subseteq \mathbf{P}^n_k$$

be a nonsingular hypersurface defined by a homogeneous polynomial of degree d. Then

$$0 \longrightarrow \mathscr{O}_X(-d) \xrightarrow{dF} \Omega^1_{\mathbf{P}^n/k}|_X \longrightarrow \Omega^1_{X/k} \longrightarrow 0$$

is exact, and

$$\boxed{\omega_X \cong \mathscr{O}_X(d-n-1).}$$

Proof. The ideal sheaf of X is $\mathscr{O}_{\mathbf{P}^n}(-d)$, so

$$\mathscr{I}/\mathscr{I}^2 \cong \mathscr{O}_X(-d).$$

Both X and $\mathbf{P}^n$ are nonsingular, hence theorem 8.30 makes the conormal sequence short exact.

Taking determinants and using proposition 8.44 gives

$$\omega_{\mathbf{P}^n}|_X \cong \mathscr{O}_X(-d) \otimes \omega_X.$$

By corollary 8.45, the left side is $\mathscr{O}_X(-n-1)$. Tensoring with $\mathscr{O}_X(d)$ gives

$$\omega_X \cong \mathscr{O}_X(d-n-1).$$

□

Corollary 8.48 (Smooth complete intersections)**.** *Let $X \subseteq \mathbf{P}^n_k$ be a nonsingular complete intersection cut out by a regular sequence of homogeneous polynomials of degrees $d_1, \dots, d_r$. Then*

$$\omega_X \cong \mathscr{O}_X(d_1 + \cdots + d_r - n - 1).$$

Proof. For a complete intersection,

$$\mathscr{I}/\mathscr{I}^2 \cong \bigoplus_{i=1}^r \mathscr{O}_X(-d_i).$$

Apply theorem 8.30, take determinants, and use corollary 8.45. □

The geometric genus is birational. Before proving this, we extend the codimension-one intersection theorem of Chapter 7 from functions to line bundles.

Lemma 8.49 (Sections of a line bundle are determined in codimension one)**.** *Let X be a normal Noetherian integral scheme with generic point η, and let $\mathscr{L}$ be invertible. Inside the one-dimensional $K(X)$-vector space $\mathscr{L}_\eta$,*

$$H^0(X, \mathscr{L}) = \bigcap_{x \in X^{(1)}} \mathscr{L}_x.$$

More precisely, a rational section of $\mathscr{L}$ is regular on X if and only if it belongs to $\mathscr{L}_x$ for every codimension-one point x.

Proof. Regular sections clearly belong to every stalk. Conversely, let $s \in \mathscr{L}_\eta$ belong to every $\mathscr{L}_x$ for $x \in X^{(1)}$. Cover X by affine opens $U = \operatorname{Spec} A$ on which $\mathscr{L}$ is trivial, with frame e. Write

$$s = fe, \qquad f \in K(X).$$

For every height-one prime $\mathfrak{p} \subseteq A$, the hypothesis $s \in \mathscr{L}_\mathfrak{p}$ says $f \in A_\mathfrak{p}$. Since A is a normal Noetherian domain, lemma 7.5 gives

$$f \in \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_\mathfrak{p} = A.$$

Thus s is regular on every trivializing affine open, and these local sections agree because they are restrictions of the same rational section. □

Theorem 8.50 (Birational invariance of regular top forms)**.** *Let k be algebraically closed and let X and Y be nonsingular projective varieties which are birational over k. Under the induced identification of function fields, restriction to the generic point identifies*

$$H^0(X, \omega_X) \cong H^0(Y, \omega_Y).$$

Consequently,

$$\boxed{p_g(X) = p_g(Y).}$$

Proof. Let

$$\varphi : X \dashrightarrow Y$$

be a birational map and identify $K(X)$ and $K(Y)$ with a common field K by proposition 3.22. Then both spaces of global top forms sit naturally inside

$$\bigwedge^n \Omega_{K/k}, \qquad n = \dim X = \dim Y.$$

We prove that a form regular on X is regular on Y; symmetry then gives equality.

Take

$$\alpha \in H^0(X, \omega_X)$$

and let $y \in Y^{(1)}$. Since Y is nonsingular, it is normal by corollary 4.9, and its local ring

$$R = \mathscr{O}_{Y,y}$$

is a DVR by corollary 4.22; its fraction field is K. Consider the inverse rational map

$$\varphi^{-1} : Y \dashrightarrow X.$$

Its generic point gives a morphism $\operatorname{Spec} K \to X$. Because X is projective and hence proper over k by corollary 3.63, the valuative criterion theorem 3.45 extends this morphism uniquely to

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} k. \end{array}$$

Call the extension $g : \operatorname{Spec} R \to X$.

Pulling back the regular form α gives a regular section of $g^*\omega_X$. The morphism on differentials

$$g^*\Omega^1_{X/k} \longrightarrow \Omega^1_{R/k}$$

induces on top exterior powers a section of

$$\bigwedge^n \Omega^1_{R/k}.$$

At the generic point this map is induced by the identity of the common function field K, so the resulting rational top form is exactly α under the identification with $\bigwedge^n \Omega_{K/k}$. Since R is the local ring of Y at y, localization of relative differentials gives

$$\bigwedge^n \Omega^1_{R/k} = (\omega_Y)_y.$$

Thus

$$\alpha \in (\omega_Y)_y$$

for every codimension-one point y of Y.

The variety Y is normal Noetherian and ω_Y is invertible. By lemma 8.49, α therefore extends to a global section of ω_Y. This gives

$$H^0(X, \omega_X) \subseteq H^0(Y, \omega_Y)$$

inside $\bigwedge^n \Omega_{K/k}$. Applying the same argument to the inverse birational map gives the reverse inclusion. □

Remark 8.51. The proof uses no resolution of indeterminacy. Properness extends a rational map to every codimension-one valuation ring, while normality says that a section of a line bundle which is regular at all codimension-one points is already regular globally. This is precisely where the valuative language of Chapter 3 and the codimension-one intersection theorem of Chapter 7 meet the theory of differentials.

8.5 Higher Direct Images and Relative Duality

Cohomology in Chapter 6 was attached to a single space. For a morphism $f : X \to Y$, the same construction can be performed fiberwise over open subsets of Y and retained as a sheaf. This is the natural language in which duality becomes relative.

Definition 8.52 (Higher direct images)**.** Let $f : X \to Y$ be a morphism of ringed spaces and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Choose an injective resolution

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots .$$

The *i-th higher direct image* is

$$R^i f_* \mathscr{F} := H^i(f_* \mathscr{I}^\bullet).$$

Thus $R^0 f_* = f_*$.

This definition uses only the ordinary derived-functor formalism already used in Chapter 6; no derived category is required.

Proposition 8.53 (Local description of higher direct images)**.** *The sheaf $R^i f_* \mathscr{F}$ is the sheaf associated to the presheaf*

$$V \longmapsto H^i(f^{-1}(V), \mathscr{F}), \qquad V \subseteq Y \text{ open.}$$

Equivalently, for $y \in Y$,

$$(R^i f_* \mathscr{F})_y \cong \varinjlim_{y \in V} H^i(f^{-1}(V), \mathscr{F}).$$

A short exact sequence of sheaves on X gives a natural long exact sequence of higher direct images.

Proof. Let $\mathscr{I}^\bullet$ be an injective resolution. Injective sheaves are flasque by proposition 6.6, and their restrictions to open subsets remain flasque. Hence for every open $V \subseteq Y$ the cohomology of the complex of sections

$$\Gamma(V, f_*\mathscr{I}^\bullet) = \Gamma(f^{-1}(V), \mathscr{I}^\bullet)$$

is $H^i(f^{-1}(V), \mathscr{F})$. On the other hand, $R^i f_*\mathscr{F}$ is the cohomology *sheaf* of $f_*\mathscr{I}^\bullet$; the cohomology sheaf of a complex is obtained by sheafifying the presheaf cohomology of its complexes of sections. This gives the first statement, and taking stalks gives the filtered-colimit formula. The long exact sequence is the standard long exact sequence of right derived functors. □

Corollary 8.54. *Let $f : X \to Y$ be an affine morphism of Noetherian separated schemes and let $\mathscr{F}$ be quasi-coherent. Then*

$$R^i f_*\mathscr{F} = 0 \qquad (i > 0).$$

Proof. For every affine open $V \subseteq Y$, the inverse image $f^{-1}(V)$ is affine. By theorem 6.14,

$$H^i(f^{-1}(V), \mathscr{F}) = 0$$

for $i > 0$. Thus the presheaf whose sheafification is $R^i f_*\mathscr{F}$ vanishes on an affine basis of Y. By proposition 8.53, its associated sheaf is zero. □

For projective morphisms, the higher direct images of coherent sheaves remain coherent. This is the relative form of the finiteness theorem of Chapter 6.

Theorem 8.55 (Projective direct image theorem)**.** *Let $f : X \to Y$ be a projective morphism of Noetherian schemes and let $\mathscr{F}$ be coherent on X. Then each*

$$R^i f_*\mathscr{F}$$

is coherent on Y.

Proof. The assertion is local on Y, so let $V = \operatorname{Spec} A \subseteq Y$ be affine and replace X by $X_V = f^{-1}(V)$. Then X is projective over A. Put

$$M_i := H^i(X, \mathscr{F}).$$

By theorem 6.31, M_i is a finitely generated A-module.

We claim that on V,

$$R^i f_*\mathscr{F} \cong \widetilde{M}_i.$$

It is enough to verify the localization formula on a principal open $D(g) \subseteq V$:

$$M_{ig} \xrightarrow{\sim} H^i(X_{D(g)}, \mathscr{F}|_{X_{D(g)}}).$$

Choose a finite affine cover of X obtained by intersecting a projective embedding with the standard affine charts. Since X is separated, all finite intersections are affine, so theorem 6.23 computes cohomology by the finite Čech complex of sections on these intersections. After base change to $D(g)$, quasi-coherence identifies every term with its localization at g. Localization is exact and commutes with finite direct sums, hence it commutes with the cohomology of this finite complex. This proves the localization formula.

The localization formula says that on the basis of principal opens of V, the presheaf $D(g) \mapsto H^i(X_{D(g)}, \mathscr{F})$ is exactly the module presheaf $D(g) \mapsto (M_i)_g$. After sheafification, proposition 8.53 therefore gives

$$R^i f_*\mathscr{F}|_V \cong \widetilde{M}_i.$$

Since M_i is finite over the Noetherian ring A, this sheaf is coherent by theorem 5.16. The affine opens V cover Y. □

The duality theorem is organized around a trace. We first isolate the top-degree, or order-zero, form of the idea.

Definition 8.56 (Top dualizing sheaf)**.** Let $f : X \to Y$ be proper, with fibers of dimension at most n. An *n-dualizing sheaf* for f is a coherent sheaf ω_f together with a trace morphism

$$\mathrm{Tr}_f : R^n f_* \omega_f \longrightarrow \mathscr{O}_Y$$

such that for every coherent $\mathscr{F}$ the pairing obtained by composition,

$$f_* \mathscr{H}om_X(\mathscr{F}, \omega_f) \otimes R^n f_* \mathscr{F} \longrightarrow R^n f_* \omega_f \xrightarrow{\mathrm{Tr}_f} \mathscr{O}_Y,$$

induces an isomorphism

$$f_* \mathscr{H}om_X(\mathscr{F}, \omega_f) \xrightarrow{\sim} \mathscr{H}om_Y(R^n f_* \mathscr{F}, \mathscr{O}_Y).$$

When $Y = \operatorname{Spec} k$, this says simply

$$\mathrm{Hom}_X(\mathscr{F}, \omega_f) \cong H^n(X, \mathscr{F})^\vee.$$

The representing property also shows uniqueness: two dualizing sheaves with trace represent the same contravariant functor and are uniquely isomorphic in a way compatible with trace.

Projective space provides a completely explicit model, using only the cohomology already computed in Chapter 6.

Theorem 8.57 (Top duality on projective space)**.** *Let k be a field, let*

$$\pi : \mathbf{P}^n_k \longrightarrow \operatorname{Spec} k,$$

and put

$$\omega = \mathscr{O}_{\mathbf{P}^n}(-n-1).$$

Let

$$\mathrm{Tr} : H^n(\mathbf{P}^n, \omega) \longrightarrow k$$

be the map sending the Čech class of

$$\frac{1}{x_0 \cdots x_n}$$

to 1*. Then for every coherent sheaf $\mathscr{F}$ the natural map*

$$\mathrm{Hom}_{\mathbf{P}^n}(\mathscr{F}, \omega) \xrightarrow{\sim} H^n(\mathbf{P}^n, \mathscr{F})^\vee$$

is an isomorphism. Hence (ω, Tr) *is an n-dualizing sheaf for π.*

Proof. For $\mathscr{F} = \mathscr{O}(m)$, the left side is

$$H^0(\mathbf{P}^n, \mathscr{O}(-m-n-1)).$$

By corollary 6.30, multiplication followed by Tr gives a perfect pairing

$$H^n(\mathbf{P}^n, \mathscr{O}(m)) \times H^0(\mathbf{P}^n, \mathscr{O}(-m-n-1)) \longrightarrow k.$$

Thus the assertion holds for every twist and hence for every finite direct sum of twists.

Let $\mathscr{F}$ be coherent. By corollary 5.28, choose an exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{L} \longrightarrow \mathscr{F} \longrightarrow 0$$

with $\mathscr{L}$ a finite direct sum of twists. Since projective n-space has no cohomology above degree n, the top of the long exact sequence is

$$H^n(\mathscr{G}) \longrightarrow H^n(\mathscr{L}) \longrightarrow H^n(\mathscr{F}) \longrightarrow 0.$$

Dualizing gives an exact row

$$0 \longrightarrow H^n(\mathscr{F})^\vee \longrightarrow H^n(\mathscr{L})^\vee \longrightarrow H^n(\mathscr{G})^\vee.$$

Left exactness of $\mathrm{Hom}(-,\omega)$ gives another exact row

$$0 \longrightarrow \mathrm{Hom}(\mathscr{F},\omega) \longrightarrow \mathrm{Hom}(\mathscr{L},\omega) \longrightarrow \mathrm{Hom}(\mathscr{G},\omega).$$

Naturality of cup product and trace gives the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathscr{F},\omega) \arrow[r] \arrow[d, "\psi_{\mathscr{F}}"] & \mathrm{Hom}(\mathscr{L},\omega) \arrow[r] \arrow[d, "\psi_{\mathscr{L}}", "\sim"'] & \mathrm{Hom}(\mathscr{G},\omega) \arrow[d, "\psi_{\mathscr{G}}"] \\
0 \arrow[r] & H^n(\mathscr{F})^\vee \arrow[r] & H^n(\mathscr{L})^\vee \arrow[r] & H^n(\mathscr{G})^\vee.
\end{tikzcd}$$

Since $\psi_{\mathscr{L}}$ is an isomorphism, exactness immediately makes $\psi_{\mathscr{F}}$ injective.

The same argument, applied to a presentation of the coherent sheaf $\mathscr{G}$ by a finite sum of twists, shows that $\psi_{\mathscr{G}}$ is injective. Now take $\lambda \in H^n(\mathscr{F})^\vee$. Its image in $H^n(\mathscr{L})^\vee$ has a unique preimage $u \in \mathrm{Hom}(\mathscr{L},\omega)$ under $\psi_{\mathscr{L}}$. The image of u in $\mathrm{Hom}(\mathscr{G},\omega)$ maps to zero in $H^n(\mathscr{G})^\vee$ by commutativity; injectivity of $\psi_{\mathscr{G}}$ makes this image zero. Thus u descends to an element of $\mathrm{Hom}(\mathscr{F},\omega)$ mapping to λ. Hence $\psi_{\mathscr{F}}$ is surjective as well. □

The previous theorem is the top-degree prototype of relative duality. For a smooth proper morphism the dualizing sheaf is the canonical sheaf, and cup products give pairings in complementary degrees.

Theorem 8.58 (Higher relative duality)**.** *Let*

$$f : X \longrightarrow Y$$

be a smooth projective morphism of pure relative dimension n between locally Noetherian schemes, and let $\mathscr{E}$ be finite locally free. There is a canonical trace

$$\mathrm{Tr}_f : R^n f_*\omega_{X/Y} \longrightarrow \mathscr{O}_Y$$

and natural cup-product pairings

$$R^i f_*\mathscr{E} \otimes R^{n-i} f_*(\mathscr{E}^\vee \otimes \omega_{X/Y}) \longrightarrow R^n f_*\omega_{X/Y} \xrightarrow{\mathrm{Tr}_f} \mathscr{O}_Y.$$

On every open subset of Y on which the two higher direct images in complementary degrees are locally free and their formation commutes with base change, this pairing is perfect. Equivalently, the induced morphism

$$R^{n-i} f_*(\mathscr{E}^\vee \otimes \omega_{X/Y}) \xrightarrow{\sim} \mathscr{H}om_Y(R^i f_*\mathscr{E}, \mathscr{O}_Y)$$

is an isomorphism there.

Proof. This is the higher-order relative duality theorem. Its construction extends the order-zero trace formalism of definition 8.56: one first constructs the trace in top degree and the cup-product morphism, then proves perfectness by the cohomology-and-base-change theorem and the higher-order duality argument. A complete proof requires the flat-base-change machinery for higher direct images and a systematic treatment of Ext beyond what has been developed in these notes so far. We therefore take the higher-order theorem as a standard input at this point.

What is important for the remainder of the chapter is that the statement itself uses only the ordinary sheaves $R^i f_* \mathscr{F}$, tensor products, and the trace map. No derived-category language is needed. The absolute consequence below requires no additional base-change hypothesis, since the base is a point. □

Remark 8.59. The formulation above follows the relative viewpoint in which the trace and the top dualizing property come first and the familiar vector-space duality is obtained only after passing to a point. The distinction between the elementary top-order prototype and the deeper higher-order theorem is deliberate: it keeps the logical cost of the imported theorem visible instead of hiding it inside the notation.

Now take $Y = \operatorname{Spec} k$. Higher direct images are ordinary cohomology groups, so relative duality immediately becomes Serre duality.

Corollary 8.60 (Serre duality for vector bundles)**.** *Let X be a smooth projective variety of dimension n over a field k, and let $\mathscr{E}$ be finite locally free. There is a natural perfect pairing*

$$H^i(X, \mathscr{E}) \times H^{n-i}(X, \mathscr{E}^\vee \otimes \omega_X) \longrightarrow H^n(X, \omega_X) \xrightarrow{\mathrm{Tr}} k.$$

Equivalently,

$$\boxed{H^i(X, \mathscr{E})^\vee \cong H^{n-i}(X, \mathscr{E}^\vee \otimes \omega_X).}$$

Proof. Apply theorem 8.58 to the structure morphism

$$f : X \longrightarrow \operatorname{Spec} k.$$

Since the base is the one-point scheme $\operatorname{Spec} k$, the presheaf in proposition 8.53 has only its global value, and hence

$$R^i f_* \mathscr{E} = H^i(X, \mathscr{E}),$$

and $\omega_{X/k} = \omega_X$. Every finite-dimensional vector space over a field is locally free on $\operatorname{Spec} k$, and base change is vacuous. The relative perfect pairing is therefore exactly the displayed pairing. □

For a general coherent sheaf the usual formulation replaces the complementary cohomology group by Ext.

Theorem 8.61 (Serre duality for coherent sheaves)**.** *Let X be a smooth projective variety of dimension n over a field k. For every coherent sheaf $\mathscr{F}$ there are natural isomorphisms*

$$\boxed{H^i(X, \mathscr{F})^\vee \cong \operatorname{Ext}_X^{n-i}(\mathscr{F}, \omega_X).}$$

If $\mathscr{F}$ is locally free, this reduces to corollary 8.60 because

$$\operatorname{Ext}_X^q(\mathscr{F}, \omega_X) \cong H^q(X, \mathscr{F}^\vee \otimes \omega_X).$$

Proof. We use one standard homological fact about regular Noetherian schemes: on a regular scheme of dimension n, every coherent sheaf has locally free dimension at most n. Since X is projective, corollary 5.28 lets us choose the successive locally free surjections globally. Thus there is a finite locally free resolution

$$0 \longrightarrow \mathscr{E}_m \longrightarrow \cdots \longrightarrow \mathscr{E}_0 \longrightarrow \mathscr{F} \longrightarrow 0, \qquad m \leq n.$$

Put $\mathscr{F}_0 = \mathscr{F}$ and let $\mathscr{F}_{j+1}$ be the kernel of $\mathscr{E}_j \twoheadrightarrow \mathscr{F}_j$. We have short exact sequences

$$0 \longrightarrow \mathscr{F}_{j+1} \longrightarrow \mathscr{E}_j \longrightarrow \mathscr{F}_j \longrightarrow 0.$$

For the final syzygy, which is locally free, the assertion is corollary 8.60. Suppose the duality isomorphisms have been established for $\mathscr{F}_{j+1}$. The short exact sequence above gives a long exact sequence in cohomology; dualizing it is exact because the groups are finite-dimensional k-vector spaces. Applying $\mathrm{Hom}_X(-, \omega_X)$ and its ordinary right derived functors gives the corresponding long exact Ext sequence. Naturality of the vector-bundle duality for $\mathscr{E}_j$ makes the two long exact sequences into a commutative ladder. The already known isomorphisms for $\mathscr{F}_{j+1}$ and $\mathscr{E}_j$, together with the five lemma applied successively along the ladder, give

$$H^i(X, \mathscr{F}_j)^\vee \cong \mathrm{Ext}_X^{n-i}(\mathscr{F}_j, \omega_X)$$

for every i. Descending induction on j reaches $\mathscr{F}_0 = \mathscr{F}$.

The only imported ingredient here is the finite locally free dimension of coherent sheaves on a regular Noetherian scheme; the argument itself uses ordinary long exact sequences of Ext and cohomology. When $\mathscr{F}$ is locally free, the internal Hom is $\mathscr{F}^\vee \otimes \omega_X$ and higher sheaf Ext vanishes, so the result reduces to the vector-bundle form. □

Finally, projective space closes the circle with Chapter 6. Taking

$$\mathscr{E} = \mathscr{O}_{\mathbf{P}^n}(d)$$

and using corollary 8.45, we obtain

$$\mathscr{E}^\vee \otimes \omega_{\mathbf{P}^n} \cong \mathscr{O}_{\mathbf{P}^n}(-d-n-1).$$

Hence Serre duality gives

$$\boxed{H^i(\mathbf{P}^n, \mathscr{O}(d))^\vee \cong H^{n-i}(\mathbf{P}^n, \mathscr{O}(-d-n-1)).}$$

This is exactly the symmetry first discovered by the explicit Čech calculation in corollary 6.30. The calculation showed that the symmetry is true; the canonical sheaf and the trace explain why the shift by $-n-1$ is inevitable.

The chapter may therefore be read as two chains which eventually meet:

$$\Omega_{B/A} \longrightarrow \Omega^1_{X/S} \longrightarrow T_{X/S,x} \longrightarrow \text{smoothness} \longrightarrow \text{infinitesimal lifting},$$

and

$$\Omega^1_{X/S} \to \omega_{X/S} \to K_X,\ p_g(X) \to R^i f_* \to \text{relative duality} \to \text{Serre duality}.$$

Differentials thus connect the infinitesimal geometry of a morphism with the global cohomological symmetry of a projective variety.

8.6 The Cohomology Class of a Smooth Subvariety

The trace map and the exterior powers of differentials give one further construction which will be needed for surfaces. Throughout this section, let X be a smooth projective integral variety of dimension n over an algebraically closed field k, and let

$$i : Y \hookrightarrow X$$

be a smooth closed integral subvariety of codimension p. Thus Y has dimension $n-p$ and $\omega_Y = \Omega_Y^{n-p}$.

Restriction of differential forms, followed by the natural quotient to relative differentials on Y, gives

$$i^*\Omega_X^{n-p} \longrightarrow \Omega_Y^{n-p} = \omega_Y.$$

Taking cohomology and then the trace of Serre duality on Y produces a functional

$$H^{n-p}(X, \Omega_X^{n-p}) \longrightarrow H^{n-p}(Y, \omega_Y) \xrightarrow{\mathrm{Tr}_Y} k. \tag{8.1}$$

Definition 8.62 (Cohomology class of a smooth subvariety)**.** The wedge product is a perfect pairing of vector bundles

$$\Omega_X^p \otimes \Omega_X^{n-p} \longrightarrow \omega_X,$$

and hence identifies

$$\Omega_X^p \simeq \mathcal{H}om_{\mathscr{O}_X}(\Omega_X^{n-p}, \omega_X).$$

By Serre duality on X, the dual of $H^{n-p}(X, \Omega_X^{n-p})$ is therefore $H^p(X, \Omega_X^p)$. The class

$$\boxed{[Y] \in H^p(X, \Omega_X^p)}$$

is the unique element corresponding to the functional (8.1).

Equivalently, $[Y]$ is characterized by the identity

$$\mathrm{Tr}_X([Y] \smile \alpha) = \mathrm{Tr}_Y(i^*\alpha) \qquad (\alpha \in H^{n-p}(X, \Omega_X^{n-p})). \tag{8.2}$$

Every arrow in this characterization is canonical, so the construction is independent of local equations, coordinates, and choices of a canonical form.

Proposition 8.63 (Functorial normalization)**.** *If $p = 0$ and $Y = X$, then $[X] = 1 \in H^0(X, \mathscr{O}_X)$. If $p = n$ and $Y = \{y\}$ is a closed point, then*

$$\mathrm{Tr}_X([y]) = 1.$$

For a finite disjoint union of smooth closed subvarieties of the same codimension, the associated class is the sum of their classes.

Proof. For $Y = X$, equation (8.2) says that multiplication by $[X]$ leaves the trace functional unchanged, so $[X] = 1$. For a closed point, $H^0(Y, \omega_Y) = k$ and its trace is the identity, giving the second assertion. For a disjoint union, both restriction and trace are direct sums; the defining functional is therefore the sum of the component functionals, and Serre duality is linear. □

The construction in this section is deliberately limited to smooth subvarieties. It does not require Chow groups or a general intersection product. In section 11.5 we will return to the case $p = 1$ and identify the class of a smooth divisor with its first Chern class; on a surface, the cup product of two such classes recovers the intersection number.

Chapter 9

Flat Families and Smooth Morphisms

Flatness is the condition that makes a varying scheme behave like a family rather than a collection of unrelated fibres. The algebraic definition is expressed through tensor products, but its geometric force is seen in three places: cohomology survives flat changes of base, dimensions of fibres obey an exact formula, and in projective families the Hilbert polynomial is constant. Smoothness adds a geometric condition on the fibres and thereby turns a flat family into a family without singularities.

The organization of this chapter follows the geometric progression of Hartshorne, Chapter III, §§9–10. We shall freely use the standard commutative-algebra theory of flat modules: locality, transitivity, stability under base change, the local criterion for flatness, finite flat modules over local rings being free, and torsion-free modules over a discrete valuation ring being flat. These facts are used as algebraic input rather than redeveloped here.

9.1 Flat Morphisms and Flat Base Change

Let $f : X \to Y$ be a morphism and let $\mathscr{F}$ be an $\mathscr{O}_X$-module. Flatness over the base is measured stalkwise.

Definition 9.1. Let $x \in X$ and put $y = f(x)$.

1. The sheaf $\mathscr{F}$ is *flat over Y at x* if the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$, regarded as an $\mathscr{O}_{Y,y}$-module through
$$\mathscr{O}_{Y,y} \longrightarrow \mathscr{O}_{X,x},$$
is flat.

2. It is *flat over Y* if it is flat over Y at every point of X.

3. The morphism f is *flat* if $\mathscr{O}_X$ is flat over Y.

For an affine morphism the definition is exactly the usual module-theoretic one.

Proposition 9.2 (Basic properties of flat morphisms)**.** *Let $f : X \to Y$ be a morphism of schemes.*

1. *Every open immersion is flat.*

2. *Suppose*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

is Cartesian. If $\mathscr{F}$ is flat over Y, then $v^*\mathscr{F}$ is flat over Y'. In particular, a base change of a flat morphism is flat.

3. *If $X \xrightarrow{f} Y \xrightarrow{g} Z$, if $\mathscr{F}$ is flat over Y, and if Y is flat over Z, then $\mathscr{F}$ is flat over Z. In particular, compositions of flat morphisms are flat.*

4. *If $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, and $\mathscr{F} = \widetilde{M}$ for a B-module M, then $\mathscr{F}$ is flat over Y if and only if M is flat over A.*

Proof. The assertions are local on X and Y. For an open immersion, every local-ring map is an isomorphism, hence flat.

For base change, take affine neighborhoods $Y = \operatorname{Spec} A$, $Y' = \operatorname{Spec} A'$, $X = \operatorname{Spec} B$, so that

$$X' = \operatorname{Spec}(B \otimes_A A').$$

If M represents $\mathscr{F}$ on X, then $v^*\mathscr{F}$ is represented by

$$M \otimes_A A'.$$

Flatness of M over A implies flatness of $M \otimes_A A'$ over A'. The stalkwise statement follows after localization.

Transitivity is the corresponding transitivity theorem for flat modules, applied to

$$\mathscr{O}_{Z,g(y)} \longrightarrow \mathscr{O}_{Y,y} \longrightarrow \mathscr{O}_{X,x}.$$

Finally, if $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, then

$$(\widetilde{M})_{\mathfrak{q}} = M_{\mathfrak{q}}$$

for $\mathfrak{q} \in \operatorname{Spec} B$. Flatness of M over A implies flatness of every localization $M_{\mathfrak{q}}$ over $A_{\mathfrak{p}}$, where $\mathfrak{p} = \mathfrak{q} \cap A$. Conversely, if all these stalks are flat, then for each $\mathfrak{p} \in \operatorname{Spec} A$ the localizations of $M_{\mathfrak{p}}$ at all primes of $B \otimes_A A_{\mathfrak{p}}$ are flat over $A_{\mathfrak{p}}$. Flatness is local on the source module, so $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for every $\mathfrak{p}$, and hence M is flat over A. $\square$

The first substantial geometric use of flatness is that higher direct images commute with a flat change of base. This is precisely the kind of base-change input that was left implicit in the relative duality discussion of Chapter 8.

Theorem 9.3 (Flat base change). *Let $f : X \to Y$ be a separated morphism of finite type between Noetherian schemes, let $\mathscr{F}$ be quasi-coherent on X, and let $u : Y' \to Y$ be a flat morphism with Y' Noetherian. Form the Cartesian square*

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y. \end{tikzcd}$$

Then for every $i \geq 0$ the natural base-change morphism

$$u^* R^i f_* \mathscr{F} \longrightarrow R^i g_* v^* \mathscr{F}$$

is an isomorphism.

Proof. The assertion is local on Y and Y'. Choose affine opens

$$Y = \operatorname{Spec} A, \qquad Y' = \operatorname{Spec} A'$$

with A' flat over A. Since f is separated and of finite type over the Noetherian affine scheme Y, the scheme X is Noetherian, separated, and quasi-compact. Choose a finite affine open cover

$$\mathfrak{U} = \{U_1, \dots, U_r\}$$

of X. Because X is separated, every finite intersection

$$U_{i_0 \cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$$

is affine. By theorem 6.23, the finite Čech complex

$$C^\bullet(\mathfrak{U}, \mathscr{F})$$

computes $H^i(X, \mathscr{F})$.

Let $U'_{i_0 \cdots i_p} = U_{i_0 \cdots i_p} \times_A A'$. These opens form the pulled-back affine cover $\mathfrak{U}'$ of X'. Quasi-coherence and affine base change give

$$\Gamma(U'_{i_0 \cdots i_p}, v^* \mathscr{F}) \cong \Gamma(U_{i_0 \cdots i_p}, \mathscr{F}) \otimes_A A'.$$

Hence there is a natural isomorphism of complexes

$$C^\bullet(\mathfrak{U}', v^* \mathscr{F}) \cong C^\bullet(\mathfrak{U}, \mathscr{F}) \otimes_A A'.$$

Because A' is flat over A, tensoring with A' is exact and therefore commutes with the cohomology of this complex. Thus

$$H^i(X', v^* \mathscr{F}) \cong H^i(X, \mathscr{F}) \otimes_A A'.$$

It remains to identify the corresponding sheaves on the affine base. If $a \in A$, the same Čech complex localized at a computes the cohomology over $D(a)$, and exactness of localization gives

$$H^i(f^{-1} D(a), \mathscr{F}) \cong H^i(X, \mathscr{F})_a.$$

Hence, on $Y = \operatorname{Spec} A$,

$$R^i f_* \mathscr{F} \cong \widetilde{H^i(X, \mathscr{F})}.$$

The identical argument on $Y' = \operatorname{Spec} A'$ gives

$$R^i g_* v^* \mathscr{F} \cong \widetilde{H^i(X', v^* \mathscr{F})}.$$

Under these identifications the base-change morphism is the sheaf associated with

$$H^i(X, \mathscr{F}) \otimes_A A' \xrightarrow{\sim} H^i(X', v^* \mathscr{F}),$$

so it is an isomorphism. □

Corollary 9.4 (Localization of cohomology under a flat change of base)**.** *Let $A \to A'$ be a flat homomorphism of Noetherian rings, let X be a separated scheme of finite type over A, and let $\mathscr{F}$ be quasi-coherent. Then*

$$H^i(X, \mathscr{F}) \otimes_A A' \xrightarrow{\sim} H^i(X_{A'}, \mathscr{F}_{A'})$$

for all $i \geq 0$.

Proof. Apply theorem 9.3 to

$$\operatorname{Spec} A' \longrightarrow \operatorname{Spec} A$$

and then take global sections over the affine base. □

9.2 Dimension and Flat Families

The point of flatness becomes visible when one compares fibres. A nonflat family can gain an entire component at a special parameter, while a flat family may change its geometry without changing its numerical size.

Definition 9.5. A *flat family over T* is a flat morphism

$$f : X \longrightarrow T.$$

When $t \in T$, the member of the family over t is the fibre

$$X_t = X \times_T \operatorname{Spec} \kappa(t).$$

Example 9.6 (A dimension jump)**.** Consider

$$X = \operatorname{Spec} k[t, x]/(tx) \longrightarrow \mathbb{A}^1_k = \operatorname{Spec} k[t].$$

For $a \neq 0$, the fibre over $t = a$ is the single reduced point defined by $x = 0$. The fibre over 0 is

$$\operatorname{Spec} k[x] = \mathbb{A}^1_k.$$

The element t is a zero divisor in $k[t, x]/(tx)$, so this algebra is not torsion-free over the PID $k[t]$, hence not flat. The failure of flatness is accompanied by a jump in fibre dimension.

Example 9.7 (A flat family with a nonreduced special fibre)**.** Assume $\operatorname{char} k \neq 2$. The morphism

$$\operatorname{Spec} k[t, x]/(x^2 - t) \longrightarrow \operatorname{Spec} k[t]$$

is finite flat of rank 2, because

$$k[t, x]/(x^2 - t) \cong k[t] \oplus k[t]x$$

as a $k[t]$-module. For $a \neq 0$ the geometric fibre consists of two reduced points, while the fibre over 0 is

$$\operatorname{Spec} k[x]/(x^2),$$

a nonreduced double point. Flatness does not require the fibres to be isomorphic or reduced; it preserves a subtler algebraic size.

Example 9.8 (A flat degeneration of conics)**.** Inside $\mathbb{P}^2_{k[t]}$ consider

$$\mathscr{X} = V(XY - tZ^2).$$

The homogeneous coordinate ring

$$S = k[t, X, Y, Z]/(XY - tZ^2)$$

is a domain and hence torsion-free, therefore flat, over the PID $k[t]$. On each standard projective chart the coordinate ring is the degree-zero part of a homogeneous localization of S; this is a direct summand of a flat $k[t]$-module and is therefore flat. Hence $\mathscr{X} \to \mathbb{A}^1_k$ is flat. For $t \neq 0$ the fibre is a smooth conic, while the special fibre is the reducible conic

$$V(XY) = V(X) \cup V(Y).$$

Thus irreducibility and smoothness may fail under flat specialization even when the Hilbert polynomial remains unchanged.

The first general control on the fibres is the dimension formula.

Proposition 9.9 (Dimension formula for a flat morphism)**.** *Let*

$$f : X \longrightarrow Y$$

be a flat morphism of finite type between schemes of finite type over a field k. For $x \in X$ and $y = f(x)$,

$$\boxed{\dim_x X_y = \dim_x X - \dim_y Y.}$$

Here $\dim_x X = \dim \mathscr{O}_{X,x}$ and similarly for the fibre and the base.

Proof. The assertion is unchanged after replacing Y by the local scheme

$$Y' = \operatorname{Spec} \mathscr{O}_{Y,y}$$

and X by $X' = X \times_Y Y'$. Indeed, x has a canonical lift to X', the fibre over the closed point of Y' is still X_y, and the relevant local rings at the lifted point are unchanged. We may also replace Y' by its reduction. If N is the nilradical of the Noetherian local ring $\mathscr{O}_{Y,y}$, then N is nilpotent, and its extension to the local ring on X' is again nilpotent. Passing to the quotients therefore changes neither Krull dimension; flatness is preserved because this is a base change. We may consequently assume that $Y = \operatorname{Spec} A$ is reduced local with closed point y, and write

$$d = \dim A.$$

We argue by induction on d.

If $d = 0$, the reduced local ring A is a field. Then Y has one point and $X_y = X$, so the formula is immediate.

Assume $d > 0$. Since A is reduced Noetherian, its zero divisors form the union of its finitely many minimal primes. The maximal ideal cannot be the union of the minimal primes, because $d > 0$. By prime avoidance choose

$$a \in \mathfrak{m}_y$$

which lies in no minimal prime. Then a is a nonzerodivisor on A. Flatness of $\mathscr{O}_{X,x}$ over A implies that multiplication by a remains injective on $\mathscr{O}_{X,x}$: tensor the exact sequence

$$0 \longrightarrow A \xrightarrow{\cdot a} A$$

with $\mathscr{O}_{X,x}$. Thus the image of a is also a nonzerodivisor on $\mathscr{O}_{X,x}$.

Let

$$Y_1 = V(a) \subseteq Y, \qquad X_1 = X \times_Y Y_1.$$

By base change, $X_1 \to Y_1$ is flat. Since a belongs to the maximal ideals and is a nonzero-divisor in both local rings,

$$\dim_y Y_1 = \dim_y Y - 1, \qquad \dim_x X_1 = \dim_x X - 1.$$

The fibre of $X_1 \to Y_1$ at y is still X_y. The induction hypothesis therefore gives

$$\dim_x X_y = \dim_x X_1 - \dim_y Y_1 = (\dim_x X - 1) - (\dim_y Y - 1),$$

which is the desired formula. □

Lemma 9.10 (Local dimension on an equidimensional variety)**.** *Let Z be an equidimensional scheme of finite type over a field k, of dimension d. For every point $z \in Z$,*

$$\dim_z Z = d - \dim \overline{\{z\}}.$$

If Z is irreducible, this is the familiar identity

$$\mathrm{ht}(\mathfrak{p}) + \dim A/\mathfrak{p} = \dim A$$

on an affine neighborhood $\operatorname{Spec} A$ *of z.*

Proof. Choose an affine neighborhood $\operatorname{Spec} A$ of z, and let $\mathfrak{p} \subseteq A$ correspond to z. Let

$$\mathfrak{p}_1, \dots, \mathfrak{p}_s$$

be the minimal primes of A contained in $\mathfrak{p}$. They correspond exactly to the irreducible components through z. After shrinking the affine neighborhood if necessary, each such component still has dimension d, so

$$\dim A/\mathfrak{p}_i = d.$$

For each i, the finitely generated integral k-algebra $A/\mathfrak{p}_i$ satisfies the dimension formula from Noether normalization:

$$\mathrm{ht}(\mathfrak{p}/\mathfrak{p}_i) + \dim A/\mathfrak{p} = \dim A/\mathfrak{p}_i = d.$$

Thus every component through z gives the same value

$$\mathrm{ht}(\mathfrak{p}/\mathfrak{p}_i) = d - \dim A/\mathfrak{p}.$$

The dimension of the local ring is the supremum of these heights, hence

$$\dim \mathscr{O}_{Z,z} = d - \dim A/\mathfrak{p}.$$

Finally, $\dim A/\mathfrak{p} = \dim \overline{\{z\}}$, which proves the assertion. □

Corollary 9.11 (Equidimensionality in a flat family)**.** *Let $f : X \to Y$ be a flat morphism of finite type over a field k, and assume that Y is irreducible. Fix $n \geq 0$. The following are equivalent:*

1. *every irreducible component of X has dimension* $\dim Y + n$*;*
2. *for every $y \in Y$, every irreducible component of X_y has dimension n.*

Proof. Assume (1). Fix $y \in Y$ and let W be an irreducible component of X_y. Choose a closed point $x \in W$ which lies on no other irreducible component of X_y. Then

$$\dim_x X_y = \dim W.$$

The dimension formula proposition 9.9 gives

$$\dim W = \dim_x X - \dim_y Y.$$

Since X is equidimensional of dimension $\dim Y + n$, lemma 9.10 gives

$$\dim_x X = \dim Y + n - \dim \overline{\{x\}},$$

while irreducibility of Y gives

$$\dim_y Y = \dim Y - \dim \overline{\{y\}}.$$

Because x is closed in the fibre X_y, the residue extension

$$\kappa(y) \subseteq \kappa(x)$$

is finite. Therefore

$$\dim \overline{\{x\}} = \operatorname{trdeg}_k \kappa(x) = \operatorname{trdeg}_k \kappa(y) = \dim \overline{\{y\}}.$$

Substituting gives $\dim W = n$. Hence (2) holds.

Conversely assume (2), and let Z be an irreducible component of X. Choose a closed point $x \in Z$ which lies on no other irreducible component of X, and put $y = f(x)$. Since x is closed in the finite-type k-scheme X, its image y is closed in Y. Thus

$$\dim_x X = \dim Z, \qquad \dim_y Y = \dim Y.$$

The point x lies on an irreducible component of the fibre X_y, and by (2) every such component has dimension n; because x was chosen away from the other components of X, the local fibre dimension at x is n. Hence proposition 9.9 gives

$$n = \dim_x X_y = \dim Z - \dim Y,$$

so $\dim Z = \dim Y + n$. This proves (1). $\square$

For a one-dimensional regular base, flatness has a particularly transparent description in terms of associated points.

Definition 9.12. Let X be a Noetherian scheme. A point $\xi \in X$ is an *associated point* if, on some affine open $U = \operatorname{Spec} A$ containing ξ, the corresponding prime belongs to

$$\operatorname{Ass}(A).$$

The set of associated points is denoted $\operatorname{Ass}(X)$.

This is independent of the chosen affine neighborhood because associated primes localize. For a reduced Noetherian scheme the associated points are exactly the generic points of its irreducible components.

Proposition 9.13 (Flatness over a regular curve)**.** *Let*

$$f : X \longrightarrow Y$$

be a morphism of finite type between Noetherian schemes, and suppose that Y is regular, integral, and one-dimensional. Then

$$\boxed{f \textit{ is flat} \iff \textit{every point of } \operatorname{Ass}(X) \textit{ maps to } \eta_Y.}$$

If X is reduced, this says that f is flat if and only if every irreducible component of X dominates Y.

Proof. Fix $x \in X$ and put $y = f(x)$. If $y = \eta_Y$, then $\mathscr{O}_{Y,y} = K(Y)$ is a field, so $\mathscr{O}_{X,x}$ is automatically flat over it.

Suppose now that y is closed. The local ring

$$R = \mathscr{O}_{Y,y}$$

is a discrete valuation ring; choose a uniformizer π. A module over a DVR is flat if and only if it is torsion-free. Thus $\mathscr{O}_{X,x}$ is flat over R exactly when multiplication by π is injective on $\mathscr{O}_{X,x}$.

In a Noetherian ring, an element is a zero divisor if and only if it belongs to an associated prime. Therefore multiplication by π fails to be injective on $\mathscr{O}_{X,x}$ precisely when some associated prime of the local ring $\mathscr{O}_{X,x}$ contains the image of π. Associated primes of $\mathscr{O}_{X,x}$ are the localizations of associated points $\xi \in \operatorname{Ass}(X)$ specializing to x. The condition $\pi \in \mathfrak{p}_\xi$ means exactly that $f(\xi) = y$, since π vanishes precisely at the closed point of the local curve.

Consequently, f is flat at every point above y if and only if no associated point of X maps to y. Repeating this for all closed points of Y gives the stated criterion.

If X is reduced, its associated points are the generic points of its irreducible components. Such a generic point maps to η_Y exactly when the corresponding component dominates Y. □

Remark 9.14. The one-dimensional hypothesis is essential. For instance, the blow-up

$$\operatorname{Bl}_0 \mathbb{A}^2_k \longrightarrow \mathbb{A}^2_k$$

is dominant between integral nonsingular surfaces, but it is not flat: the generic fibre is a point while the exceptional fibre has dimension one, contradicting proposition 9.9.

9.3 Flat Limits and Hilbert Polynomials

Over a regular curve, flatness determines the correct scheme structure on a limiting fibre. Projectively, the same principle is measured by the Hilbert polynomial.

Proposition 9.15 (Unique flat closure over a punctured regular curve)**.** *Let Y be a regular integral scheme of dimension one, let $P \in Y$ be a closed point, and put*

$$U = Y \setminus \{P\}.$$

Suppose

$$X \hookrightarrow \mathbb{P}^n_U$$

is a closed subscheme flat over U. Let

$$\overline{X} \hookrightarrow \mathbb{P}^n_Y$$

be its scheme-theoretic closure. Then $\overline{X}$ is flat over Y. Moreover it is the unique closed subscheme of $\mathbb{P}^n_Y$ which is flat over Y and restricts to X over U.

Proof. Flatness and the assertion about closure are local near P, so replace Y by

$$\operatorname{Spec} R,$$

where $R = \mathscr{O}_{Y,P}$ is a DVR with uniformizer π. Cover $\mathbb{P}^n_R$ by its standard affine opens. On one such affine open write the coordinate ring as an R-algebra A. The closed subscheme X over the punctured spectrum is defined by an ideal

$$I \subseteq A[1/\pi].$$

The scheme-theoretic closure is defined on this chart by the contracted ideal

$$J = I \cap A.$$

By construction the natural map

$$A/J \longrightarrow (A/J)[1/\pi]$$

is injective: if the class of $a \in A$ becomes zero after inverting π, then $\pi^m a \in J$ for some m, hence $\pi^m a \in I \cap A$; but then $a \in I$ in $A[1/\pi]$, so $a \in I \cap A = J$. Thus A/J has no π-torsion. Over the DVR R, torsion-free modules are flat. Hence every affine chart of $\overline{X}$ is flat over R, so $\overline{X}$ is flat over Y.

For uniqueness, let $Z \subseteq \mathbb{P}^n_R$ be another flat closed extension. On the same affine chart let $J_Z \subseteq A$ be its ideal. Since $Z|_U = X$, the localized ideals agree:

$$(J_Z)[1/\pi] = I.$$

Certainly $J_Z \subseteq I \cap A = J$. Conversely, take $a \in J$. Since $a/1 \in I = (J_Z)[1/\pi]$, some power $\pi^m a$ lies in J_Z. Flatness of A/J_Z over R makes multiplication by π injective, so from

$$\pi^m \overline{a} = 0 \quad \text{in } A/J_Z$$

we obtain $\overline{a} = 0$. Hence $a \in J_Z$, so $J = J_Z$. The two closed subschemes agree on every standard affine chart and are therefore equal. $\square$

The proposition explains why a flat limit may carry nilpotents or embedded structure which is invisible in a set-theoretic limit: the scheme structure is forced by the requirement that no torsion appear over the parameter curve.

We now turn to the numerical invariant which characterizes flatness for projective families. The Hilbert polynomial for a projective scheme over a field was introduced in Chapter 6 through the Euler characteristic.

Theorem 9.16 (Hilbert polynomial criterion for flatness)**.** *Let T be an integral Noetherian scheme, let*

$$\pi : \mathbb{P}^n_T \longrightarrow T$$

be projective space, and let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n_T$. For $t \in T$, write

$$\mathscr{F}_t = \mathscr{F}|_{\mathbb{P}^n_{\kappa(t)}}$$

and let

$$P_t(m) = \chi(\mathbb{P}^n_{\kappa(t)}, \mathscr{F}_t(m))$$

be its Hilbert polynomial. Then the following are equivalent:

1. *$\mathscr{F}$ is flat over T;*
2. *the polynomial P_t is independent of $t \in T$.*

Proof. Flatness is local on T. Since T is integral, to prove or test constancy it is enough to compare each point with the generic point. Fix $t \in T$, replace T by the local scheme

$$\operatorname{Spec} A := \operatorname{Spec} \mathscr{O}_{T,t},$$

and write $K = \operatorname{Frac}(A)$ and $k = A/\mathfrak{m}$. We may therefore assume throughout the proof that A is a Noetherian local domain. Put

$$X = \mathbb{P}^n_A.$$

We prove the equivalence through the intermediate condition

$$H^0(X, \mathscr{F}(m)) \text{ is a finite free } A\text{-module for every } m \gg 0. \tag{9.1}$$

Flatness implies (9.1)*.* Choose the standard affine cover

$$\mathfrak{U} = \{D_+(X_0), \dots, D_+(X_n)\}.$$

All finite intersections are affine. Since $\mathscr{F}$ is flat over A, proposition 9.2 implies that the module of sections of $\mathscr{F}(m)$ on every such affine intersection is flat over A. Thus every term of the finite Čech complex

$$C^\bullet(\mathfrak{U}, \mathscr{F}(m))$$

is flat over A.

By Serre vanishing, theorem 6.31, for $m \gg 0$ we have

$$H^i(X, \mathscr{F}(m)) = 0 \qquad (i > 0).$$

By theorem 6.23, the augmented Čech complex is then exact:

$$0 \longrightarrow H^0(X, \mathscr{F}(m)) \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0.$$

Starting at the right and using the standard flat-module fact that in a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

flatness of M and M'' implies flatness of M', we conclude inductively that $H^0(X, \mathscr{F}(m))$ is flat over A. It is finite over A by Serre finiteness, hence finite free because A is local. This proves (9.1).

Condition (9.1) *implies flatness.* Choose m_0 so that all the modules

$$M_m := H^0(X, \mathscr{F}(m))$$

are finite free for $m \geq m_0$, and form the truncated graded module

$$M = \bigoplus_{m \geq m_0} M_m$$

over $A[X_0, \dots, X_n]$. As an A-module, M is a direct sum of free modules and therefore flat.

The natural morphism from the truncation M to the full graded module $\Gamma_*(\mathscr{F})$ has cokernel concentrated in degrees bounded above (and zero kernel). A graded module concentrated in degrees bounded above has zero associated sheaf on projective space, because after localizing at any X_i every homogeneous element is killed by a sufficiently large power of X_i. Hence

$$\widetilde{M} \cong \widetilde{\Gamma_*(\mathscr{F})} \cong \mathscr{F}$$

by theorem 5.25.

On the standard affine chart $D_+(X_i)$, the sheaf $\widetilde{M}$ corresponds to the degree-zero part

$$(M_{X_i})_0.$$

The localization M_{X_i} is a filtered colimit of copies of the flat A-module M, hence is flat over A. Its grading gives a direct-sum decomposition as an A-module, so $(M_{X_i})_0$ is a direct summand and is therefore flat over A. Thus $\mathscr{F}$ is flat over A on the standard affine cover, and hence flat over T.

Condition (9.1) *implies constancy of the Hilbert polynomial.* Let $\mathfrak{p} \in \operatorname{Spec} A$. Replace A by the flat localization $A_{\mathfrak{p}}$; condition (9.1) is preserved. We may therefore suppose that $\mathfrak{p}$ is the maximal ideal and its residue field is k.

Choose a finite presentation

$$A^q \longrightarrow A \longrightarrow k \longrightarrow 0.$$

Tensoring with $\mathscr{F}$ gives a right exact sequence

$$\mathscr{F}^{\oplus q} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_k \longrightarrow 0.$$

Let $\mathscr{G}$ be the image of the first arrow and $\mathscr{K}$ its kernel, so that

$$0 \to \mathscr{K} \to \mathscr{F}^{\oplus q} \to \mathscr{G} \to 0, \qquad 0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{F}_k \to 0.$$

Both $\mathscr{G}$ and $\mathscr{K}$ are coherent. For $m \gg 0$, Serre vanishing gives

$$H^1(X, \mathscr{G}(m)) = H^1(X, \mathscr{K}(m)) = 0.$$

Taking global sections in the two short exact sequences therefore shows that

$$H^0(X, \mathscr{F}_k(m)) \cong \operatorname{coker}\big(H^0(X, \mathscr{F}(m))^{\oplus q} \to H^0(X, \mathscr{F}(m))\big).$$

The right-hand side is precisely

$$H^0(X, \mathscr{F}(m)) \otimes_A k.$$

Hence, for $m \gg 0$,

$$H^0(X_k, \mathscr{F}_k(m)) \cong H^0(X, \mathscr{F}(m)) \otimes_A k.$$

The same statement over the generic point follows by the flat base change theorem theorem 9.3. If (9.1) holds, both vector-space dimensions are the common rank of the free module $H^0(X, \mathscr{F}(m))$. Since for large m the Hilbert polynomial equals the dimension of global sections, P_t is independent of t.

Constancy of the Hilbert polynomial implies (9.1). The preceding base-change calculation did not use flatness of $\mathscr{F}$; it used only coherence and Serre vanishing. Thus, for every sufficiently large m,

$$\dim_k\big(H^0(X, \mathscr{F}(m)) \otimes_A k\big) = P_k(m),$$

while

$$\dim_K\big(H^0(X, \mathscr{F}(m)) \otimes_A K\big) = P_K(m).$$

If the Hilbert polynomial is constant, these dimensions are equal. Put

$$N = H^0(X, \mathscr{F}(m)), \qquad r = \dim_K(N \otimes_A K).$$

The equality

$$\dim_k(N/\mathfrak{m}N) = r$$

shows, by Nakayama's lemma, that N is generated by r elements. Choose a surjection

$$A^r \twoheadrightarrow N$$

and let L be its kernel. After tensoring with K, both source and target have dimension r, so

$$L \otimes_A K = 0.$$

But $L \subseteq A^r$ is torsion-free over the domain A. A torsion-free module whose localization at the fraction field is zero must itself be zero. Therefore $L = 0$ and

$$N \cong A^r.$$

Thus $H^0(X, \mathscr{F}(m))$ is finite free for all $m \gg 0$, establishing (9.1) and completing the proof. □

Corollary 9.17 (Numerical invariants in a projective flat family)**.** *Let T be a connected Noetherian scheme and let*

$$X \hookrightarrow \mathbb{P}^n_T$$

be a closed subscheme flat over T. Then the Hilbert polynomial of the fibre X_t is independent of $t \in T$. Consequently the following invariants are constant on the fibres:

$$\dim X_t, \qquad \deg X_t, \qquad p_a(X_t),$$

whenever the arithmetic genus is defined in the pure-dimensional convention of definition 6.39.

Proof. After base change to the reduced structure on any irreducible component $T_i \subseteq T$, flatness is preserved. The restriction of X to T_i therefore has constant Hilbert polynomial by theorem 9.16. In a connected Noetherian space, the irreducible components can be joined by a chain of components with nonempty consecutive intersections. At a point of intersection the two restrictions have the same fibre, so their Hilbert polynomials agree. Thus a single polynomial works on all of T.

For a projective scheme over a field, the degree of the Hilbert polynomial is the dimension of the support, and if the pure dimension is r, its leading coefficient is $\deg(X_t)/r!$. The arithmetic genus is obtained from the constant term through

$$p_a(X_t) = (-1)^r \big(P_{X_t}(0) - 1\big).$$

Hence all three invariants are constant. □

9.4 Smooth Families and Geometric Fibres

Chapter 8 approached smoothness through standard smooth presentations, tangent spaces, and infinitesimal lifting. Flatness supplies the complementary fibrewise description. The bridge is the following standard local algebra theorem for finitely presented algebras, which we use in the same way as the local criterion for flatness: an A-algebra of finite presentation is smooth at a prime if and only if it is flat there and its geometric fibre is regular at the corresponding point. This is the algebraic content of the Jacobian-flatness criterion and is the local theorem underlying Hartshorne III.10.2.

Theorem 9.18 (Flat-fibre criterion for smoothness). *Let*

$$f : X \longrightarrow Y$$

be locally of finite presentation. Then the following are equivalent:

1. *f is smooth in the sense of the standard smooth presentations of definition 8.23;*
2. *f is flat and every geometric fibre*

$$X_{\bar{y}} = X \times_Y \operatorname{Spec} \overline{\kappa(y)}$$

   is regular.

If these conditions hold and the fibres are pure of dimension n, then f is smooth of relative dimension n.

Proof. The assertion is local on X and Y. Write $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ with B finitely presented over A. The standard algebraic smoothness criterion quoted above says precisely that, for every prime $\mathfrak{q} \subseteq B$ with image $\mathfrak{p} \subseteq A$, the localization $B_{\mathfrak{q}}$ is standard smooth over $A_{\mathfrak{p}}$ if and only if it is flat over $A_{\mathfrak{p}}$ and the local ring of

$$B \otimes_A \overline{\kappa(\mathfrak{p})}$$

at every geometric point over $\mathfrak{q}$ is regular. Applying this criterion at every point gives the equivalence.

If the geometric fibres are pure of dimension n, then on a standard smooth neighborhood the rank of $\Omega^1_{X/Y}$ equals the dimension of the geometric fibre by theorem 8.24 and base change for differentials, proposition 8.10. Hence the relative dimension is n. □

Remark 9.19. The word *geometric* cannot be dropped. Over an imperfect field a regular finite-type scheme may cease to be regular after a purely inseparable extension. The phenomenon was already visible in example 8.11: nonzero Kähler differentials can survive in a purely inseparable algebraic extension even though ordinary Krull dimension does not detect them.

Smoothness is compatible with the constructions that one expects of a family.

Proposition 9.20 (Permanence of smoothness). *Smooth morphisms are stable under arbitrary base change and under composition. If $X \to Z$ and $Y \to Z$ are smooth of relative dimensions m and n, then*

$$X \times_Z Y \longrightarrow Z$$

is smooth of relative dimension $m + n$.

Proof. For base change, work directly with the standard smooth presentations of definition 8.23. If

$$B = (A[x_1, \dots, x_r]/(f_1, \dots, f_a))_g$$

is standard smooth over A, then after any homomorphism $A \to A'$ its base change is

$$B \otimes_A A' \cong (A'[x_1, \dots, x_r]/(f'_1, \dots, f'_a))_{g'},$$

and the same Jacobian minor remains invertible after applying $A \to A'$. Hence a base change of a standard smooth morphism is standard smooth, with the same relative dimension.

For composition, again work with standard smooth presentations. Suppose locally

$$B = (A[x_1, \dots, x_r]/(f_1, \dots, f_a))_g$$

is standard smooth over A, and

$$C = (B[y_1, \dots, y_s]/(h_1, \dots, h_b))_q$$

is standard smooth over B. Substituting the presentation of B gives a presentation of C over A. With the variables ordered as x followed by y, the Jacobian matrix contains a block lower-triangular square minor whose diagonal blocks are the invertible Jacobian minors for the two presentations. Its determinant is the product of two units. Hence the composite is standard smooth, and the relative dimensions add.

Finally, the projection $X \times_Z Y \to Y$ is a base change of $X \to Z$, so it is smooth of relative dimension m. Composing with $Y \to Z$, which is smooth of relative dimension n, gives the product statement. □

For morphisms between nonsingular varieties, smoothness can be read entirely from the differential on tangent spaces. This is the relative analogue of the Jacobian criterion developed in Chapter 8.

Proposition 9.21 (Tangent-space criterion for smoothness)**.** *Let k be algebraically closed, let*

$$f : X \longrightarrow Y$$

be a morphism of nonsingular varieties, and put

$$n = \dim X - \dim Y.$$

Assume $n \geq 0$. The following are equivalent:

1. *f is smooth of relative dimension n;*
2. *$\Omega^1_{X/Y}$ is locally free of rank n;*
3. *for every closed point $x \in X$, with $y = f(x)$, the tangent map*

$$df_x : T_xX \longrightarrow T_yY$$

is surjective.

Proof. The implication (1)⇒(2) is theorem 8.24.

Assume (2). The transitivity sequence of theorem 8.14, restricted to a closed point x, gives a right exact sequence of k-vector spaces

$$\Omega^1_{Y/k,y} \otimes k \longrightarrow \Omega^1_{X/k,x} \otimes k \longrightarrow \Omega^1_{X/Y,x} \otimes k \longrightarrow 0.$$

Since X and Y are nonsingular, the middle and left spaces have dimensions $\dim X$ and $\dim Y$ by theorem 8.28, while the final space has dimension n. Hence the first map is injective. Dualizing gives the surjective tangent map df_x. Thus (2)⇒(3).

Assume (3). Fix a closed point $x \in X$, put $y = f(x)$, and write

$$A = \mathscr{O}_{Y,y}, \qquad B = \mathscr{O}_{X,x}.$$

Surjectivity of df_x is dual to injectivity of

$$\mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2.$$

Choose a regular system of parameters

$$t_1, \dots, t_m$$

for the regular local ring A, where $m = \dim Y$. Their images in $\mathfrak{m}_B/\mathfrak{m}_B^2$ are linearly independent. Since B is regular local, they extend to a regular system of parameters of B; in particular $t_1, \dots, t_m$ form a B-regular sequence.

We first recover the flatness which is implicit in the tangent condition. Put

$$A_i = A/(t_1, \dots, t_i), \qquad B_i = B/(t_1, \dots, t_i)B.$$

The local criterion for flatness applied successively to the nonzerodivisors t_{i+1} gives

$$B_i \text{ flat over } A_i \iff B_{i+1} \text{ flat over } A_{i+1}.$$

At the end $A_m = \kappa(y)$ is a field, so B_m is flat over A_m. Induction upward proves that B is flat over A.

The same argument proves flatness at every closed point of X. If $\xi \in X$ is arbitrary, choose a closed specialization $x \in \overline{\{\xi\}}$. Then $f(x)$ is a specialization of $f(\xi)$, and the maps of local rings at ξ are obtained from those at x by localization. Since flatness is preserved under localization, f is flat at ξ. Thus f is flat everywhere.

To prove that f is standard smooth near x, choose affine neighborhoods

$$U = \operatorname{Spec} C \subseteq X, \qquad V = \operatorname{Spec} R \subseteq Y$$

of x and y such that $f(U) \subseteq V$, and choose generators of the finite-type R-algebra C. They give a closed immersion over V

$$i : U \hookrightarrow Z := \mathbb{A}^N_V.$$

After shrinking U and V, both U and V are nonsingular, and therefore so is Z. Let $\mathscr{I}$ be the ideal sheaf of U in Z. By theorem 8.30, the absolute conormal sequence is short exact:

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow i^*\Omega^1_{Z/k} \longrightarrow \Omega^1_{U/k} \longrightarrow 0.$$

The tangent surjectivity at x says that

$$f^*\Omega^1_{V/k} \otimes k(x) \longrightarrow \Omega^1_{U/k} \otimes k(x)$$

is injective. Since both sheaves are locally free, some maximal minor is a unit at x; after shrinking U, the map

$$f^*\Omega^1_{V/k} \longrightarrow \Omega^1_{U/k}$$

is a subbundle inclusion.

Consider now the relative conormal map

$$\mathscr{I}/\mathscr{I}^2 \longrightarrow i^*\Omega^1_{Z/V}.$$

It is injective near x. Indeed, suppose a local section α maps to zero in $i^*\Omega^1_{Z/V}$. Its image in $i^*\Omega^1_{Z/k}$ then comes from a section β of $f^*\Omega^1_{V/k}$. Since α belongs to the absolute conormal sheaf, its image in $\Omega^1_{U/k}$ is zero. The subbundle injection just established forces $\beta = 0$, and the injectivity of the absolute conormal map then forces $\alpha = 0$.

Thus, after shrinking,

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow i^*\Omega^1_{Z/V} \longrightarrow \Omega^1_{U/V} \longrightarrow 0$$

is exact. Since

$$\mathrm{rk}(\mathscr{I}/\mathscr{I}^2) = \dim Z - \dim U = N - n,$$

choose local equations

$$g_1, \dots, g_{N-n} \in \mathscr{I}$$

whose classes form a basis of $\mathscr{I}/\mathscr{I}^2$ at x. Nakayama's lemma shows that these equations generate $\mathscr{I}$ after shrinking: their classes generate $\mathscr{I}/\mathscr{I}^2$, so locally

$$\mathscr{I} = (g_1, \dots, g_{N-n}) + \mathscr{I}^2,$$

and Nakayama applied to $\mathscr{I}/(g_1, \dots, g_{N-n})$ gives equality with $(g_1, \dots, g_{N-n})$.

On $Z = \mathbb{A}^N_V$, the sheaf $\Omega^1_{Z/V}$ is free with basis $dT_1, \dots, dT_N$. The injective conormal map sends the classes of the g_i to their relative differentials. Their linear independence at x means that some $(N-n) \times (N-n)$ minor of

$$\left(\frac{\partial g_i}{\partial T_j}\right)$$

is nonzero at x, hence a unit after shrinking. This is exactly a standard smooth presentation of $U \to V$ of relative dimension n. Therefore f is smooth near every closed point. These neighborhoods cover X, because every nonempty closed subset of a finite-type k-scheme contains a closed point. Hence f is smooth everywhere. Thus (3)⇒(1). □

Remark 9.22. In the last proof, the flatness step is the essential new ingredient that was unavailable in Chapter 8. Surjectivity on tangent spaces supplies a regular sequence in the source local ring, and the local criterion for flatness converts that regular sequence into flatness over the target.

9.5 Generic Smoothness and Ètale Morphisms

Over a field of characteristic zero, separability forces a dominant morphism to become smooth on a dense open subset. The statement fails in positive characteristic precisely because purely inseparable maps can have identically vanishing differential.

Lemma 9.23 (Dominant morphisms are generically smooth in characteristic zero)**.** *Let k be algebraically closed of characteristic zero, and let*

$$f : X \longrightarrow Y$$

be a dominant morphism of integral k-schemes of finite type. Then there is a nonempty open subset

$$U \subseteq X$$

such that

$$f|_U : U \longrightarrow Y$$

is smooth.

Proof. Shrink X and Y to nonempty nonsingular open subsets, replacing X by the inverse image of the chosen open in Y. Dominance is preserved. Let

$$K = K(Y), \qquad L = K(X), \qquad r = \mathrm{trdeg}_K L.$$

Since $\operatorname{char} k = 0$, the finitely generated extension L/K is separably generated. Therefore

$$\dim_L \Omega_{L/K} = r.$$

The coherent sheaf $\Omega^1_{X/Y}$ has generic fibre $\Omega_{L/K}$, so after shrinking X once more it is locally free of rank r. On the nonsingular varieties X and Y, proposition 9.21 then shows that f is smooth on this open subset. $\square$

Example 9.24 (Frobenius)**.** Let k have characteristic $p > 0$. The Frobenius morphism

$$F : \mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k, \qquad t \longmapsto t^p$$

is dominant, but

$$d(t^p) = pt^{p-1}dt = 0.$$

Thus the induced tangent map has rank zero everywhere. The morphism is not smooth on any nonempty open subset. This is the basic obstruction which the characteristic-zero hypothesis excludes.

For a general morphism, the critical values occupy a subset of controlled dimension.

Proposition 9.25 (Rank-deficient locus)**.** *Let k be algebraically closed of characteristic zero and let*

$$f : X \longrightarrow Y$$

be a morphism of varieties. For an integer $r \geq 0$, let

$$X_r = \{x \in X(k) : \operatorname{rk}(df_x) \leq r\}.$$

Then

$$\dim \overline{f(X_r)} \leq r.$$

Proof. Let Z be an irreducible component of $\overline{f(X_r)}$. Since X is Noetherian, the closure $\overline{X_r}$ has finitely many irreducible components; choose one, say W, for which $X_r \cap W$ is dense in W and whose image is dense in Z. Give W and Z their reduced structures.

Shrink W and Z to their nonsingular loci and then apply lemma 9.23 to the induced dominant morphism

$$W \longrightarrow Z.$$

After shrinking W further, this morphism is smooth. Since X_r is dense in W, we can choose a closed point

$$x \in X_r \cap W$$

in this smooth open, with $y = f(x)$ lying in the chosen nonsingular open of Z.

The locally closed immersions $W \hookrightarrow X$ and $Z \hookrightarrow Y$ induce injective maps on tangent spaces, and we have a commutative square

$$\begin{tikzcd} T_xW \arrow[r, "d(f|_W)_x"] \arrow[d, hook] & T_yZ \arrow[d, hook] \\ T_xX \arrow[r, "df_x"'] & T_yY. \end{tikzcd}$$

The top arrow is surjective because $W \to Z$ is smooth and both are nonsingular; see proposition 9.21. Hence

$$\dim Z = \dim T_yZ \leq \operatorname{rk}(df_x) \leq r.$$

Since Z was an arbitrary irreducible component of $\overline{f(X_r)}$, the required dimension bound follows. $\square$

Corollary 9.26 (Generic smoothness)**.** *Let k be algebraically closed of characteristic zero, let X be a nonsingular variety, and let*

$$f : X \longrightarrow Y$$

be a morphism of varieties. Then there exists a nonempty open subset $V \subseteq Y$ such that

$$f^{-1}(V) \longrightarrow V$$

is smooth. If f is dominant, V may be chosen dense in Y.

Proof. If f is not dominant, remove the proper closed subset $\overline{f(X)}$ from Y; over the resulting nonempty open the inverse image is empty, hence smooth. Assume therefore that f is dominant.

Shrink Y to its nonsingular locus, which is nonempty because Y is a variety. Put

$$r = \dim Y.$$

By proposition 9.25, the closure of the image of

$$X_{r-1} = \{x : \mathrm{rk}(df_x) \leq r - 1\}$$

has dimension at most $r - 1$, hence is a proper closed subset of Y. Let

$$V = Y \setminus \overline{f(X_{r-1})}.$$

For every closed $x \in f^{-1}(V)$ the tangent map

$$df_x : T_x X \to T_{f(x)} Y$$

has rank $r = \dim Y$ and is therefore surjective. The source and target are nonsingular on these opens, so proposition 9.21 shows that $f^{-1}(V) \to V$ is smooth. □

We finish by isolating the relative-dimension-zero case. Chapter 8 already showed that the vanishing of relative differentials is exactly the infinitesimal uniqueness condition. With flatness available, this becomes the geometric notion of an ètale morphism.

Definition 9.27. Let $f : X \to Y$ be a morphism.

1. If f is locally of finite type, it is *unramified* if

$$\Omega^1_{X/Y} = 0.$$

2. If f is locally of finite presentation, it is *ètale* if it is flat and unramified.

Proposition 9.28 (Diagonal criterion for unramified morphisms)**.** *Let $f : X \to Y$ be locally of finite type. Then the following are equivalent:*

1. *f is unramified;*
2. *the diagonal*

$$\Delta_{X/Y} : X \longrightarrow X \times_Y X$$

is an open immersion;

3. *f is formally unramified in the sense of definition 8.32.*

Proof. The equivalence of (1) and (3) is proposition 8.34, sheafified locally on X and Y.

We prove the equivalence with (2). By proposition 3.32, the diagonal is always an immersion. Hence near every point of its image there is an open subset

$$W \subseteq X \times_Y X$$

on which the diagonal is a closed immersion. Let $\mathscr{I}$ be its ideal sheaf on W. The conormal sheaf of the diagonal is naturally

$$\mathscr{I}/\mathscr{I}^2 \cong \Omega^1_{X/Y}$$

by the affine diagonal calculation of theorem 8.5; the identification is local and therefore does not require the whole diagonal to be closed.

Assume $\Omega^1_{X/Y} = 0$. Then $\mathscr{I}/\mathscr{I}^2 = 0$. The ideal $\mathscr{I}$ is locally finitely generated. Indeed, affinely write $A \to B$ with B generated over A by $b_1, \ldots, b_r$. The ideal of the multiplication map

$$B \otimes_A B \longrightarrow B$$

is generated by

$$b_i \otimes 1 - 1 \otimes b_i, \qquad 1 \leq i \leq r.$$

Thus the diagonal of a locally finite-type morphism is locally of finite presentation as an immersion. At a point of the diagonal, Nakayama's lemma applied to the local ideal gives

$$\mathscr{I} = \mathscr{I}^2 \subseteq \mathfrak{m}\mathscr{I} \quad \Longrightarrow \quad \mathscr{I} = 0$$

after shrinking. Thus the diagonal identifies X locally with an open neighborhood of its image in W. Since this holds at every point, $\Delta_{X/Y}$ is an open immersion.

Conversely, if the diagonal is an open immersion, its conormal sheaf is zero. The same local identification with $\Omega^1_{X/Y}$ gives $\Omega^1_{X/Y} = 0$, so f is unramified. □

Theorem 9.29 (Equivalent forms of ètaleness)**.** *Let $f : X \to Y$ be locally of finite presentation. The following are equivalent:*

1. *f is ètale;*
2. *f is smooth of relative dimension 0;*
3. *f is flat and formally unramified;*
4. *f is formally ètale.*

Proof. By definition, (1) means that f is flat and $\Omega^1_{X/Y} = 0$. By proposition 8.34, this is equivalent to (3).

Assume (1). After any base change to a geometric point $\bar{y} \to Y$, flatness is preserved and differentials commute with base change by proposition 8.10; hence

$$\Omega^1_{X_{\bar{y}}/\overline{\kappa(y)}} = 0.$$

At a closed point x of the finite-type scheme $X_{\bar{y}}$, the cotangent space therefore has dimension zero. The standard embedding-dimension inequality

$$\dim \mathscr{O}_{X_{\bar{y}},x} \leq \dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2$$

forces the local dimension to be zero, so the local ring is regular. If the geometric fibre had a positive-dimensional irreducible component, that component would contain a closed point whose local dimension is positive, a contradiction. Hence the fibre is zero-dimensional; a Noetherian zero-dimensional scheme has only closed points. We have therefore shown that every local ring of every geometric fibre is regular, and the fibres are pure of dimension zero. The flat-fibre criterion theorem 9.18 gives (2).

Conversely, if (2) holds, then f is flat by theorem 9.18, and theorem 8.24 gives

$$\Omega^1_{X/Y} = 0.$$

Thus (2)⇒(1).

A smooth morphism is formally smooth by theorem 8.36, while relative dimension zero gives $\Omega^1_{X/Y} = 0$ and hence formal unramifiedness. Therefore (2)⇒(4).

Finally, formally ètale means formally smooth and formally unramified. The standard comparison theorem recalled in remark 8.39 turns local finite presentation plus formal smoothness into smoothness; formal unramifiedness gives $\Omega^1_{X/Y} = 0$, so the smooth relative dimension is zero. Hence (4)⇒(2). □

Example 9.30 (Finite separable extensions)**.** Let L/K be a finite field extension. The morphism

$$\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$$

is finite flat. By example 8.11,

$$\Omega_{L/K} = 0$$

if L/K is separable. Hence a finite separable extension gives a finite ètale morphism. Conversely, if the morphism is ètale, then $\Omega_{L/K} = 0$. We use here the standard separability criterion for algebraic field extensions: for a finite extension L/K, one has $\Omega_{L/K} = 0$ if and only if L/K is separable. Hence ètaleness forces separability.

Example 9.31 (A monogenic ètale algebra)**.** Let A be a ring, let $f(T) \in A[T]$ be monic, and put

$$B = A[T]/(f).$$

The conormal sequence gives

$$\Omega_{B/A} \cong B/(f'(T))\, dT.$$

Since B is finite free over A, the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat. On the open subset of $\operatorname{Spec} B$ where $f'(T)$ is invertible, the relative differential module vanishes, so the morphism is ètale there.

Example 9.32 (Power maps on the multiplicative group)**.** Let $n \geq 1$ be invertible in the base ring A. The map

$$[n] : \mathbb{G}_{m,A} \longrightarrow \mathbb{G}_{m,A}, \qquad t \longmapsto t^n$$

corresponds to

$$A[s, s^{-1}] \longrightarrow A[t, t^{-1}], \qquad s \longmapsto t^n.$$

The target is finite free of rank n over the source with basis

$$1, t, \ldots, t^{n-1}.$$

Moreover

$$ds = d(t^n) = nt^{n-1}dt,$$

and nt^{n-1} is a unit. Hence

$$\Omega^1_{\mathbb{G}_m/\mathbb{G}_m} = 0,$$

so the power map is finite ètale.

Chapter 10

Algebraic Curves

The preceding chapters developed divisors, linear systems, cohomology, differentials, flatness, and étale morphisms in a relative setting. On a smooth projective curve these theories become unusually concrete: codimension-one points are ordinary closed points, every local ring at such a point is a discrete valuation ring, line bundles are controlled by divisors, and the canonical sheaf is itself a line bundle of differentials. This chapter uses that one-dimensional simplification to turn the general machinery into explicit geometry.

The organization is guided by Hartshorne, Chapter IV, §§1–3, together with Exercises IV.2.1–2.2. Unless explicitly stated otherwise, throughout this chapter k is an algebraically closed field and a *curve* means a smooth projective integral curve over k.

10.1 Divisors and Linear Systems on Curves

Let C be a curve. Since C is regular of dimension one, every closed point $P \in C$ determines a prime divisor and the local ring $\mathscr{O}_{C,P}$ is a discrete valuation ring. Thus every divisor has a unique expression

$$D = \sum_{P \in C} n_P P,$$

with $n_P \in \mathbb{Z}$ and only finitely many nonzero coefficients. By corollary 7.37, Weil and Cartier divisor classes agree on C:

$$\operatorname{Pic}(C) \cong \operatorname{Cl}(C).$$

The first invariant special to curves is the degree.

Definition 10.1 (Degree of a divisor)**.** For

$$D = \sum_P n_P P \in \operatorname{Div}(C),$$

define

$$\deg D := \sum_P n_P.$$

If $\mathscr{L} \cong \mathscr{O}_C(D)$ is an invertible sheaf, we will eventually write $\deg \mathscr{L} := \deg D$; this is well defined once we know that principal divisors have degree zero.

A rational function on a complete nonsingular curve defines an honest morphism to the projective line.

Lemma 10.2 (Rational functions and maps to the projective line)**.** *Let $f \in K(C)$ be nonconstant. Then the rational map*

$$[1 : f] : C \dashrightarrow \mathbf{P}^1_k$$

extends uniquely to a nonconstant morphism

$$\varphi_f : C \longrightarrow \mathbf{P}^1_k.$$

Moreover, φ_f is finite.

Proof. Let $P \in C$. Since $\mathscr{O}_{C,P}$ is a discrete valuation ring, either

$$f \in \mathscr{O}_{C,P} \qquad \text{or} \qquad f^{-1} \in \mathscr{O}_{C,P}.$$

In the first case the generic map extends near P by the affine coordinate f on the chart $X_0 \neq 0$; in the second it extends by the affine coordinate f^{-1} on the chart $X_1 \neq 0$. These local extensions agree on their common dense generic point, and $\mathbf{P}^1_k$ is separated, so they glue uniquely to a morphism φ_f.

The map is nonconstant because it induces the nontrivial inclusion

$$k(t) \hookrightarrow K(C), \qquad t \longmapsto f.$$

Since C is projective, φ_f is proper. A fibre of a nonconstant morphism from an integral curve cannot contain a one-dimensional irreducible component, for then it would contain the generic point and the map would be constant. Hence every fibre is finite, so φ_f is quasi-finite. We use the standard theorem that a proper quasi-finite morphism is finite. Therefore φ_f is finite. □

The same argument applies to any nonconstant morphism between curves.

Corollary 10.3. *Every nonconstant morphism*

$$\varphi : C \longrightarrow C'$$

between curves is finite and surjective.

Proof. The image of a proper morphism is closed. Since C is irreducible and the map is nonconstant, its image is an irreducible closed subset of the one-dimensional curve C' which is not a point; therefore the image is all of C'. The proof of lemma 10.2 then shows that φ is proper and quasi-finite, hence finite. □

A finite map of smooth curves is automatically flat. This will be used repeatedly.

Proposition 10.4 (Finite maps of smooth curves are flat)**.** *Let*

$$\varphi : C \longrightarrow C'$$

be a nonconstant morphism of curves and put

$$d = [K(C) : K(C')].$$

Then φ is finite flat of constant rank d. For every closed point $Q \in C'$, the scheme-theoretic fibre C_Q has length d over k.

Proof. By corollary 10.3, the morphism is finite. Let $P \in C$ and $Q = \varphi(P)$. The induced map

$$\mathscr{O}_{C',Q} \longrightarrow \mathscr{O}_{C,P}$$

is an inclusion of domains because the map on function fields is injective. Hence $\mathscr{O}_{C,P}$ is torsion-free as a module over the discrete valuation ring $\mathscr{O}_{C',Q}$. Torsion-free modules over a DVR are flat, so φ is flat at every point.

A finite flat morphism is finite locally free. Since C' is connected, its rank is constant, and at the generic point that rank is

$$\dim_{K(C')} K(C) = d.$$

Thus $\varphi_*\mathscr{O}_C$ is locally free of rank d. Base change to the closed point Q gives a k-vector space of dimension d, namely the coordinate algebra of the finite fibre. Therefore

$$\mathrm{length}_k(C_Q) = d.$$

□

Lemma 10.5 (Finite degree-one maps of curves)**.** *Let $f : C \to D$ be a finite morphism between normal integral curves. If*

$$[K(C) : K(D)] = 1,$$

then f is an isomorphism.

Proof. The assertion is affine on D. Let $U = \operatorname{Spec} A \subseteq D$ be affine. Since f is finite,

$$f^{-1}(U) = \operatorname{Spec} B$$

with B finite over A. Both A and B embed into the common function field

$$K = K(C) = K(D).$$

Every element of B is integral over A, while A is integrally closed because D is normal. Hence $B \subseteq A$. The structural inclusion $A \subseteq B$ gives $A = B$. Thus f is an isomorphism over every affine open of D, and therefore globally. □

We can now prove the basic degree theorem without importing a product formula for valuations.

Theorem 10.6 (Principal divisors have degree zero)**.** *For every $f \in K(C)^\times$,*

$$\boxed{\deg \operatorname{div}(f) = 0.}$$

Consequently degree descends to homomorphisms

$$\deg : \mathrm{Cl}(C) \longrightarrow \mathbb{Z}, \qquad \deg : \mathrm{Pic}(C) \longrightarrow \mathbb{Z}.$$

Proof. If $f \in k^\times$, the divisor is zero. Suppose f is nonconstant and let

$$\varphi_f : C \longrightarrow \mathbf{P}^1$$

be the finite morphism of lemma 10.2. Let $0, \infty \in \mathbf{P}^1$ denote the zero and pole of the standard coordinate t. Then

$$f = \varphi_f^* t$$

and therefore, as Cartier divisors,

$$\operatorname{div}(f) = \varphi_f^*(0) - \varphi_f^*(\infty).$$

Let $d = \deg \varphi_f = [K(C) : k(t)]$. By proposition 10.4, each scheme-theoretic fibre has length d. The divisor $\varphi_f^*(0)$ records the local lengths of the fibre over 0, so

$$\deg \varphi_f^*(0) = d,$$

and similarly

$$\deg \varphi_f^*(\infty) = d.$$

Hence

$$\deg \operatorname{div}(f) = d - d = 0.$$

Thus linearly equivalent divisors have the same degree, and the degree map descends to both divisor classes and line bundles. $\square$

Definition 10.7. For $d \in \mathbb{Z}$, write

$$\operatorname{Pic}^d(C) := \{[\mathscr{L}] \in \operatorname{Pic}(C) \mid \deg \mathscr{L} = d\}, \qquad \operatorname{Pic}^0(C) := \ker(\deg : \operatorname{Pic}(C) \to \mathbb{Z}).$$

The general description of sections of a divisor sheaf from proposition 7.50 takes its classical form on a curve.

Definition 10.8 (Riemann–Roch space)**.** For a divisor D on C, define

$$L(D) := \{0\} \cup \{f \in K(C)^\times \mid \operatorname{div}(f) + D \geq 0\},$$

and write

$$\ell(D) := \dim_k L(D).$$

By proposition 7.50,

$$\boxed{L(D) = H^0(C, \mathscr{O}_C(D)).}$$

The complete linear system of D is the projective space

$$|D| = \mathbf{P}(L(D)),$$

whenever $L(D) \neq 0$, by theorem 7.52. The curve case allows a useful numerical criterion for base points and embeddings.

Proposition 10.9 (Linear systems on a curve)**.** *Let D be a divisor on C.*

1. *The complete linear system $|D|$ has no base points if and only if*

$$\ell(D - P) = \ell(D) - 1$$

for every $P \in C$.

2. *The divisor D is very ample if and only if*

$$\boxed{\ell(D - P - Q) = \ell(D) - 2}$$

for every pair $P, Q \in C$, allowing $P = Q$.

Proof. For a point P, the quotient

$$\mathscr{O}_C(D)/\mathscr{O}_C(D-P)$$

is the skyscraper sheaf $k(P) \cong k$. Hence there is an exact sequence

$$0 \longrightarrow \mathscr{O}_C(D-P) \longrightarrow \mathscr{O}_C(D) \longrightarrow k(P) \longrightarrow 0.$$

Taking global sections gives

$$0 \longrightarrow L(D-P) \longrightarrow L(D) \xrightarrow{\mathrm{ev}_P} k.$$

The point P is not a base point precisely when some section is nonzero at P, that is, when ev_P is surjective. Since the target has dimension one, this is equivalent to

$$\ell(D-P) = \ell(D) - 1.$$

This proves (1).

For (2), let $Z = P + Q$ be the length-two effective divisor, with $Z = 2P$ when $P = Q$. Then

$$0 \longrightarrow \mathscr{O}_C(D-Z) \longrightarrow \mathscr{O}_C(D) \longrightarrow \mathscr{O}_C(D)|_Z \longrightarrow 0,$$

and

$$\dim_k H^0(Z, \mathscr{O}_C(D)|_Z) = 2.$$

Thus

$$\ell(D-P-Q) = \ell(D) - 2$$

if and only if the restriction map

$$H^0(C, \mathscr{O}_C(D)) \longrightarrow H^0(Z, \mathscr{O}_C(D)|_Z)$$

is surjective for every length-two divisor Z.

Notice first that this condition already implies global generation. Fix P and choose any Q. Since removing one point can lower the dimension of sections by at most one,

$$\ell(D) - 1 \le \ell(D-P) \le \ell(D),$$

while

$$\ell(D-P-Q) \ge \ell(D-P) - 1.$$

The assumed equality $\ell(D-P-Q) = \ell(D) - 2$ forces

$$\ell(D-P) = \ell(D) - 1.$$

By part (1), P is not a base point; hence $\mathscr{O}_C(D)$ is globally generated.

When $P \neq Q$, surjectivity on $P+Q$ is exactly the condition that global sections separate the two points. When $P = Q$, surjectivity on $2P$ says that sections separate the first infinitesimal neighborhood of P, equivalently the associated morphism separates the tangent direction at P. The standard closed-immersion criterion for a globally generated line bundle says that its associated morphism is a closed immersion precisely when sections separate points and tangent directions. In dimension one these are exactly the length-two subschemes above. Hence the displayed equality for all P, Q is equivalent to very ampleness. ∎

Definition 10.10 (Linear series)**.** A *linear series* g^r_d on C is a pair $(\mathscr{L}, V)$ where

$$\deg \mathscr{L} = d, \qquad V \subseteq H^0(C, \mathscr{L}), \qquad \dim_k V = r + 1.$$

Equivalently, after choosing a divisor D with $\mathscr{L} \cong \mathscr{O}_C(D)$, it is an r-dimensional projective linear subsystem of $|D|$. A g^1_d is called a *pencil*.

If V is base-point free, proposition 7.49 gives a morphism

$$\varphi_V : C \longrightarrow \mathbf{P}(V^\vee) \cong \mathbf{P}^r.$$

For a base-point-free pencil this map lands in $\mathbf{P}^1$, and its degree is the degree of the pencil.

Proposition 10.11 (Pencils and maps to $\mathbf{P}^1$)**.** *Let $(\mathscr{L}, V)$ be a base-point-free g^1_d on C. If the associated morphism*

$$\varphi_V : C \longrightarrow \mathbf{P}^1$$

is nonconstant, then it is finite of degree d and

$$\mathscr{L} \cong \varphi_V^* \mathscr{O}_{\mathbf{P}^1}(1).$$

Conversely, every nonconstant morphism $\varphi : C \to \mathbf{P}^1$ of degree d determines a base-point-free g^1_d by pulling back the two-dimensional space $H^0(\mathbf{P}^1, \mathscr{O}(1))$.

Proof. The pullback identity is proposition 7.49. Let $Q \in \mathbf{P}^1$ be the zero divisor of a nonzero section of $\mathscr{O}(1)$. Its pullback is the zero divisor of the corresponding section of $\mathscr{L}$. Hence

$$\deg \mathscr{L} = \deg \varphi_V^* Q.$$

By proposition 10.4, the latter is the length of a fibre and therefore equals $\deg \varphi_V$. Thus $d = \deg \varphi_V$.

Conversely, the two standard sections of $\mathscr{O}_{\mathbf{P}^1}(1)$ have no common zero. Their pullbacks generate $\varphi^* \mathscr{O}(1)$ and therefore form a base-point-free pencil. The degree calculation just proved gives

$$\deg \varphi^* \mathscr{O}(1) = \deg \varphi = d.$$

□

10.2 The Riemann–Roch Theorem

The genus of a curve is naturally cohomological.

Definition 10.12 (Genus)**.** The *genus* of C is

$$g = g(C) := h^1(C, \mathscr{O}_C).$$

Since C is proper and integral,

$$H^0(C, \mathscr{O}_C) = k.$$

Serre duality, corollary 8.60, gives

$$H^1(C, \mathscr{O}_C)^\vee \cong H^0(C, \omega_C),$$

so

$$g = h^0(C, \omega_C).$$

Choose a nonzero rational differential on C. By definition 8.46, it determines a canonical divisor K_C satisfying

$$\omega_C \cong \mathscr{O}_C(K_C).$$

The key numerical identity is first an identity of Euler characteristics.

Proposition 10.13 (Euler characteristic and degree)**.** *For every divisor D on C,*

$$\boxed{\chi(\mathscr{O}_C(D)) = \deg D + 1 - g.}$$

Proof. Since C has dimension one, Grothendieck vanishing gives

$$H^i(C, \mathscr{O}_C) = 0 \qquad (i > 1).$$

Hence

$$\chi(\mathscr{O}_C) = h^0(C, \mathscr{O}_C) - h^1(C, \mathscr{O}_C) = 1 - g.$$

Let $P \in C$. As in the proof of proposition 10.9, there is a short exact sequence

$$0 \longrightarrow \mathscr{O}_C(D) \longrightarrow \mathscr{O}_C(D + P) \longrightarrow k(P) \longrightarrow 0.$$

Additivity of Euler characteristic gives

$$\chi(\mathscr{O}_C(D + P)) = \chi(\mathscr{O}_C(D)) + 1.$$

At the same time

$$\deg(D + P) = \deg D + 1.$$

Thus the quantity

$$\chi(\mathscr{O}_C(D)) - \deg D$$

is unchanged when a point is added or removed. Every divisor is obtained from the zero divisor by finitely many additions and subtractions of closed points, so

$$\chi(\mathscr{O}_C(D)) - \deg D = \chi(\mathscr{O}_C) = 1 - g.$$

□

Serre duality turns the H^1 term into another space of sections.

Theorem 10.14 (Riemann–Roch)**.** *Let D be a divisor on a curve C of genus g. Then*

$$\boxed{\ell(D) - \ell(K_C - D) = \deg D + 1 - g.}$$

Proof. By definition,

$$\ell(D) = h^0(C, \mathscr{O}_C(D)).$$

Serre duality gives

$$H^1(C, \mathscr{O}_C(D))^\vee \cong H^0(C, \mathscr{O}_C(-D) \otimes \omega_C) \cong H^0(C, \mathscr{O}_C(K_C - D)).$$

Therefore

$$h^1(C, \mathscr{O}_C(D)) = \ell(K_C - D).$$

Using proposition 10.13,

$$\ell(D) - \ell(K_C - D) = \chi(\mathscr{O}_C(D)) = \deg D + 1 - g.$$

□

Several classical facts now become immediate consequences.

Corollary 10.15 (Degree of the canonical divisor)**.** *For a curve of genus g,*

$$\boxed{\deg K_C = 2g - 2.}$$

Proof. Apply Riemann–Roch to $D = K_C$. Since

$$\ell(K_C) = h^0(C, \omega_C) = g$$

and

$$\ell(0) = h^0(C, \mathscr{O}_C) = 1,$$

we obtain

$$g - 1 = \deg K_C + 1 - g.$$

Thus $\deg K_C = 2g - 2$. □

Definition 10.16 (Special divisor)**.** A divisor D is *special* if

$$\ell(K_C - D) > 0.$$

Equivalently,

$$H^1(C, \mathscr{O}_C(D)) \neq 0.$$

Corollary 10.17 (Large-degree divisors)**.** *If*

$$\deg D > 2g - 2,$$

then D is nonspecial and

$$\boxed{\ell(D) = \deg D + 1 - g.}$$

Proof. By corollary 10.15,

$$\deg(K_C - D) < 0.$$

If a divisor E has negative degree, then $L(E) = 0$: a nonzero section would make E linearly equivalent to an effective divisor, which has nonnegative degree. Hence

$$\ell(K_C - D) = 0,$$

and Riemann–Roch gives the formula. □

The numerical criteria of proposition 10.9 now turn positivity into projective geometry.

Theorem 10.18 (Base-point freeness and very ampleness)**.** *Let D be a divisor on a curve of genus g.*

1. *If*

$$\deg D \geq 2g,$$

then $|D|$ has no base points.

2. *If*

$$\deg D \geq 2g + 1,$$

then D is very ample.

Proof. Assume first that $\deg D \geq 2g$ and let $P \in C$. Then

$$\deg(K_C - D + P) \leq (2g-2) - 2g + 1 = -1.$$

Hence

$$\ell(K_C - D + P) = 0.$$

Also $\deg(K_C - D) < 0$, so $\ell(K_C - D) = 0$. Applying Riemann–Roch to D and $D - P$ gives

$$\ell(D) = \deg D + 1 - g$$

and

$$\ell(D - P) = \deg D - g.$$

Therefore

$$\ell(D - P) = \ell(D) - 1.$$

By proposition 10.9, $|D|$ is base-point free.

Now assume $\deg D \geq 2g + 1$ and let $P, Q \in C$, allowing $P = Q$. Then

$$\deg(K_C - D + P + Q) \leq (2g-2) - (2g+1) + 2 = -1.$$

Thus $D - P - Q$ is nonspecial, as is D. Riemann–Roch gives

$$\ell(D - P - Q) = \deg D - 2 + 1 - g = \ell(D) - 2.$$

The second criterion in proposition 10.9 shows that D is very ample. □

Corollary 10.19 (Genus zero)**.** *If* $g(C) = 0$*, then*

$$C \cong \mathbf{P}^1_k.$$

Proof. Choose $P \in C$. Since $\deg K_C = -2$, the divisor $K_C - P$ has negative degree, so Riemann–Roch gives

$$\ell(P) = \deg P + 1 = 2.$$

The canonical section $1 \in H^0(C, \mathscr{O}_C(P))$ vanishes only at P, so it already excludes every point $Q \neq P$ from the base locus. Since

$$\ell(P - P) = \ell(0) = 1 = \ell(P) - 1,$$

proposition 10.9 shows that P is not a base point either. Hence $|P|$ is a base-point-free g^1_1 and defines a degree-one morphism

$$C \longrightarrow \mathbf{P}^1$$

by proposition 10.11. A finite degree-one morphism between normal integral curves is an isomorphism by lemma 10.5. Therefore $C \cong \mathbf{P}^1$. □

Corollary 10.20 (Genus one canonical class)**.** *If* $g(C) = 1$*, then*

$$\omega_C \cong \mathscr{O}_C.$$

Proof. The canonical divisor has degree zero and

$$\ell(K_C) = g = 1.$$

Thus K_C is linearly equivalent to an effective divisor of degree zero. The only such divisor is zero, so $K_C \sim 0$. □

10.3 Hurwitz, Étale Covers, and the Algebraic Fundamental Group

Let

$$f : C \longrightarrow D$$

be a nonconstant morphism of curves. By corollary 10.3, it is finite and surjective. Put

$$n = \deg f = [K(C) : K(D)].$$

For the differential theory we first assume that the field extension $K(C)/K(D)$ is separable.

Definition 10.21 (Ramification index)**.** Let $P \in C$ and $Q = f(P)$. Choose a uniformizer $t_Q \in \mathscr{O}_{D,Q}$. The *ramification index* of f at P is

$$e_P := v_P(f^* t_Q).$$

This does not depend on the chosen uniformizer. The point P is *ramified* if $e_P > 1$ and *unramified* if $e_P = 1$.

Since k is algebraically closed, all residue fields are k. The degree of a fibre is therefore measured only by the ramification indices.

Proposition 10.22 (Degree formula for a fibre)**.** *For every $Q \in D$,*

$$\boxed{\sum_{P \in f^{-1}(Q)} e_P = n.}$$

Equivalently,

$$f^* Q = \sum_{P \mapsto Q} e_P P$$

has degree n.

Proof. By proposition 10.4, the scheme-theoretic fibre over Q has length n. At $P \in f^{-1}(Q)$, a local parameter t_Q cuts out the fibre, so the local contribution is

$$\operatorname{length}_{\mathscr{O}_{C,P}} \mathscr{O}_{C,P}/(f^* t_Q).$$

Since $\mathscr{O}_{C,P}$ is a DVR and

$$v_P(f^* t_Q) = e_P,$$

this length is e_P. Summing the local lengths gives the result. □

Differentials record ramification intrinsically.

Proposition 10.23 (Relative differentials of a separable curve map)**.** *Let $f : C \to D$ be finite and separable. Then there is a short exact sequence*

$$\boxed{0 \longrightarrow f^* \omega_D \longrightarrow \omega_C \longrightarrow \Omega^1_{C/D} \longrightarrow 0.}$$

The sheaf $\Omega^1_{C/D}$ is a torsion sheaf supported exactly at the ramification points.

Proof. The transitivity sequence of theorem 8.14 gives

$$f^*\Omega^1_{D/k} \longrightarrow \Omega^1_{C/k} \longrightarrow \Omega^1_{C/D} \longrightarrow 0.$$

Because C and D are smooth curves,

$$\Omega^1_{C/k} = \omega_C, \qquad \Omega^1_{D/k} = \omega_D.$$

At the generic point the first map is

$$\Omega^1_{K(D)/k} \otimes_{K(D)} K(C) \longrightarrow \Omega^1_{K(C)/k}.$$

The exact sequence of field differentials has cokernel $\Omega^1_{K(C)/K(D)}$, which is zero because the finite field extension is separable. Thus the generic map between the two one-dimensional $K(C)$-vector spaces is an isomorphism. Since $f^*\omega_D$ is invertible, it is torsion-free, so a morphism which is injective at the generic point is injective everywhere. Hence the displayed sequence is short exact and the cokernel is torsion.

Let $P \in C$, $Q = f(P)$, and choose local parameters u at P and t at Q. Then

$$f^*t = au^{e_P}$$

for a unit $a \in \mathscr{O}^\times_{C,P}$. Locally the map of differentials sends

$$dt \longmapsto d(au^{e_P}).$$

This is a unit multiple of du precisely when $e_P = 1$; equivalently the cokernel vanishes at P precisely when f is unramified there. Thus the support is the ramification locus. □

Definition 10.24 (Ramification divisor)**.** For a finite separable morphism $f : C \to D$, define

$$d_P := \mathrm{length}_{\mathscr{O}_{C,P}} \Omega^1_{C/D,P}$$

and the *ramification divisor*

$$R_f := \sum_{P \in C} d_P P.$$

Only finitely many d_P are nonzero because $\Omega^1_{C/D}$ is a coherent torsion sheaf.

The coefficient d_P is the different exponent. It agrees with $e_P - 1$ in the tame case and can be larger in wild characteristic.

Proposition 10.25 (Tame ramification)**.** *Let $P \in C$, $Q = f(P)$, and let $e = e_P$.*

1. *One always has*

$$d_P \geq e - 1.$$

2. *If* $\operatorname{char} k \nmid e$, *then*

$$\boxed{d_P = e - 1.}$$

Proof. Choose uniformizers u at P and t at Q and write

$$f^*t = au^e$$

with a a unit. Since ω_C is generated locally by du and $f^*\omega_D$ by dt,

$$dt = d(au^e) = u^{e-1}(ea + ua')\, du,$$

where $da = a'\, du$. Therefore

$$d_P = v_P(u^{e-1}(ea + ua')) \geq e - 1.$$

If e is nonzero in k, then ea is a unit and ua' lies in the maximal ideal. Hence $ea + ua'$ is a unit, so its valuation is zero and $d_P = e - 1$. □

The exact sequence of differentials is an inclusion of line bundles with a torsion quotient. On a regular curve such an inclusion is measured by a divisor.

Proposition 10.26 (Canonical divisors under a finite map)**.** *Let $f : C \to D$ be finite and separable. Then*

$$\boxed{\omega_C \cong f^*\omega_D \otimes \mathscr{O}_C(R_f).}$$

Equivalently,

$$\boxed{K_C \sim f^*K_D + R_f.}$$

Proof. The injection

$$f^*\omega_D \hookrightarrow \omega_C$$

from proposition 10.23 is locally, after choosing generators, multiplication by a nonzero element $a_P \in \mathscr{O}_{C,P}$. Its cokernel is

$$\mathscr{O}_{C,P}/(a_P),$$

whose length is $v_P(a_P) = d_P$. Thus the divisor of the corresponding rational section of

$$\omega_C \otimes (f^*\omega_D)^{-1}$$

is precisely

$$\sum_P d_P P = R_f.$$

By the divisor–line-bundle correspondence of section 7.2, this gives

$$\omega_C \otimes (f^*\omega_D)^{-1} \cong \mathscr{O}_C(R_f),$$

which is the desired identity. □

Taking degrees gives Hurwitz's theorem.

Theorem 10.27 (Riemann–Hurwitz)**.** *Let*

$$f : C \longrightarrow D$$

be a finite separable morphism of degree n between curves of genera $g(C)$ and $g(D)$. Then

$$\boxed{2g(C) - 2 = n(2g(D) - 2) + \deg R_f.}$$

If every ramification point is tame, then

$$\boxed{2g(C) - 2 = n(2g(D) - 2) + \sum_{P \in C} (e_P - 1).}$$

Proof. By corollary 10.15,

$$\deg K_C = 2g(C) - 2, \qquad \deg K_D = 2g(D) - 2.$$

For any divisor E on D, proposition 10.22 and additivity give

$$\deg f^*E = n \deg E.$$

Taking degrees in

$$K_C \sim f^*K_D + R_f$$

therefore yields

$$2g(C) - 2 = n(2g(D) - 2) + \deg R_f.$$

The tame formula follows from proposition 10.25. □

Chapter 9 identified étale morphisms as flat unramified morphisms of finite presentation. For finite maps of smooth curves, this is exactly the absence of ramification.

Definition 10.28 (Finite étale cover)**.** A *finite étale cover* of a scheme X is a finite étale morphism

$$Y \longrightarrow X.$$

It is *connected* if Y is connected. The trivial cover of degree d is

$$\coprod_{i=1}^{d} X \longrightarrow X.$$

Corollary 10.29 (Étale covers of curves)**.** *Let $f : C \to D$ be a finite separable morphism of curves. Then*

$$f \text{ is étale} \iff \Omega^1_{C/D} = 0 \iff R_f = 0.$$

For a finite étale cover of degree n,

$$\boxed{2g(C) - 2 = n(2g(D) - 2).}$$

Proof. A finite map of curves is flat by proposition 10.4. By definition 9.27, it is therefore étale precisely when it is unramified, which is equivalent to $\Omega^1_{C/D} = 0$. By definition this is equivalent to $R_f = 0$. The formula is the Riemann–Hurwitz theorem with vanishing ramification divisor. □

Corollary 10.30 (The projective line has no nontrivial connected étale cover)**.** *Every connected finite étale cover*

$$C \longrightarrow \mathbf{P}^1_k$$

has degree one and is an isomorphism.

Proof. A connected finite étale cover of the smooth projective curve $\mathbf{P}^1$ is itself smooth and projective. Since a regular connected scheme has only one irreducible component, the source is a curve in our sense. Let the degree be n. Since $g(\mathbf{P}^1) = 0$, corollary 10.29 gives

$$2g(C) - 2 = -2n.$$

Hence

$$g(C) = 1 - n.$$

The genus is nonnegative, so $n \leq 1$. Since the cover is nonempty, $n = 1$. A finite flat morphism of rank one is an isomorphism: locally its coordinate algebra is a free rank-one module and the unit section is a basis by Nakayama's lemma. Thus the cover is trivial. □

Finite étale covers are organized by a profinite group. We introduce only the part of Grothendieck's theory needed here.

Definition 10.31 (Geometric point and fibre functor)**.** Let X be a connected scheme. A *geometric point* is a morphism

$$\bar{x} : \operatorname{Spec} \Omega \longrightarrow X$$

with Ω separably closed. Let $\mathrm{FÉt}(X)$ denote the category of finite étale covers of X. The *fibre functor* at $\bar{x}$ is

$$F_{\bar{x}} : \mathrm{FÉt}(X) \longrightarrow \textbf{FiniteSets}, \qquad (Y \to X) \longmapsto \operatorname{Hom}_X(\bar{x}, Y).$$

Definition 10.32 (Algebraic fundamental group)**.** The *étale fundamental group* of the connected scheme X based at $\bar{x}$ is

$$\boxed{\pi_1^{\text{ét}}(X, \bar{x}) := \operatorname{Aut}(F_{\bar{x}}),}$$

endowed with its canonical profinite topology.

The structural theorem behind this definition is the Galois correspondence for finite étale covers. Its proof belongs to the theory of Galois categories; we record the precise statement and use it as a standard input rather than developing that theory here.

Theorem 10.33 (Grothendieck's Galois correspondence)**.** *Let X be connected and let $\bar{x}$ be a geometric point. The fibre functor induces an equivalence*

$$\mathrm{F\acute{E}t}(X) \xrightarrow{\sim} \{\textit{finite continuous } \pi_1^{\text{ét}}(X, \bar{x})\textit{-sets}\}.$$

Under this equivalence, connected finite étale covers correspond to transitive finite continuous $\pi_1^{\text{ét}}(X, \bar{x})$-sets.

Definition 10.34 (Algebraically simply connected)**.** A connected scheme X is *algebraically simply connected* if every connected finite étale cover of X is an isomorphism. By theorem 10.33, this is equivalent to

$$\pi_1^{\text{ét}}(X, \bar{x}) = 1.$$

Thus corollary 10.30 says

$$\pi_1^{\text{ét}}(\mathbf{P}_k^1) = 1.$$

To pass to higher-dimensional projective space we need one connectedness theorem. It is a standard consequence of the connectedness results in Serre duality theory; in Hartshorne it appears as Chapter III, Corollary 7.9.

Theorem 10.35 (Connectedness of ample divisors)**.** *Let X be a connected projective Cohen–Macaulay scheme over a field, with*

$$\dim X \geq 2.$$

If D is an effective ample Cartier divisor on X, then the support $|D|$ is connected.

Remark 10.36. We use theorem 10.35 only for smooth projective varieties. In that case the Cohen–Macaulay hypothesis is automatic because regular local rings are Cohen–Macaulay. The theorem is logically separate from the elementary curve theory above; stating it explicitly prevents the induction below from hiding a connectedness input.

Lemma 10.37 (Finite pullback preserves ampleness)**.** *Let $f : X \to Y$ be a finite morphism of projective schemes over a Noetherian ring, and let $\mathscr{L}$ be ample on Y. Then $f^*\mathscr{L}$ is ample on X.*

Proof. Let $\mathscr{F}$ be coherent on X. Since f is finite, $f_*\mathscr{F}$ is coherent on Y. For $m \gg 0$, ampleness of $\mathscr{L}$ makes

$$f_*\mathscr{F} \otimes \mathscr{L}^{\otimes m}$$

generated by finitely many global sections. Pulling those sections back and using the projection formula gives a surjection onto

$$f^*(f_*\mathscr{F}) \otimes f^*\mathscr{L}^{\otimes m}.$$

Because f is affine, the adjunction evaluation map

$$f^*f_*\mathscr{F} \longrightarrow \mathscr{F}$$

is surjective: on an affine piece it is the multiplication map $B \otimes_A M \to M$. Hence

$$\mathscr{F} \otimes f^*\mathscr{L}^{\otimes m}$$

is generated by finitely many global sections for $m \gg 0$. By definition 7.44, $f^*\mathscr{L}$ is ample. $\square$

Theorem 10.38 (Projective space is algebraically simply connected)**.** *For every $n \geq 1$,*

$$\boxed{\pi_1^{\text{ét}}(\mathbf{P}^n_k) = 1.}$$

Equivalently, every connected finite étale cover of $\mathbf{P}^n_k$ is an isomorphism.

Proof. We argue by induction on n. The case $n = 1$ is corollary 10.30.

Assume $n \geq 2$ and let

$$f : X \longrightarrow \mathbf{P}^n_k$$

be a connected finite étale cover of degree d. Since étale morphisms preserve smoothness under composition and $\mathbf{P}^n$ is smooth, X is smooth. It is projective because f is finite and $\mathbf{P}^n$ is projective. A smooth connected scheme is regular, and distinct irreducible components of a regular scheme cannot meet; hence connectedness forces X to be integral.

Let

$$H \cong \mathbf{P}^{n-1}_k$$

be a hyperplane and put

$$D = f^{-1}(H).$$

Since H is an effective Cartier divisor and étale maps are flat, D is an effective Cartier divisor on X. Moreover

$$\mathscr{O}_X(D) \cong f^*\mathscr{O}_{\mathbf{P}^n}(1).$$

By lemma 10.37, the line bundle $f^*\mathscr{O}_{\mathbf{P}^n}(1)$ is ample, so D is ample. Since X is smooth projective of dimension $n \geq 2$, theorem 10.35 shows that D is connected.

By base-change stability of étale morphisms, the restriction

$$D = X \times_{\mathbf{P}^n} H \longrightarrow H \cong \mathbf{P}^{n-1}$$

is a connected finite étale cover. Its degree is still d, because finite locally free rank is preserved by base change. By the induction hypothesis, every connected finite étale cover of H has degree one. Therefore $d = 1$.

Finally, a finite étale morphism of degree one is finite locally free of rank one and hence an isomorphism, as in corollary 10.30. Thus f is trivial. $\square$

10.4 Curves of Genus Two

We now apply the preceding theory to the first genus for which every curve has a nontrivial canonical map to the projective line. Throughout this section assume

$$\operatorname{char} k \neq 2.$$

Let C be a curve of genus two. Then

$$\deg K_C = 2, \qquad \ell(K_C) = 2.$$

Thus the canonical system has projective dimension one. The first point is that it has no base points.

Lemma 10.39 (The canonical pencil in genus two)**.** *Let C have genus 2. Then $|K_C|$ is a base-point-free g^1_2.*

Proof. We already know

$$\deg K_C = 2, \qquad \ell(K_C) = 2.$$

Fix $P \in C$. We claim that

$$\ell(P) = 1.$$

If $\ell(P) \geq 2$, choose a nonconstant $f \in L(P)$. The only possible pole of f is P, and its order is at most one. A nonconstant rational function on a projective curve must have a pole, so

$$(f)_\infty = P.$$

The morphism $\varphi_f : C \to \mathbf{P}^1$ of lemma 10.2 therefore has degree one: the fibre over ∞ has degree one. By lemma 10.5, $C \cong \mathbf{P}^1$, contradicting $g(C) = 2$. Hence $\ell(P) = 1$.

Apply Riemann–Roch to $K_C - P$:

$$\ell(K_C - P) - \ell(P) = \deg(K_C - P) + 1 - g = 1 + 1 - 2 = 0.$$

Hence

$$\ell(K_C - P) = \ell(P) = 1 = \ell(K_C) - 1.$$

By proposition 10.9, P is not a base point. Since P was arbitrary, $|K_C|$ is base-point free. Its degree is 2 and its projective dimension is 1, so it is a g^1_2. □

Theorem 10.40 (Every genus-two curve is hyperelliptic)**.** *Let C be a curve of genus 2 over an algebraically closed field of characteristic different from 2. The canonical linear system defines a finite morphism*

$$\varphi_K : C \longrightarrow \mathbf{P}^1$$

of degree 2. It is ramified at exactly six points, each with ramification index 2.

Proof. By lemma 10.39, the canonical system is a base-point-free g^1_2. By proposition 10.11, its associated morphism has degree 2. Since $\operatorname{char} k \neq 2$, the quadratic extension of function fields is separable and every ramification index $e_P = 2$ is tame.

Apply Riemann–Hurwitz to

$$\varphi_K : C \to \mathbf{P}^1.$$

We obtain

$$2g(C) - 2 = 2\big(2g(\mathbf{P}^1) - 2\big) + \deg R_{\varphi_K},$$

so

$$2 = 2(-2) + \deg R_{\varphi_K}.$$

Therefore

$$\deg R_{\varphi_K} = 6.$$

In the tame quadratic case each ramified point contributes

$$e_P - 1 = 1.$$

Hence there are exactly six ramification points. Over a branch point the fibre-degree formula

$$\sum_{P \mapsto Q} e_P = 2$$

shows that there is a unique point above Q, with $e_P = 2$. Thus the six ramification points map to six distinct branch points on $\mathbf{P}^1$. □

Definition 10.41 (Hyperelliptic and Weierstrass points)**.** A curve of genus at least two is *hyperelliptic* if it admits a degree-two morphism to $\mathbf{P}^1$. For a genus-two curve, the six ramification points of the canonical double cover are called its *Weierstrass points.*

The six branch points also reconstruct the curve.

Proposition 10.42 (Constructing a genus-two curve from six points)**.** *Let*

$$B = \{a_1, \ldots, a_6\} \subset \mathbf{P}^1(k)$$

be six distinct points. There exists a smooth projective curve C_B and a degree-two morphism

$$f_B : C_B \longrightarrow \mathbf{P}^1$$

branched exactly over B. The curve C_B has genus 2, and f_B is the canonical morphism.

Proof. After a projective change of coordinate we may suppose that none of the six points is ∞. Write their affine coordinates as $a_1, \ldots, a_6 \in k$ and set

$$h(x) = \prod_{i=1}^{6} (x - a_i).$$

Since the roots are distinct and $\operatorname{char} k \neq 2$, h is squarefree. Consider the quadratic extension

$$k(x) \subset k(x, y), \qquad y^2 = h(x).$$

Let C_B be the normalization of $\mathbf{P}^1$ in this field. Normalization is finite in this finite extension, so we obtain a finite morphism

$$f_B : C_B \longrightarrow \mathbf{P}^1$$

of degree two. The curve C_B is normal and one-dimensional; hence it is regular. Over the algebraically closed field k, a regular finite-type curve is smooth. Since it is finite over the projective line, it is projective.

The local ramification can be read directly from the quadratic equation. Over a DVR with uniformizer π, write

$$h = \pi^m u, \qquad u \in A^\times.$$

Multiplying y by a suitable power of π changes m by an even integer. If m is odd, the normalized local equation has the form

$$z^2 = \pi u,$$

so the ramification index is 2. If m is even, it has the form $z^2 = u$; because the residue field is the algebraically closed field k, the residue of u is a square, and the derivative $2z$ is invertible at the two residue roots. Hence the extension is étale there.

At a zero $x = a_i$ one has $v_{a_i}(h) = 1$, so the quadratic extension is ramified with index 2. Away from the zeros of h and from infinity the valuation is 0, hence the extension is unramified. At infinity,

$$v_\infty(h) = -6$$

is even, so infinity is also unramified. Thus the branch locus is exactly B, and there are six tame ramification points.

Riemann–Hurwitz gives

$$2g(C_B) - 2 = 2(-2) + 6 = 2,$$

so $g(C_B) = 2$.

Finally put

$$\mathscr{L} = f_B^* \mathscr{O}_{\mathbf{P}^1}(1).$$

Then $\deg \mathscr{L} = 2$ and the pullback of $H^0(\mathbf{P}^1, \mathscr{O}(1))$ gives at least two independent sections, so

$$h^0(C_B, \mathscr{L}) \geq 2.$$

Riemann–Roch on the genus-two curve gives

$$h^0(C_B, \mathscr{L}) - h^0(C_B, \omega_{C_B} \otimes \mathscr{L}^{-1}) = 2 + 1 - 2 = 1.$$

Thus

$$h^0(C_B, \omega_{C_B} \otimes \mathscr{L}^{-1}) \geq 1.$$

The line bundle $\omega_{C_B} \otimes \mathscr{L}^{-1}$ has degree

$$2 - 2 = 0.$$

A degree-zero line bundle with a nonzero section is trivial, because its zero divisor is effective of degree zero. Therefore

$$\omega_{C_B} \cong \mathscr{L}.$$

Since $h^0(C_B, \omega_{C_B}) = 2$, the two pulled-back sections are the full canonical system. Hence f_B is the canonical morphism. □

To identify when two branch configurations give the same curve, we need the elementary classification of quadratic covers of $\mathbf{P}^1$ by their branch divisor.

Lemma 10.43 (A quadratic cover is determined by its branch set)**.** *Assume* $\operatorname{char} k \neq 2$. *Let*

$$K = k(\mathbf{P}^1)$$

and let

$$L_1 = K(\sqrt{f_1}), \qquad L_2 = K(\sqrt{f_2})$$

be separable quadratic extensions whose associated covers of $\mathbf{P}^1$ *have the same branch set. Then*

$$L_1 \cong L_2$$

as extensions of K*.*

Proof. A place of K is ramified in $K(\sqrt{f})$ precisely when the valuation of f is odd. Since the two extensions have the same branch set,

$$v_P(f_1/f_2) \in 2\mathbb{Z}$$

for every point $P \in \mathbf{P}^1$. Hence

$$\operatorname{div}(f_1/f_2) = 2D$$

for a divisor D on $\mathbf{P}^1$. The left side has degree zero, so $\deg D = 0$. Since

$$\operatorname{Pic}^0(\mathbf{P}^1) = 0,$$

there exists $g \in K^\times$ with

$$D = \operatorname{div}(g).$$

Therefore

$$\operatorname{div}(f_1/(f_2 g^2)) = 0.$$

A rational function on $\mathbf{P}^1$ with zero divisor is a scalar, so

$$f_1 = c g^2 f_2$$

for some $c \in k^\times$. Since k is algebraically closed, c is a square. Thus f_1/f_2 is a square in K, and the two quadratic extensions are isomorphic over K. □

Theorem 10.44 (Classification of genus-two curves)**.** *Let k be algebraically closed of characteristic different from 2. There is a natural bijection*

$$\boxed{\left\{\begin{array}{c}\textit{genus-two curves over } k\\ \textit{up to isomorphism}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{unordered reduced six-point subsets}\\ B \subset \mathbf{P}^1(k)\end{array}\right\} / \operatorname{PGL}_2(k).}$$

Proof. A genus-two curve C determines its canonical degree-two map

$$\varphi_K : C \to \mathbf{P}^1$$

and hence its unordered six-point branch set B_C by theorem 10.40. The target of the canonical map is intrinsically

$$\mathbf{P}(H^0(C, \omega_C)^\vee),$$

which is a projective line but has no preferred coordinate. Therefore the branch set is defined only up to $\operatorname{PGL}_2(k)$.

If

$$\alpha : C \xrightarrow{\sim} C'$$

is an isomorphism, pullback identifies the canonical spaces

$$H^0(C', \omega_{C'}) \xrightarrow{\sim} H^0(C, \omega_C).$$

Projectivizing gives an automorphism of the target projective lines which intertwines the canonical maps. Hence it carries B_C to $B_{C'}$. Thus the construction is well defined on isomorphism classes.

Conversely, proposition 10.42 constructs a genus-two curve from every six-point set. If two branch sets are related by an automorphism

$$\sigma \in \operatorname{PGL}_2(k),$$

pull one of the covers back along σ; the resulting two quadratic covers have the same branch set. By lemma 10.43, their function fields are isomorphic over $k(\mathbf{P}^1)$. Taking normalizations therefore gives isomorphic curves. The two constructions are inverse. □

Remark 10.45 (Three parameters). An ordered triple of distinct points of $\mathbf{P}^1$ can be sent uniquely to

$$0, \quad 1, \quad \infty$$

by an element of $\mathrm{PGL}_2(k)$. Thus, after ordering three of the six branch points, a genus-two curve can be represented by a branch set

$$\{0, 1, \infty, \lambda_1, \lambda_2, \lambda_3\},$$

where the λ_i are distinct and avoid $0, 1$. Forgetting the ordering introduces only a finite action coming from permutations of the six branch points. This is the concrete form of the three-dimensional parameter count in Hartshorne, Exercise IV.2.2.

10.5 Embeddings in Projective Space

Riemann–Roch already gives many projective embeddings: every divisor of degree at least $2g + 1$ is very ample. We now ask a different question. Once a curve is embedded in a high-dimensional projective space, how far can the ambient dimension be lowered without changing the curve? The answer is strikingly uniform: dimension three always suffices.

We first reinterpret the projection criterion in the language of secant and tangent lines. Let

$$C \hookrightarrow \mathbf{P}^N$$

be a closed immersion. For distinct points $P, Q \in C$, write

$$\overline{PQ} \subset \mathbf{P}^N$$

for their secant line. At P, the embedded tangent space is a projective line

$$T_P C \subset \mathbf{P}^N$$

because C is smooth of dimension one.

Proposition 10.46 (Projection criterion). *Let $O \in \mathbf{P}^N \setminus C$, and let*

$$\pi_O : \mathbf{P}^N \dashrightarrow \mathbf{P}^{N-1}$$

be linear projection from O. The restriction

$$\pi_O|_C : C \longrightarrow \mathbf{P}^{N-1}$$

is a closed immersion if and only if

1. *O lies on no secant line $\overline{PQ}$ with $P \neq Q$;*
2. *O lies on no tangent line $T_P C$.*

Proof. Hyperplanes of $\mathbf{P}^N$ containing O form an N-dimensional vector subspace

$$V \subseteq H^0(\mathbf{P}^N, \mathscr{O}(1)).$$

Since $O \notin C$, their restrictions have no common zero on C, and the associated morphism of the restricted linear system is exactly the projection

$$\pi_O|_C : C \to \mathbf{P}(V^\vee) \cong \mathbf{P}^{N-1}.$$

For two distinct points $P, Q \in C$, the images $\pi_O(P)$ and $\pi_O(Q)$ coincide precisely when P, Q, O are collinear, that is, precisely when

$$O \in \overline{PQ}.$$

Thus the restricted system separates distinct points exactly under condition (1).

At a point P, the differential of projection fails to be injective on the one-dimensional tangent space precisely when the projection center lies on the embedded tangent line $T_P C$. Equivalently, the system fails to separate the length-two subscheme $2P$ exactly in that case. Hence condition (2) is precisely tangent-direction separation.

By the length-two very-ampleness criterion of proposition 10.9, the restricted system gives a closed immersion if and only if it separates every pair of distinct points and every tangent direction. This is exactly (1) and (2). □

We now estimate how many bad centers there are.

Lemma 10.47 (Secant and tangent loci). *Let $C \subseteq \mathbf{P}^N$ be a smooth projective curve. Let $\mathrm{Sec}(C)$ be the Zariski closure of the union of all secant lines $\overline{PQ}$ with $P \neq Q$, and let $\mathrm{Tan}(C)$ be the union of all embedded tangent lines. Then*

$$\dim \mathrm{Sec}(C) \leq 3, \qquad \dim \mathrm{Tan}(C) \leq 2.$$

Moreover both are closed subsets of $\mathbf{P}^N$ after taking the indicated closure for the secant locus.

Proof. Consider the open subset

$$U = (C \times C) \setminus \Delta_C.$$

Over $(P, Q) \in U$, the line $\overline{PQ}$ is a copy of $\mathbf{P}^1$. The corresponding incidence space

$$I_{\mathrm{sec}} := \{(P, Q, R) \in U \times \mathbf{P}^N \mid R \in \overline{PQ}\}$$

is a $\mathbf{P}^1$-bundle over U. Hence

$$\dim I_{\mathrm{sec}} = \dim U + 1 = 3.$$

Its image in $\mathbf{P}^N$ is the union of secant lines. The dimension of the closure of an image is at most the dimension of the source, so

$$\dim \mathrm{Sec}(C) \leq 3.$$

For tangent lines, the embedded tangent line depends algebraically on P: the differential of the closed immersion gives the Gauss map from C to the Grassmannian of lines in $\mathbf{P}^N$. Pull back the universal line over that Grassmannian. We obtain a $\mathbf{P}^1$-bundle

$$I_{\mathrm{tan}} \longrightarrow C$$

of dimension 2, together with a proper morphism

$$I_{\mathrm{tan}} \longrightarrow \mathbf{P}^N$$

whose image is $\mathrm{Tan}(C)$. Proper images are closed, and

$$\dim \mathrm{Tan}(C) \leq 2.$$

□

Theorem 10.48 (Generic projection preserves a curve)**.** *Let*

$$C \hookrightarrow \mathbf{P}^N$$

be a smooth projective curve with $N \geq 4$. Then there exists a point

$$O \in \mathbf{P}^N \setminus C$$

such that projection from O restricts to a closed immersion

$$C \hookrightarrow \mathbf{P}^{N-1}.$$

Proof. By lemma 10.47,

$$\dim(\mathrm{Sec}(C) \cup \mathrm{Tan}(C)) \leq 3.$$

Since $N \geq 4$, this union is a proper closed subset of $\mathbf{P}^N$. Choose

$$O \notin \mathrm{Sec}(C) \cup \mathrm{Tan}(C).$$

Then $O \notin C$ and lies on neither a secant nor a tangent line. By proposition 10.46, projection from O restricts to a closed immersion into $\mathbf{P}^{N-1}$. □

Corollary 10.49 (Every curve embeds in $\mathbf{P}^3$)**.** *Every smooth projective curve over an algebraically closed field admits a closed immersion*

$$\boxed{C \hookrightarrow \mathbf{P}^3_k.}$$

Proof. By definition C is projective, so choose some closed immersion

$$C \hookrightarrow \mathbf{P}^N.$$

If $N \leq 3$, compose with a linear closed immersion $\mathbf{P}^N \hookrightarrow \mathbf{P}^3$ and we are done. If $N \geq 4$, apply theorem 10.48 to obtain an embedding

$$C \hookrightarrow \mathbf{P}^{N-1}.$$

Repeating finitely many times reduces the ambient dimension to 3. □

The chapter began with divisors on an abstract curve and ended with a uniform projective realization. The chain of ideas can be summarized as

$$\text{divisors} \longrightarrow \text{linear systems} \longrightarrow \text{Riemann–Roch} \longrightarrow \text{maps of curves},$$
$$\text{maps of curves} \longrightarrow \text{ramification and étale covers} \longrightarrow \text{projective geometry}.$$

The genus-two classification shows how these tools can describe an entire class of curves, while generic projection shows that every smooth projective curve can ultimately be studied as a space curve in $\mathbf{P}^3$.

Chapter 11

Blow-ups and Algebraic Surfaces

The final chapter brings together the constructions developed throughout the book. A blow-up replaces an ideal by a Cartier divisor; on a smooth surface that divisor contributes a new negative direction to the intersection form. Intersection numbers then enter adjunction and Riemann–Roch, Riemann–Roch leads to the Hodge index theorem and positivity, and the Hodge index theorem on $C \times C$ gives the Weil bound. The last section translates that bound into the zeta function and the distribution of closed points.

11.1 Blow-ups and Birational Modifications

Let X be a Noetherian scheme and let $\mathscr{I} \subseteq \mathscr{O}_X$ be a coherent ideal sheaf. Its Rees algebra is the graded quasi-coherent algebra

$$\mathscr{R}(\mathscr{I}) = \bigoplus_{m \geq 0} \mathscr{I}^m, \qquad \mathscr{I}^0 = \mathscr{O}_X.$$

Definition 11.1 (Blow-up)**.** The blow-up of X along $\mathscr{I}$, or along the closed subscheme $Z = V(\mathscr{I})$, is

$$\boxed{\operatorname{Bl}_Z X = \operatorname{Bl}_{\mathscr{I}} X := \underline{\operatorname{Proj}}_X \mathscr{R}(\mathscr{I}).}$$

Its structure morphism is denoted $\pi : \operatorname{Bl}_{\mathscr{I}} X \to X$.

The definition is independent of the chosen generators. On an affine open $U = \operatorname{Spec} A$ on which $\mathscr{I}$ corresponds to $I = (f_0, \ldots, f_r)$, the standard open attached to f_i is

$$U_i = \operatorname{Spec} A\left[\frac{I}{f_i}\right] = \operatorname{Spec} A\left[\frac{f_0}{f_i}, \ldots, \frac{f_r}{f_i}\right] \subseteq \operatorname{Spec} A_{f_i}. \tag{11.1}$$

Here the displayed algebra means the image of the indicated subalgebra of A_{f_i}; it does not require f_i to be a nonzerodivisor.

Example 11.2 (The blow-up of the affine plane)**.** Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ and $I = (x, y)$. The two charts are

$$U_x = \operatorname{Spec} k[x, v], \qquad y = xv,$$

and

$$U_y = \operatorname{Spec} k[u, y], \qquad x = uy.$$

On the overlap, $u = v^{-1}$ and $y = xv$. Thus

$$\operatorname{Bl}_0 \mathbb{A}^2 = \{((x, y), [s : t]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xt = ys\}.$$

Over the origin the equation disappears and the fibre is $\mathbb{P}^1$; away from the origin, $[s : t] = [x : y]$ is forced.

The preceding equations have a useful global interpretation. They explain why the blow-up separates not points of the centre, but directions of approach to the centre.

Proposition 11.3 (Blow-up as the closure of a graph)**.** *Let $X = \operatorname{Spec} A$ be integral, let $0 \neq I = (f_0, \dots, f_r)$, and put $U = X \setminus V(I)$. The rational map*

$$\varphi_I : X \dashrightarrow \mathbb{P}^r, \qquad x \longmapsto [f_0(x) : \cdots : f_r(x)]$$

is defined on U. The graded surjection

$$A[T_0, \dots, T_r] \longrightarrow \mathscr{R}(I), \qquad T_i \longmapsto f_i t,$$

induces a closed immersion

$$\operatorname{Bl}_I X \hookrightarrow X \times \mathbb{P}^r,$$

whose image is the scheme-theoretic closure of the graph of $\varphi_I|_U$.

Proof. The surjection gives the closed immersion after applying Proj. Over U one of the f_i is a unit, and the chart description (11.1) identifies the blow-up with U; under the closed immersion its second coordinate is $[f_0 : \cdots : f_r]$. Thus the inverse image of U is the graph. Since the blow-up is integral and this inverse image is dense, its image is the scheme-theoretic closure of that graph. $\square$

The equations $f_i T_j - f_j T_i$ always vanish on this closure. They need not generate its ideal: higher syzygies among the powers of I may contribute additional equations. The Rees algebra, rather than only the evident bilinear relations, is what records the correct scheme structure.

Proposition 11.4 (Exceptional divisor over a smooth centre)**.** *Suppose that X is smooth and that $Z \hookrightarrow X$ is a smooth closed subvariety of codimension c, with ideal sheaf $\mathscr{I}$. Then*

$$E \simeq \underline{\operatorname{Proj}}_Z \operatorname{Sym}_{\mathscr{O}_Z}(\mathscr{I}/\mathscr{I}^2) = \mathbb{P}(N_{Z/X}),$$

where the last equality uses the convention that $\mathbb{P}(N_{Z/X})$ parametrizes one-dimensional normal directions. Moreover

$$\mathscr{O}_E(-E) \simeq \mathscr{O}_{\mathbb{P}(N_{Z/X})}(1).$$

In particular every fibre of $E \to Z$ is $\mathbb{P}^{c-1}$.

Proof. The exceptional divisor is the relative Proj of the associated graded algebra

$$\operatorname{gr}_{\mathscr{I}}(\mathscr{O}_X) = \bigoplus_{m \geq 0} \mathscr{I}^m/\mathscr{I}^{m+1}.$$

Because a smooth closed immersion is regular, multiplication induces an isomorphism

$$\operatorname{Sym}_{\mathscr{O}_Z}(\mathscr{I}/\mathscr{I}^2) \xrightarrow{\sim} \operatorname{gr}_{\mathscr{I}}(\mathscr{O}_X).$$

The assertion about the tautological bundle is the restriction of $\mathscr{O}_{\operatorname{Bl}_Z X}(-E) = \mathscr{I}\mathscr{O}_{\operatorname{Bl}_Z X} = \mathscr{O}_{\operatorname{Bl}_Z X}(1)$. $\square$

Proposition 11.5 (Flat base change)**.** *For a flat morphism $g : X' \to X$ and $Z' = Z \times_X X'$, the canonical morphism*

$$\operatorname{Bl}_{Z'} X' \longrightarrow \operatorname{Bl}_Z X \times_X X'$$

is an isomorphism.

Proof. Flatness identifies

$$\mathscr{I}^m \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} = (\mathscr{I}\mathscr{O}_{X'})^m$$

for every m. Hence the pulled-back Rees algebra is the Rees algebra of the extended ideal. Relative Proj commutes with base change, which proves the claim. Without flatness the displayed identification of powers can fail, so the hypothesis is essential. □

Proposition 11.6 (Strict transform as a blow-up)**.** *Let $Y \hookrightarrow X$ be an integral closed subscheme not contained in $Z = V(\mathscr{I})$. The strict transform of Y, defined as the closure of*

$$Y \setminus Z \subseteq \mathrm{Bl}_Z X,$$

is canonically isomorphic to

$$\mathrm{Bl}_{\mathscr{I}\mathscr{O}_Y} Y.$$

In particular, if $Y \cap Z$ is an effective Cartier divisor on Y, then the strict transform is isomorphic to Y.

Proof. Restriction gives a surjection of graded algebras

$$\bigoplus_{m\geq 0} \mathscr{I}^m \longrightarrow \bigoplus_{m\geq 0} (\mathscr{I}\mathscr{O}_Y)^m.$$

Its relative Proj is therefore a closed subscheme of $\mathrm{Bl}_Z X$. Away from Z it is exactly $Y \setminus Z$, and this open is dense because Y is integral and is not contained in the centre. The image is consequently the strict transform. If the restricted ideal is invertible, its blow-up is Y itself by the universal property. □

Example 11.7 (Blowing up a Cartier divisor does nothing)**.** If $Z \subset X$ is already an effective Cartier divisor, then $\mathscr{I}_Z$ is invertible and

$$\mathscr{R}(\mathscr{I}_Z) \simeq \mathscr{O}_X[T]$$

locally after choosing a generator. Hence $\mathrm{Bl}_Z X \simeq X$. A blow-up changes the geometry only where the centre fails to be Cartier. This explains why a closed point on a smooth curve has trivial blow-up, while a closed point on a smooth surface produces an exceptional $\mathbb{P}^1$.

Example 11.8 (The affine-plane blow-up as a line bundle)**.** The two charts of $\mathrm{Bl}_0 \mathbb{A}^2$ also identify it with the total space of the tautological line bundle $\mathscr{O}_{\mathbb{P}^1}(-1)$. On the chart with base coordinate $v = y/x$, use x as fibre coordinate; on the chart with base coordinate $u = x/y = v^{-1}$, use y as fibre coordinate. On the overlap

$$y = vx,$$

which is the transition function of $\mathscr{O}_{\mathbb{P}^1}(-1)$. The zero section is the exceptional curve. Its normal bundle is therefore $\mathscr{O}_{\mathbb{P}^1}(-1)$, giving a geometric preview of the calculation $E^2 = -1$ in the next section.

Example 11.9 (A node and its tangent directions)**.** Assume $\operatorname{char} k \neq 2$ and consider

$$C : y^2 - x^2 - x^3 = 0$$

at the origin. In the chart $y = xv$ the total transform is

$$x^2(v^2 - 1 - x) = 0,$$

so the strict transform is $v^2 = 1+x$. It meets the exceptional line $x = 0$ at the two reduced points $v = 1$ and $v = -1$, precisely the two lines in the tangent cone $(y-x)(y+x) = 0$. The strict transform is smooth at both points. Thus one blow-up separates the two branches of the node.

Proposition 11.10 (The exceptional Cartier divisor)**.** *Let $\widetilde{X} = \mathrm{Bl}_{\mathscr{I}} X$. Then*

$$\mathscr{I}\mathscr{O}_{\widetilde{X}} \simeq \mathscr{O}_{\widetilde{X}}(1)$$

is an invertible ideal. Its zero scheme

$$E = V(\mathscr{I}\mathscr{O}_{\widetilde{X}}) = \pi^{-1}(Z)$$

is an effective Cartier divisor, and

$$\mathscr{O}_{\widetilde{X}}(-E) = \mathscr{I}\mathscr{O}_{\widetilde{X}}.$$

Moreover, π is an isomorphism over $X \setminus Z$.

Proof. On the chart (11.1), the extended ideal is generated by f_i, since every $f_j = f_i(f_j/f_i)$. The transition function from the f_i-generator to the f_j-generator is the unit f_j/f_i on the overlap. The ideal is therefore invertible and defines an effective Cartier divisor. If $\mathscr{I} = \mathscr{O}_X$ on an open set, its Rees algebra is $\mathscr{O}_X[T]$, whose relative Proj is that open set itself. This applies to $X \setminus Z$. □

The invertibility just proved is the universal reason for blowing up.

Theorem 11.11 (Universal property of the blow-up)**.** *Let $g : T \to X$ be a morphism for which $\mathscr{I}\mathscr{O}_T$ is an invertible ideal; equivalently, assume that $g^{-1}(Z)$ is an effective Cartier divisor. Then there exists a unique factorization*

$$\begin{tikzcd} T \arrow[rr, "g"] \arrow[dr, dashed, "\widetilde{g}"'] & & X \\ & \mathrm{Bl}_{\mathscr{I}} X \arrow[ur, "\pi"'] & \end{tikzcd}$$

Thus $\mathrm{Bl}_{\mathscr{I}} X$ is terminal among X-schemes on which the extended ideal is invertible.

Proof. The canonical invertible quotient

$$g^*\mathscr{I} \twoheadrightarrow \mathscr{I}\mathscr{O}_T$$

induces compatible surjections $g^*\mathscr{I}^m \twoheadrightarrow (\mathscr{I}\mathscr{O}_T)^{\otimes m}$. The universal property of relative Proj therefore gives an X-morphism $T \to \underline{\mathrm{Proj}}_X \mathscr{R}(\mathscr{I})$. Locally, if $\mathscr{I}\mathscr{O}_T$ is generated by g^*f_i, the morphism lands in U_i and sends f_j/f_i to g^*f_j/g^*f_i; this also proves uniqueness and shows that the local maps glue. □

Remark 11.12. An arbitrary morphism $T \to \mathrm{Bl}_{\mathscr{I}} X$ need not pull the exceptional divisor back to a Cartier divisor: for example, a point may map into E. The universal property is the terminal statement above, not an "if and only if" test for all possible factorizations. More generally, relative Proj represents invertible quotients of $g^*\mathscr{I}$; the invertible ideal $\mathscr{I}\mathscr{O}_T$ supplies the distinguished quotient used here.

Proposition 11.13 (Blowing up an integral scheme)**.** *Let X be an integral Noetherian scheme and let $0 \neq \mathscr{I} \subsetneq \mathscr{O}_X$. Then $\mathrm{Bl}_{\mathscr{I}} X$ is integral and $\pi : \mathrm{Bl}_{\mathscr{I}} X \to X$ is projective, birational, and surjective. If X is a variety over a field, so is $\mathrm{Bl}_{\mathscr{I}} X$.*

Proof. On an affine open $\operatorname{Spec} A$, each chart $A[I/f_i]$ is a subring of the fraction field of the domain A, hence is a domain. The charts have the common dense open lying over $X \setminus Z$, so their union is integral. Relative Proj of a finitely generated graded algebra is projective. The isomorphism over the dense open $X \setminus Z$ proves birationality. The image of a proper morphism is closed and contains this dense open, so it is all of X. Finite type over the ground field follows from the affine chart description. □

Theorem 11.14 (Birational morphisms as blow-ups)**.** *Let X be a quasi-projective integral variety and let $f : Y \to X$ be a projective birational morphism from an integral variety. Then there is a coherent ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_X$ such that*

$$Y \simeq \operatorname{Bl}_{\mathscr{I}} X$$

over X.

Proof. Choose an f-very ample invertible sheaf $\mathscr{L}$ on Y. Its relative section algebra

$$\mathscr{A} = \bigoplus_{m \geq 0} f_* \mathscr{L}^{\otimes m}$$

is of finite type and $Y \simeq \underline{\operatorname{Proj}}_X \mathscr{A}$. Replacing $\mathscr{L}$ by a positive power replaces $\mathscr{A}$ by a Veronese algebra and does not change this relative Proj; we may therefore assume that $\mathscr{A}$ is generated by $\mathscr{A}_1$.

Over the dense open on which f is an isomorphism, choose a rational generator of $\mathscr{L}$. It embeds every $f_*\mathscr{L}^{\otimes m}$ as a rank-one coherent subsheaf of the constant sheaf $K(X)$. Quasi-projectivity supplies an effective Cartier divisor A which clears the finitely many poles of a set of generators of $\mathscr{A}_1$. Replacing $\mathscr{L}$ by $\mathscr{L} \otimes f^*\mathscr{O}_X(-A)$ tensors the degree-m piece by $\mathscr{O}_X(-mA)$ and therefore does not change relative Proj. After this twist we obtain an inclusion

$$\mathscr{A}_1 = \mathscr{I} \hookrightarrow \mathscr{O}_X$$

for a coherent ideal $\mathscr{I}$. Multiplication is computed inside $K(X)$, and the assumption that $\mathscr{A}$ is generated in degree one now gives

$$\mathscr{A}_m = \mathscr{I}^m \qquad (m \geq 0).$$

Consequently

$$Y \simeq \underline{\operatorname{Proj}}_X \mathscr{A} = \underline{\operatorname{Proj}}_X \bigoplus_{m \geq 0} \mathscr{I}^m = \operatorname{Bl}_{\mathscr{I}} X.$$

The quasi-projectivity hypothesis is used precisely in the pole-clearing step. □

Corollary 11.15 (Normalization of a curve as a blow-up)**.** *Let C be a quasi-projective integral curve and $\nu : C^\nu \to C$ its normalization. If $C^\nu \to C$ is finite, then $C^\nu \simeq \operatorname{Bl}_{\mathscr{I}} C$ for some coherent ideal $\mathscr{I}$.*

Proof. The normalization is finite birational, hence projective birational. Apply theorem 11.14. □

Example 11.16 (The cusp)**.** For $C = V(y^2 - x^3) \subseteq \mathbb{A}^2$, blow up the origin. In the chart $x = u$, $y = uv$, the total transform has equation

$$u^2(v^2 - u) = 0.$$

The strict transform is $u = v^2$, hence is the affine line with parameter v; the map to C is

$$v \longmapsto (v^2, v^3).$$

The other chart adds no further point. Thus the strict transform is the normalization of the cusp.

11.2 Intersection Theory and Blow-ups on Smooth Surfaces

From now through section 11.5, let X be a smooth projective integral surface over an algebraically closed field k. Every Weil divisor on X is Cartier.

Definition 11.17 (Local intersection multiplicity)**.** Let $C, D \subset X$ be integral curves with no common component and let $P \in C \cap D$. If $f, g \in \mathscr{O}_{X,P}$ are local equations, define

$$i_P(C, D) = \operatorname{length}_{\mathscr{O}_{X,P}} \frac{\mathscr{O}_{X,P}}{(f, g)}.$$

This number is finite because the regular local ring $\mathscr{O}_{X,P}$ has dimension two and (f, g) is primary to its maximal ideal. Replacing either equation by a unit multiple does not change its ideal; changing a trivialization of the Cartier divisor does exactly that. Thus the definition is independent of all choices.

For an integral projective curve C, define the degree of a line bundle by

$$\deg_C \mathscr{L} = \chi(C, \mathscr{L}) - \chi(C, \mathscr{O}_C).$$

This agrees with the degree in Chapter 10 when C is smooth.

Definition 11.18 (Global intersection number)**.** For a divisor D and an integral curve C, put

$$D \cdot C = \deg_C(\mathscr{O}_X(D)|_C),$$

and extend linearly in the curve variable. We write $C^2 = C \cdot C$.

Lemma 11.19 (Moving divisors away from components)**.** *Let A be a divisor and let $C_1, \ldots, C_s$ be finitely many integral curves on X. For a sufficiently positive very ample divisor H, there are effective divisors*

$$A' \in |A + mH|, \qquad B \in |mH|$$

for some $m > 0$ such that no C_i is a component of either divisor. In particular $A \sim A' - B$ with both effective terms avoiding the prescribed components.

Proof. For $m \gg 0$, both line bundles are globally generated. The sections vanishing identically on a fixed C_i form a proper linear subspace of the space of global sections: a globally generating section can be chosen nonzero at the generic point of C_i. Since k is infinite, a finite union of proper linear subspaces cannot fill the whole space. Choose sections outside that union and take their zero divisors. □

Theorem 11.20 (Basic properties of the intersection pairing)**.** *The pairing* $\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ *is bilinear, symmetric, and depends only on linear equivalence classes. If C and D are effective curves without a common component, then*

$$\boxed{C \cdot D = \sum_{P \in C \cap D} i_P(C, D).}$$

Consequently $C \cdot D$ is the set-theoretic number of intersection points only when every intersection is transverse.

Proof. Additivity in the first variable follows from tensor products of line bundles; additivity in the second follows from additivity of Euler characteristic in the restriction sequences. Principal divisors give the trivial line bundle, so the pairing factors through linear equivalence.

For effective D not containing C, restriction of its defining section gives

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_X(D)|_C \longrightarrow \mathscr{O}_{C\cap D}(D) \longrightarrow 0.$$

The last sheaf has at P the length of $\mathscr{O}_{X,P}/(f,g)$; twisting does not change the length. Taking Euler characteristics proves the displayed formula, which is symmetric.

For arbitrary divisors, apply lemma 11.19 successively to both arguments, enlarging the prescribed list after each choice. Each divisor becomes a difference of effective divisors, and every pair occurring in the bilinear expansion has no common component. Symmetry therefore reduces to the case just proved. □

Proposition 11.21 (Projection formula for curves)**.** *Let $f : Y \to X$ be a proper morphism of smooth projective surfaces, let D be a divisor on X, and let $C \subset Y$ be an integral curve. Then*

$$\boxed{(f^*D)\cdot C = D \cdot f_*C.}$$

Here $f_*C = 0$ when C is contracted, and otherwise it is the image curve multiplied by the degree of $C \to f(C)$.

Proof. If C is contracted, $f^*\mathscr{O}_X(D)|_C$ is pulled back from a point and has degree zero. Otherwise, for the finite map $g : C \to f(C)$ obtained on the curves, degree of line bundles satisfies

$$\deg_C g^*L = (\deg g)\deg_{f(C)} L.$$

Apply this to $L = \mathscr{O}_X(D)|_{f(C)}$. □

Proposition 11.22 (Normal bundle and self-intersection)**.** *If $C \subset X$ is a smooth curve, then*

$$\mathscr{O}_X(C)|_C \simeq N_{C/X}, \qquad C^2 = \deg N_{C/X}.$$

Proof. The conormal sequence identifies $\mathscr{I}_C/\mathscr{I}_C^2$ with $\mathscr{O}_X(-C)|_C$. Dualizing gives the first isomorphism; the second is the definition of intersection. □

Now let $P \in X$ and

$$\pi : \widetilde{X} = \mathrm{Bl}_P\, X \longrightarrow X$$

be the blow-up, with exceptional curve E.

Proposition 11.23 (The exceptional curve)**.** *There are isomorphisms*

$$E \simeq \mathbb{P}(T_PX) \simeq \mathbb{P}^1, \qquad \mathscr{O}_{\widetilde{X}}(E)|_E \simeq \mathscr{O}_{\mathbb{P}^1}(-1).$$

In particular,

$$\boxed{E^2 = -1.}$$

Proof. The associated graded algebra of the maximal ideal $\mathfrak{m}_P$ is $\mathrm{Sym}_k(\mathfrak{m}_P/\mathfrak{m}_P^2)$ because X is smooth at P. Hence $E = \mathrm{Proj}\,\mathrm{gr}_{\mathfrak{m}_P}\,\mathscr{O}_{X,P} \simeq \mathbb{P}^1$. By proposition 11.10, $\mathscr{O}(-E) = \mathfrak{m}_P\mathscr{O}_{\widetilde{X}} = \mathscr{O}(1)$; restricting to E gives $\mathscr{O}_E(-E) = \mathscr{O}_{\mathbb{P}^1}(1)$ and therefore the asserted formula. □

Theorem 11.24 (Picard group and intersection form of a point blow-up)**.** *Pullback is injective and*

$$\boxed{\operatorname{Pic}(\widetilde{X}) \simeq \pi^* \operatorname{Pic}(X) \oplus \mathbb{Z}[E].}$$

Moreover,

$$(\pi^* D_1) \cdot (\pi^* D_2) = D_1 \cdot D_2, \qquad (\pi^* D) \cdot E = 0, \qquad E^2 = -1.$$

Proof. Represent a line bundle on $\widetilde{X}$ by a divisor A. Push every nonexceptional prime component forward to X. Since X is smooth, the resulting Weil divisor D is Cartier. The divisors A and $\pi^* D$ agree away from E, so their difference is aE for some $a \in \mathbb{Z}$. This proves generation. If $\pi^* D + aE \sim 0$, intersection with E gives $-a = 0$; restriction to the complement and normality of X then give $D \sim 0$. Thus the sum is direct.

Choose representatives of D, D_1, D_2 whose supports avoid P. Their pullbacks are their strict transforms and do not meet E, which proves the first two intersection identities. The last is proposition 11.23. □

Proposition 11.25 (Tangent cone on the exceptional line)**.** *Let $C \subset X$ be an integral curve through P, and let $m = \operatorname{mult}_P(C)$. If $f = f_m + f_{m+1} + \cdots$ is a local equation written in regular parameters, with f_j homogeneous of degree j, then the scheme-theoretic intersection $\widetilde{C} \cap E$ is the degree-m subscheme of*

$$E = \mathbb{P}(T_P X)$$

cut out by f_m. Consequently

$$\boxed{\widetilde{C} \cdot E = m.}$$

Its support consists of the tangent directions of the branches of C at P.

Proof. On the chart $x = u$, $y = uv$, one has

$$f(u, uv) = u^m (f_m(1, v) + u f_{m+1}(1, v) + \cdots).$$

After removing the exceptional factor u^m, restriction to $E = (u = 0)$ gives $f_m(1, v)$. The other chart supplies the homogeneous description at the missing direction. Hence the intersection is the projectivized tangent cone and has degree m on $E \simeq \mathbb{P}^1$. □

Proposition 11.26 (Strict transforms)**.** *Let $C \subset X$ be an integral curve and put $m = \operatorname{mult}_P(C)$. Its strict transform satisfies*

$$\widetilde{C} \sim \pi^* C - mE, \qquad \widetilde{C}^2 = C^2 - m^2.$$

If D is another curve with multiplicity n at P, then

$$\widetilde{C} \cdot \widetilde{D} = C \cdot D - mn.$$

When C, D have no common component,

$$\boxed{i_P(C, D) = mn + \sum_{Q \in E} i_Q(\widetilde{C}, \widetilde{D}).} \tag{11.2}$$

Proof. In regular parameters x, y at P, a local equation of C has order m. On each blow-up chart its pullback has the exceptional coordinate to order m; the remaining factor defines the strict transform. Thus the total transform is $\widetilde{C} + mE$. The global intersection formulas now follow from theorem 11.24. Away from P, the blow-up is an isomorphism and local intersection multiplicities are unchanged. Comparing the global local-sum formula before and after blowing up gives (11.2). □

Corollary 11.27 (Multiplicity bound and tangent cones)**.** *If C and D have no common component, then*

$$\boxed{i_P(C, D) \geq \mathrm{mult}_P(C)\,\mathrm{mult}_P(D).}$$

Equality holds if and only if their strict transforms have no common point on E, equivalently if and only if the projectivized tangent cones of C and D have disjoint support.

Proof. Every term in the residual sum in (11.2) is a nonnegative length. It vanishes exactly when the strict transforms have no common point over P. By proposition 11.25, their intersections with E are precisely their projectivized tangent cones. □

For two smooth curves through P, both multiplicities equal one. They meet transversely exactly when their tangent directions are distinct, and then the corollary gives $i_P(C, D) = 1$. If their tangent directions agree, the residual intersection on the blow-up measures the next order of contact. Repeating the process separates successive jets rather than merely successive tangent lines.

Example 11.28 (Reading contact order by repeated blow-ups)**.** In $\mathbb{A}^2$ let

$$C : y = 0, \qquad D_m : y - x^m = 0.$$

Both curves have multiplicity one at the origin. In the chart $x = u$, $y = uv$, their strict transforms are

$$\widetilde{C} : v = 0, \qquad \widetilde{D}_m : v - u^{m-1} = 0.$$

For $m > 1$ they meet at the same point of E, and their new local intersection multiplicity is the one for the pair $(v, v - u^{m-1})$, namely $m - 1$. Thus (11.2) reads

$$i_0(C, D_m) = 1 + i_0(\widetilde{C}, \widetilde{D}_m).$$

Repeating this calculation gives

$$i_0(C, D_m) = \underbrace{1 + \cdots + 1}_{m \text{ times}} = m.$$

At the final blow-up the two strict transforms meet the exceptional line in different tangent directions, so the recursion stops.

Example 11.29 (A cusp and its tangent line)**.** Let $C : y^2 - x^3 = 0$ and $L : y = 0$. Directly,

$$i_0(C, L) = \dim_k k[[x, y]]/(y^2 - x^3, y) = 3.$$

The multiplicities at the origin are 2 and 1. After the blow-up, in the chart $x = u$, $y = uv$, the strict transforms are

$$\widetilde{C} : v^2 - u = 0, \qquad \widetilde{L} : v = 0.$$

Their remaining intersection multiplicity is one. The recursion therefore recovers the same answer geometrically:

$$i_0(C, L) = 2 \cdot 1 + 1 = 3.$$

This example also distinguishes multiplicity from tangency: the first term sees the multiplicities, while the residual term sees that the unique tangent direction is shared.

Proposition 11.30 (Intersection through infinitely near points)**.** *Let C, D be curves without a common component through $P = P_0$. Blow up P_0, and whenever their strict transforms still meet above P, blow up every such common point and continue along the resulting tree. Suppose that after finitely many steps the strict transforms are separated. Write*

$$m_j = \operatorname{mult}_{P_j}(C_j), \qquad n_j = \operatorname{mult}_{P_j}(D_j),$$

where C_j, D_j are the strict transforms at the corresponding stage. Summing over all vertices of the tree of common infinitely near points gives

$$\boxed{i_P(C, D) = \sum_j m_j n_j.}$$

For a single curve followed through a sequence of point blow-ups,

$$C_r^2 = C^2 - \sum_{j=0}^{r-1} m_j^2.$$

Proof. Apply (11.2) at P_0. Every residual intersection lies at a common point on the exceptional line; apply the same formula there and continue. Once the strict transforms are separated, the remaining residual sums vanish, leaving the sum of the products $m_j n_j$. The self-intersection formula follows independently by iterating $C_{j+1}^2 = C_j^2 - m_j^2$. □

This proposition explains the phrase *infinitely near point*: after a blow-up, a point of the exceptional curve remembers a tangent direction at the previous point; after another blow-up, it remembers a direction together with a second-order choice. Local intersection multiplicity is the total contribution accumulated along this finite tree.

Corollary 11.31 (Multiplicity form of Bezout's inequality)**.** *Let $C, D \subset \mathbb{P}^2$ be plane curves of degrees d, e without a common component. For any finite set of points P_i,*

$$\boxed{\sum_i \operatorname{mult}_{P_i}(C) \operatorname{mult}_{P_i}(D) \le de.}$$

Equality can occur only if every intersection lies among the P_i and the strict transforms have no residual common infinitely near point above them.

Proof. The global intersection number is $C \cdot D = de$. At each chosen point, corollary 11.27 gives

$$i_{P_i}(C, D) \ge \operatorname{mult}_{P_i}(C) \operatorname{mult}_{P_i}(D).$$

Sum these inequalities and compare with the sum of all local intersection multiplicities. The equality statement follows from the residual terms in the blow-up recursion. □

Theorem 11.32 (Canonical divisor under a point blow-up)**.** *For canonical divisors chosen compatibly,*

$$\boxed{K_{\widetilde{X}} = \pi^* K_X + E.}$$

Consequently $K_{\widetilde{X}}^2 = K_X^2 - 1$.

Proof. Near P, choose regular parameters x, y. In the chart $x = u$, $y = uv$,

$$dx \wedge dy = u \, du \wedge dv.$$

Thus the pullback of a local generator of ω_X vanishes to order one along E and nowhere else. The two charts give the same divisor, proving the formula. The square follows from theorem 11.24. □

Example 11.33 (Blowing up points of the projective plane)**.** Let

$$\pi : X = \mathrm{Bl}_{P_1,\dots,P_r} \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

be the blow-up of distinct points, let H be the pullback of a line, and let E_i be the exceptional curves. Repeated use of theorems 11.24 and 11.32 gives

$$\mathrm{Pic}(X) \simeq \mathbb{Z}H \oplus \bigoplus_{i=1}^{r} \mathbb{Z}E_i,$$

$$H^2 = 1, \qquad H \cdot E_i = 0, \qquad E_i \cdot E_j = -\delta_{ij},$$

and

$$\boxed{K_X = -3H + \sum_{i=1}^{r} E_i, \qquad K_X^2 = 9 - r.}$$

If a plane curve of degree d has multiplicity m_i at P_i, its strict transform has class

$$dH - \sum_i m_i E_i$$

and square $d^2 - \sum_i m_i^2$. Thus the complete intersection table of the blow-up turns geometric incidence conditions in the plane into a diagonal quadratic calculation.

Example 11.34 (Resolving a pencil of lines)**.** The pencil of lines through a point $P \in \mathbb{P}^2$ defines a rational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$

which is undefined exactly at P. Applying the affine graph-closure description of proposition 11.3 on standard charts and gluing shows that its graph closure is $X = \mathrm{Bl}_P \mathbb{P}^2$, and the second projection extends the pencil to a morphism

$$f : X \longrightarrow \mathbb{P}^1.$$

Every fibre is the strict transform of a line through P, hence has class

$$F = H - E, \qquad F^2 = 0.$$

The exceptional curve satisfies

$$E \cdot F = 1,$$

so E is a section of f. The two numerical facts have immediate geometric meanings: distinct fibres do not meet, and a section meets every fibre once.

This example is the simplest instance in which a blow-up does more than alter an intersection matrix: it resolves the base point of a linear system and turns a rational map into a morphism. Conversely, the morphism f remembers the original pencil, while the section E remembers its base point.

Example 11.35 (A pencil of conics)**.** Let two general conics span a pencil. They meet in four distinct base points $P_1, \dots, P_4$. After blowing up those points, the pencil becomes a morphism to $\mathbb{P}^1$ whose fibre class is

$$F = 2H - E_1 - E_2 - E_3 - E_4.$$

The calculation

$$F^2 = 4 - 1 - 1 - 1 - 1 = 0$$

again expresses disjointness of distinct fibres. Each E_i has $E_i \cdot F = 1$ and is a section. Thus base-point resolution, strict transforms, and the diagonal intersection form work together in a nontrivial linear system.

11.3 Riemann–Roch, Adjunction, and Positivity

Theorem 11.36 (Adjunction)**.** *If $C \subset X$ is a smooth integral curve, then*

$$\omega_C \simeq (\omega_X \otimes \mathscr{O}_X(C))|_C.$$

Equivalently,

$$\boxed{2g(C) - 2 = C \cdot (C + K_X).}$$

Proof. The conormal sequence

$$0 \to \mathscr{O}_C(-C) \to \Omega^1_X|_C \to \Omega^1_C \to 0$$

and the determinant identity give the line-bundle formula. Taking degrees and using $\deg \omega_C = 2g(C) - 2$ gives the numerical formula. □

Theorem 11.37 (Riemann–Roch for a smooth surface)**.** *For every divisor D on X,*

$$\boxed{\chi(\mathscr{O}_X(D)) = \chi(\mathscr{O}_X) + \frac{1}{2} D \cdot (D - K_X).}$$

Proof. Set

$$\Phi(D) = \chi(\mathscr{O}_X(D)) - \chi(\mathscr{O}_X) - \frac{1}{2} D(D - K_X).$$

If C is a smooth curve, the exact sequence

$$0 \to \mathscr{O}_X(D - C) \to \mathscr{O}_X(D) \to \mathscr{O}_C(D) \to 0$$

gives

$$\chi(\mathscr{O}_X(D)) - \chi(\mathscr{O}_X(D - C)) = \deg \mathscr{O}_C(D) + 1 - g(C).$$

By curve Riemann–Roch and adjunction, the right side is

$$D \cdot C - \frac{1}{2} C(C + K_X).$$

A direct expansion shows that this is exactly the change in $\frac{1}{2}D(D - K_X)$ when D is replaced by $D - C$. Hence $\Phi(D) = \Phi(D - C)$.

Choose a very ample divisor H. For $m \gg 0$, both $D + mH$ and mH are very ample, and Bertini gives smooth members $A \in |D + mH|$ and $B \in |mH|$. The connectedness theorem for effective ample divisors from Chapter 10 makes A and B connected. A smooth scheme has disjoint irreducible components, so these divisors are smooth integral curves. Applying the preceding invariance first to A and then to B gives

$$\Phi(D) = \Phi(D - A) = \Phi(-B) = \Phi(0) = 0.$$

This proves the formula without invoking Hirzebruch–Riemann–Roch. □

Corollary 11.38 (Arithmetic adjunction)**.** *Let $C \subset X$ be any effective Cartier divisor and define*

$$p_a(C) = 1 - \chi(\mathscr{O}_C).$$

Then

$$\boxed{2p_a(C) - 2 = C \cdot (C + K_X).}$$

For a smooth integral curve this agrees with theorem 11.36; the formula also applies to reducible or singular effective divisors.

Proof. The divisor sequence

$$0 \to \mathscr{O}_X(-C) \to \mathscr{O}_X \to \mathscr{O}_C \to 0$$

and surface Riemann–Roch give

$$\chi(\mathscr{O}_C) = \chi(\mathscr{O}_X) - \chi(\mathscr{O}_X(-C)) = -\frac{1}{2}C(C + K_X).$$

Substitution into the definition of $p_a(C)$ proves the assertion. □

For the exceptional curve, adjunction is a useful sign check:

$$E \cdot (E + K_{\widetilde{X}}) = E \cdot (E + \pi^* K_X + E) = -2,$$

so $p_a(E) = 0$, as required by $E \simeq \mathbb{P}^1$. Both occurrences of E are essential: one comes from the curve itself and one from the discrepancy in $K_{\widetilde{X}} = \pi^* K_X + E$.

Example 11.39 (Genus of a plane curve)**.** Let $C \subset \mathbb{P}^2$ be a smooth curve of degree d. Since $C \sim dH$ and $K_{\mathbb{P}^2} = -3H$, adjunction gives

$$2g(C) - 2 = d(d-3),$$

and therefore

$$\boxed{g(C) = \frac{(d-1)(d-2)}{2}.}$$

The same numerical expression is the arithmetic genus of every effective plane divisor of degree d, whether or not it is smooth. Singularities lower the genus of the normalization, not this arithmetic genus.

Example 11.40 (A singular cubic after blowing up)**.** A nodal or cuspidal plane cubic has arithmetic genus one and multiplicity two at its singular point P. On $X = \mathrm{Bl}_P \mathbb{P}^2$, its strict transform has class

$$\widetilde{C} = 3H - 2E.$$

Using $K_X = -3H + E$, arithmetic adjunction gives

$$\widetilde{C} \cdot (\widetilde{C} + K_X) = (3H - 2E) \cdot (-E) = -2.$$

Hence $p_a(\widetilde{C}) = 0$. In the node calculation of example 11.9 the two branches are separated and the strict transform is smooth; in the cusp calculation it is also smooth, though tangent to the exceptional curve. In either case the blow-up has converted the singular cubic into a rational smooth curve. Numerically, the lost unit of genus is visible in the multiplicity correction

$$\widetilde{C}^2 = C^2 - 2^2 = 9 - 4 = 5.$$

Lemma 11.41 (Cohomology of a point blow-up)**.** *For the blow-up of a smooth point,*

$$\pi_* \mathscr{O}_{\widetilde{X}} = \mathscr{O}_X, \qquad R^i \pi_* \mathscr{O}_{\widetilde{X}} = 0 \quad (i > 0).$$

More generally, for $m \geq 0$,

$$\pi_* \mathscr{O}_{\widetilde{X}}(-mE) = \mathfrak{m}_P^m, \qquad R^1 \pi_* \mathscr{O}_{\widetilde{X}}(-mE) = 0.$$

Consequently $\chi(\mathscr{O}_{\widetilde{X}}) = \chi(\mathscr{O}_X)$.

Proof. The assertions are local near P. On $\operatorname{Spec} A$ with regular parameters x, y, the blow-up is covered by the two affine charts $A[y/x]$ and $A[x/y]$. Their intersection is obtained by inverting the indicated ratio. In the two-term Cech complex, a Laurent polynomial in the ratio can be split into its nonnegative and nonpositive parts. It follows that degree-zero cocycles are A and that the degree-one cokernel is zero. Hence

$$\pi_* \mathscr{O}_{\widetilde{X}} = \mathscr{O}_X, \qquad R^1 \pi_* \mathscr{O}_{\widetilde{X}} = 0.$$

For the general statement use

$$0 \to \mathscr{O}(-(m+1)E) \to \mathscr{O}(-mE) \to \mathscr{O}_E(m) \to 0,$$

where $\mathscr{O}_E(-E) = \mathscr{O}_{\mathbb{P}^1}(1)$. The restriction map

$$\mathfrak{m}_P^m \longrightarrow H^0(E, \mathscr{O}_E(m)) \simeq \mathfrak{m}_P^m / \mathfrak{m}_P^{m+1}$$

is the quotient by the next power and is surjective. Since $H^1(\mathbb{P}^1, \mathscr{O}(m)) = 0$ for $m \geq 0$, induction in the direct-image long exact sequence gives

$$\pi_* \mathscr{O}(-mE) = \mathfrak{m}_P^m, \qquad R^1 \pi_* \mathscr{O}(-mE) = 0$$

for all $m \geq 0$. The fibres have dimension at most one, so there are no higher direct images. Leray then gives the equality of Euler characteristics. $\square$

Corollary 11.42 (Birational invariants of a point blow-up)**.** *For every i there are natural isomorphisms*

$$H^i(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \simeq H^i(X, \mathscr{O}_X).$$

Consequently the irregularity $q(X) = h^1(\mathscr{O}_X)$, the geometric genus $p_g(X) = h^2(\mathscr{O}_X)$, and $\chi(\mathscr{O}_X)$ are unchanged by a point blow-up, whereas

$$K_{\widetilde{X}}^2 = K_X^2 - 1.$$

Thus K^2 records the number of point blow-ups while q, p_g, and $\chi(\mathscr{O})$ do not.

Proof. Apply the Leray spectral sequence to the first two identities in lemma 11.41; the statement about K^2 is theorem 11.32. $\square$

Corollary 11.43 (Plurigenera under a point blow-up)**.** *For every integer $m \geq 1$,*

$$H^0(\widetilde{X}, \omega_{\widetilde{X}}^{\otimes m}) \simeq H^0(X, \omega_X^{\otimes m}).$$

Thus the plurigenus

$$P_m(X) = h^0(X, \omega_X^{\otimes m})$$

is unchanged by a point blow-up.

Proof. The canonical-divisor formula gives

$$mK_{\widetilde{X}} = \pi^*(mK_X) + mE.$$

For $1 \leq j \leq m$, restrict the line bundle $\mathscr{O}_{\widetilde{X}}(\pi^*(mK_X) + jE)$ to E. The pullback part is trivial on E, while

$$\mathscr{O}_E(jE) \simeq \mathscr{O}_{\mathbb{P}^1}(-j)$$

has no nonzero global section. The exact sequences obtained by increasing j therefore show that adding E successively does not change H^0. Finally

$$H^0(\widetilde{X}, \pi^* \omega_X^{\otimes m}) \simeq H^0(X, \omega_X^{\otimes m})$$

by $\pi_* \mathscr{O}_{\widetilde{X}} = \mathscr{O}_X$ and the projection formula. $\square$

The contrast is instructive. The canonical self-intersection drops by one at every point blow-up, but all spaces of pluricanonical forms remain unchanged. The number K^2 is therefore changed by the model, whereas the sequence $(P_m)_{m\geq 1}$ is stable under point blow-ups and under any sequence of such blow-ups.

Corollary 11.44 (Riemann–Roch on a blow-up)**.** *For $L = \pi^*D - mE$ with $m \geq 0$,*

$$\boxed{\chi(\mathscr{O}_{\widetilde{X}}(L)) = \chi(\mathscr{O}_X(D)) - \frac{m(m+1)}{2}.}$$

Moreover,

$$H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}(\pi^*D - mE)) \simeq H^0(X, \mathscr{O}_X(D) \otimes \mathfrak{m}_P^m).$$

Thus the exceptional divisor turns an order-m condition at P into an ordinary divisor condition.

Proof. Use theorems 11.24, 11.32 and 11.37 and lemma 11.41. The statement about sections is the projection formula together with $\pi_*\mathscr{O}(-mE) = \mathfrak{m}_P^m$. □

Corollary 11.45 (Euler characteristic for every exceptional coefficient)**.** *For $a \in \mathbb{Z}$ and a divisor D on X,*

$$\boxed{\chi(\mathscr{O}_{\widetilde{X}}(\pi^*D + aE)) = \chi(\mathscr{O}_X(D)) - \frac{a(a-1)}{2}.}$$

Taking $a = -m$ recovers the correction $m(m+1)/2$ in corollary 11.44.

Proof. Using $K_{\widetilde{X}} = \pi^*K_X + E$ and the orthogonal decomposition of the intersection form,

$$(\pi^*D + aE) \cdot (\pi^*(D - K_X) + (a-1)E) = D(D - K_X) - a(a-1).$$

Now apply surface Riemann–Roch and $\chi(\mathscr{O}_{\widetilde{X}}) = \chi(\mathscr{O}_X)$. □

Example 11.46 (Plane curves with an assigned multiple point)**.** Let $X = \mathrm{Bl}_P \mathbb{P}^2$, with classes H and E. Sections of $\mathscr{O}_X(dH - mE)$ are degree-d homogeneous polynomials whose affine expansion at P has no term of total degree below m. Therefore a point of multiplicity at least m imposes

$$1 + 2 + \cdots + m = \frac{m(m+1)}{2}$$

linear conditions, provided $d \geq m$. For example, cubics form a 10-dimensional vector space; requiring a double point at a prescribed point imposes three conditions, leaving a 7-dimensional vector space, or a 6-dimensional projective linear system. On the blow-up this is simply the complete linear system $|3H - 2E|$, whose self-intersection is

$$(3H - 2E)^2 = 9 - 4 = 5.$$

The correction $m(m+1)/2$ in corollary 11.44 is exactly the length of the jet quotient $\mathscr{O}_{X,P}/\mathfrak{m}_P^m$.

Proposition 11.47 (Riemann–Roch for assigned plane points)**.** *On $X = \mathrm{Bl}_{P_1,\dots,P_r} \mathbb{P}^2$, let*

$$L = dH - \sum_{i=1}^r m_i E_i.$$

Then

$$\boxed{\chi(\mathscr{O}_X(L)) = \binom{d+2}{2} - \sum_{i=1}^r \binom{m_i+1}{2}.}$$

When $H^1(X, \mathscr{O}_X(L)) = H^2(X, \mathscr{O}_X(L)) = 0$, this is also the dimension of the vector space of degree-d plane curves having multiplicity at least m_i at each P_i.

Proof. Use $\chi(\mathscr{O}_X) = 1$ and

$$K_X = -3H + \sum_i E_i.$$

The diagonal intersection table gives

$$\begin{aligned} L \cdot (L - K_X) &= \left(dH - \sum_i m_i E_i\right)\left((d+3)H - \sum_i (m_i+1)E_i\right) \\ &= d(d+3) - \sum_i m_i(m_i+1). \end{aligned}$$

Surface Riemann–Roch yields the displayed binomial expression. Under the vanishing hypothesis, Euler characteristic equals h^0; the identification of sections with imposed multiplicities follows from lemma 11.41 and the projection formula. □

The vanishing clause matters. Riemann–Roch always computes the Euler characteristic, but special configurations of points can create unexpected H^1 and hence more sections than the naive number of independent conditions predicts. Blow-ups do not erase that phenomenon; they isolate it in the single cohomology group $H^1(X, \mathscr{O}_X(L))$.

Example 11.48 (The anticanonical divisor after one blow-up)**.** On $X = \mathrm{Bl}_P\,\mathbb{P}^2$,

$$-K_X = 3H - E, \qquad (-K_X)^2 = 8.$$

The criterion of example 11.61 shows that $3H - E$ is ample. Its sections are plane cubics passing through P, and proposition 11.47 gives

$$\chi(\mathscr{O}_X(-K_X)) = \binom{5}{2} - \binom{2}{1} = 9.$$

Here the blow-up changes K^2 from 9 to 8 while leaving $\chi(\mathscr{O}_X) = 1$ unchanged, displaying the two different birational behaviours from corollary 11.42.

Proposition 11.49 (Effectivity from positive square)**.** *Let H be ample. If a divisor D satisfies*

$$D^2 > 0, \qquad D \cdot H > 0,$$

then for every sufficiently large n there is an effective divisor E_n with $nD \sim E_n$.

Proof. Surface Riemann–Roch gives

$$\chi(\mathscr{O}_X(nD)) = \frac{1}{2}n^2D^2 + O(n).$$

By Serre duality, $H^2(X, \mathscr{O}_X(nD))^\vee = H^0(X, \mathscr{O}_X(K_X - nD))$. If $K_X - nD$ were effective, its intersection with H would be nonnegative, but $(K_X - nD)H < 0$ for $n \gg 0$. Thus $h^2(nD) = 0$ for large n, and

$$h^0(nD) = \chi(nD) + h^1(nD) > 0.$$

A nonzero section of $\mathscr{O}_X(nD)$ defines the required effective divisor. □

11.4 Numerical Equivalence, the Hodge Index, and Ampleness

Definition 11.50 (Numerical equivalence)**.** Divisors D, D' are numerically equivalent, written $D \equiv D'$, if

$$(D - D') \cdot C = 0$$

for every integral curve $C \subset X$. Put

$$N^1(X) = \mathrm{Div}(X)/\equiv, \qquad N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The intersection form descends to $N^1(X)$.

Theorem 11.51 (Hodge index theorem)**.** *Let H be ample. If $D \cdot H = 0$, then $D^2 \leq 0$, with equality if and only if $D \equiv 0$. Equivalently, the intersection form is negative definite on $H^\perp \subset N^1(X)_{\mathbb{R}}$.*

Proof. If $D^2 > 0$, replace D by $-D$ if necessary so that $D \cdot A > 0$ for one ample divisor A sufficiently close to H in the rational divisor space. More concretely, for $r \gg 0$ the divisor $A = rH + D$ is ample and $D \cdot A = D^2 > 0$. Proposition 11.49 would then make a positive multiple of D effective. But an ample divisor H has positive intersection with every nonzero effective curve, contradicting $D \cdot H = 0$. Thus $D^2 \leq 0$.

Assume $D^2 = 0$. For any divisor B, put

$$B' = (H^2)B - (B \cdot H)H,$$

so $B' \cdot H = 0$ and $D \cdot B' = (H^2)(D \cdot B)$. If this number were nonzero, then one of the integral divisors $mD + B'$ or $mD - B'$ would have positive square for $m \gg 0$, while remaining orthogonal to H, contradicting the first paragraph. Hence $D \cdot B = 0$ for every B, so $D \equiv 0$. The converse is immediate. The same argument after clearing denominators gives nonpositivity over $\mathbb{Q}$, and continuity gives it over $\mathbb{R}$. If a real class $v \in H^\perp$ has $v^2 = 0$, then the inequality for $v + tw$ for every real t forces $v \cdot w = 0$ for every $w \in H^\perp$. Decomposing any divisor class into its H-component and an orthogonal component shows that v is numerically zero. Thus the restriction is genuinely negative definite. ☐

Corollary 11.52 (Hodge inequality)**.** *For divisors D and an ample divisor H,*

$$\boxed{(D \cdot H)^2 \geq D^2 H^2.}$$

Equality holds exactly when D is numerically proportional to H.

Proof. Apply theorem 11.51 to $H^2 D - (D \cdot H)H$, which is orthogonal to H. ☐

Definition 11.53 (Nef divisor)**.** A divisor D is *nef* if $D \cdot C \geq 0$ for every integral curve $C \subset X$. Every ample divisor is nef, but a nef divisor may have square zero or may vanish on some curve. For example, on a point blow-up the pullback of an ample divisor is nef but not ample because it has intersection zero with E.

Lemma 11.54 (The positive cone)**.** *Fix an ample class H. If real divisor classes A, B satisfy*

$$A^2 > 0, \quad B^2 \geq 0, \quad A \cdot H > 0, \quad B \cdot H > 0,$$

then $A \cdot B > 0$. Thus the classes of positive square with positive intersection against H form one of the two connected components of the positive cone.

Proof. Write

$$A = aH + A_0, \qquad B = bH + B_0, \qquad A_0, B_0 \in H^{\perp},$$

where $a, b > 0$. Negative definiteness on $H^{\perp}$ and ordinary Cauchy–Schwarz for $-(\,\cdot\,)$ give

$$A_0 \cdot B_0 \geq -\sqrt{(-A_0^2)(-B_0^2)}.$$

The assumptions $A^2 > 0$ and $B^2 \geq 0$ say

$$-A_0^2 < a^2H^2, \qquad -B_0^2 \leq b^2H^2.$$

Therefore

$$A \cdot B = abH^2 + A_0 \cdot B_0 > 0.$$

□

Corollary 11.55 (Negativity of contracted curves)**.** *Let $f : Y \to X$ be a birational morphism of smooth projective surfaces. The intersection form is negative definite on the subspace of $N^1(Y)_{\mathbb{R}}$ spanned by the numerical classes of curves contracted by f.*

Proof. Choose an ample divisor H on X and put $L = f^*H$. The projection formula gives

$$L^2 = H^2 > 0, \qquad L \cdot C = 0$$

for every contracted curve C. The Hodge index theorem gives one positive direction, so the orthogonal complement of any positive-square class in the component containing the ample cone is negative definite. The class L lies in that component, and every contracted curve lies in $L^{\perp}$. □

Example 11.56 (An infinitely near blow-up matrix)**.** Blow up $P \in X$, obtaining E_1, and then blow up a point $Q \in E_1$. Let E_1' be the strict transform and E_2 the new exceptional curve. Since the multiplicity of E_1 at Q is one,

$$(E_1')^2 = -2, \qquad E_2^2 = -1, \qquad E_1' \cdot E_2 = 1.$$

The exceptional configuration therefore has matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}.$$

Its leading principal minors have signs $-, +$, so it is negative definite, as corollary 11.55 predicts. The first exceptional curve ceases to be a (-1)-curve after the second blow-up; the new (-1)-curve is E_2.

The equality clause in the Hodge inequality is often the most useful part. If D is nef and $D^2 > 0$, then a curve satisfying $D \cdot C = 0$ must have $C^2 < 0$ unless its numerical class is zero. Thus curves obstructing ampleness of a positive nef divisor necessarily live in the negative part of the intersection form.

Lemma 11.57 (Ampleness on a projective curve)**.** *Let T be a projective scheme of pure dimension one and let M be a line bundle on T. Then M is ample if and only if*

$$\deg_C(M) > 0$$

for every irreducible component C of T_{red}.

Proof. If M is ample, a power embeds T into projective space, and its restriction to every component has positive degree. Conversely, filter an arbitrary coherent sheaf on T first by powers of the nilradical and then by coherent sheaves supported on the irreducible components. It is therefore enough to work on an integral projective curve C. Pull back to the normalization $\nu : C^\nu \to C$. The line bundle ν^*M has positive degree, so curve Riemann–Roch and Serre duality show that, for $n \gg 0$, its n-th power has vanishing first cohomology and is generated by global sections. The finite conductor quotient between C and C^ν has zero-dimensional support; after increasing n, its finitely many value conditions can also be prescribed. Devissage then gives the same vanishing and generation statements for every coherent sheaf on T. Serre's criterion implies that M is ample. $\square$

The only extra ingredient in the surface proof of Nakai–Moishezon is a base-point-free argument on an effective divisor.

Lemma 11.58 (Base-point-free step)**.** *Let L be a line bundle and suppose that a section of L cuts out an effective Cartier divisor E. If $L|_E$ is ample, then $L^{\otimes n}$ is globally generated for all sufficiently large n.*

Proof. Tensor the divisor sequence by $L^{\otimes n}$:

$$0 \longrightarrow L^{\otimes n} \longrightarrow L^{\otimes(n+1)} \longrightarrow L^{\otimes(n+1)}|_E \longrightarrow 0.$$

Since $L|_E$ is ample, its higher cohomology vanishes for large powers. Hence, for $n \gg 0$, the long exact sequence gives surjections

$$H^1(X, L^{\otimes n}) \twoheadrightarrow H^1(X, L^{\otimes(n+1)}).$$

The dimensions form an eventually nonincreasing sequence of nonnegative integers, so the maps are eventually isomorphisms. Looking one term earlier in the same long exact sequence then shows that

$$H^0(X, L^{\otimes(n+1)}) \twoheadrightarrow H^0(E, L^{\otimes(n+1)}|_E)$$

is surjective for $n \gg 0$.

Large powers of the ample line bundle $L|_E$ are globally generated. Lift a set of generating sections to X. They generate along E, while the power of the original defining section of E is nonzero and hence generates at every point of $X \setminus E$. Together these sections generate $L^{\otimes(n+1)}$ on all of X. $\square$

Theorem 11.59 (Nakai–Moishezon for surfaces)**.** *A divisor D on X is ample if and only if*

$$\boxed{D^2 > 0 \quad \textit{and} \quad D \cdot C > 0 \textit{ for every integral curve } C \subset X.}$$

Proof. If D is ample, a positive power is very ample. Two general members have positive finite intersection, so $D^2 > 0$, and restriction to every curve has positive degree.

Conversely, proposition 11.49 shows that, after replacing D by a positive multiple, we may assume $D \sim E \geq 0$. Put $L = \mathscr{O}_X(D)$. Every irreducible component C of the projective curve E satisfies

$$\deg(L|_C) = D \cdot C > 0.$$

The ampleness criterion for a line bundle on a projective curve therefore makes $L|_E$ ample. By lemma 11.58, a positive power L^n is globally generated and defines a morphism

$$f : X \longrightarrow \mathbb{P}^N, \qquad L^n \simeq f^*\mathscr{O}_{\mathbb{P}^N}(1).$$

No fibre of f can contain an integral curve C, since such a curve would have $L \cdot C = 0$. Every positive-dimensional projective scheme contains a curve, so all fibres are zero-dimensional. Thus f is proper and quasi-finite, hence finite. The pullback of an ample line bundle by a finite morphism is ample: indeed, for a coherent sheaf $\mathscr{F}$ the projection formula and exactness of f_* reduce Serre vanishing for $\mathscr{F} \otimes f^*\mathscr{O}(m)$ to Serre vanishing for $f_*\mathscr{F} \otimes \mathscr{O}(m)$ on $\mathbb{P}^N$. Hence L^n is ample, and therefore so is L. Since ampleness is unchanged by replacing a line bundle by a positive power, the original divisor D is ample.

The dependency order is

$$\text{surface Riemann–Roch} \implies \text{effectivity} \implies \text{base-point freeness} \implies \text{Nakai–Moishezon};$$

the Hodge index theorem is not used in this proof, so there is no circularity. □

Corollary 11.60 (The null-curve test for a nef and big divisor)**.** *Let D be nef with $D^2 > 0$. Then*

$$D \textit{ is ample} \iff D \cdot C > 0 \textit{ for every integral curve } C.$$

Equivalently, the only possible obstruction to ampleness is a curve in

$$\mathrm{Null}(D) = \{C : D \cdot C = 0\}.$$

Every curve in this null locus has negative self-intersection.

Proof. The equivalence is theorem 11.59; the final assertion follows from the equality case of the Hodge index theorem. □

Example 11.61 (The ample cone of a one-point plane blow-up)**.** Let $X = \mathrm{Bl}_P\,\mathbb{P}^2$ and write $D = aH - bE$. Then

$$D \cdot E = b, \qquad D \cdot (H - E) = a - b, \qquad D^2 = a^2 - b^2.$$

The classes E and $H - E$ are represented respectively by the exceptional curve and by the strict transform of a line through P. Hence ampleness forces

$$a > b > 0.$$

Conversely, an integral curve other than E is the strict transform of a plane curve of degree d and multiplicity $m \leq d$ at P. If $a > b > 0$, then

$$D \cdot \widetilde{C} = ad - bm \geq (a - b)d > 0,$$

and $D^2 > 0$. Nakai–Moishezon therefore gives the concrete criterion

$$\boxed{aH - bE \text{ is ample} \iff a > b > 0.}$$

On the boundary, H and $H - E$ are nef but not ample: the first vanishes on E, while the second has square zero.

Example 11.62 (Two blown-up points)**.** Let $X = \mathrm{Bl}_{P_1,P_2}\,\mathbb{P}^2$ for distinct points and put

$$D = aH - b_1E_1 - b_2E_2.$$

Intersection with E_1, E_2 and with the strict transform of the line through the two points shows that ampleness requires

$$b_1 > 0, \qquad b_2 > 0, \qquad a > b_1 + b_2.$$

These inequalities are also sufficient. Indeed, if an irreducible plane curve of degree d is not the line through both points and has multiplicities m_1, m_2, Bezout with that line gives $m_1 + m_2 \leq d$. Hence

$$\begin{aligned} D \cdot \widetilde{C} &= ad - b_1 m_1 - b_2 m_2 \\ &\geq (a - \max\{b_1, b_2\})d > 0. \end{aligned}$$

The distinguished line itself has intersection $a - b_1 - b_2 > 0$, and $D^2 = a^2 - b_1^2 - b_2^2 > 0$. Nakai–Moishezon gives

$$\boxed{D \text{ ample} \quad \Longleftrightarrow \quad b_1, b_2 > 0 \text{ and } a > b_1 + b_2.}$$

The extra wall $a = b_1 + b_2$ is created by the new negative curve $H - E_1 - E_2$.

Example 11.63 (New negative curves after several blow-ups)**.** If P_1, P_2 lie on a line $L \subset \mathbb{P}^2$, its strict transform on $\mathrm{Bl}_{P_1,P_2} \mathbb{P}^2$ has class

$$H - E_1 - E_2$$

and square -1. Thus negative curves on a blown-up surface are not limited to the exceptional divisors themselves: strict transforms of curves passing through the centres can become negative. This is exactly the mechanism behind the appearance of exceptional curves in birational surface geometry.

The exceptional curve of a point blow-up provides the model negative direction:

$$E \cdot \pi^* H = 0, \qquad E^2 = -1.$$

A smooth rational curve with square -1 is called an exceptional curve of the first kind. Castelnuovo's contraction theorem gives the converse over an algebraically closed field: such a curve can be contracted to a smooth point. We record this statement only; surface classification is beyond the scope of these notes.

11.5 Cohomology Classes and the Intersection Form

Section 8.6 associated to a smooth codimension-p subvariety $i : Y \hookrightarrow X$ the unique class characterized by

$$\mathrm{Tr}_X([Y] \smile \alpha) = \mathrm{Tr}_Y(i^*\alpha).$$

On a surface, divisors correspond to $p = 1$. The purpose of this section is to identify that trace-theoretic class with the elementary transition-function construction for line bundles and then compare its cup product with the integer intersection pairing.

The homomorphism of sheaves of abelian groups

$$d\log : \mathscr{O}_X^\times \longrightarrow \Omega_X^1, \qquad u \longmapsto \frac{du}{u},$$

induces

$$c_1 : \mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^\times) \longrightarrow H^1(X, \Omega_X^1).$$

For a divisor D, write $c_1(D) = c_1(\mathscr{O}_X(D))$.

Concretely, if a line bundle L has local frames e_i with $e_i = g_{ij} e_j$, then $c_1(L)$ is represented on the same cover by

$$\{d \log g_{ij}\} = \left\{ \frac{dg_{ij}}{g_{ij}} \right\}.$$

The cocycle identity for the g_{ij} becomes the additive cocycle identity after applying $d\log$. Moreover

$$c_1(L \otimes M) = c_1(L) + c_1(M), \qquad c_1(L^\vee) = -c_1(L),$$

so c_1 factors through linear equivalence exactly as the intersection pairing does.

Lemma 11.64 (Trace of the first Chern class on a curve)**.** *Let B be a smooth projective integral curve and L a line bundle on B. Then*

$$\boxed{\operatorname{Tr}_B(c_1(L)) = (\deg L)1_k.}$$

Proof. Every line bundle on B is $\mathscr{O}_B(D)$ for a divisor $D = \sum n_P P$. Additivity reduces the statement to $L = \mathscr{O}_B(P)$. For a local parameter z at P, its transition function is z^{-1}, and the corresponding boundary class is the cohomology class of the point. The normalization of the Serre trace in proposition 8.63 gives $\operatorname{Tr}_B([P]) = 1$. Summing the coefficients of D proves the formula. In positive characteristic the integer is, as usual, read through $\mathbb{Z} \to k$. ☐

Lemma 11.65 (Residue sequence for a smooth divisor)**.** *If $i : C \hookrightarrow X$ is a smooth divisor, there is an exact sequence*

$$0 \longrightarrow \Omega^2_X \longrightarrow \Omega^2_X(C) \xrightarrow{\operatorname{res}_C} \Omega^1_C \longrightarrow 0.$$

Locally, if $C = (z = 0)$ and w is a parameter along C, then

$$\operatorname{res}_C\left(a(z,w)\frac{dz}{z} \wedge dw\right) = a(0,w)\,dw.$$

Proof. Because z, w are regular parameters, a two-form with at most a simple pole along C is uniquely a regular two-form plus an expression of the displayed shape. Changing the local equation z by a unit alters dz/z by a regular one-form, so the restriction of the coefficient to C is independent of the choice. Every local one-form on C occurs as a residue, and zero residue means that a is divisible by z modulo a regular term. These observations prove surjectivity and identify the kernel with Ω^2_X. ☐

Proposition 11.66 (Functoriality of the first Chern class)**.** *For a morphism $f : Y \to X$ of smooth varieties and a line bundle L on X,*

$$c_1(f^*L) = f^*c_1(L).$$

Proof. If g_{ij} are transition functions for L, then f^*g_{ij} are transition functions for f^*L, and

$$d\log(f^*g_{ij}) = f^*(d\log g_{ij}).$$

☐

Proposition 11.67 (Divisor class and first Chern class)**.** *If $C \subset X$ is a smooth integral curve, then its class from definition 8.62 satisfies*

$$[C] = c_1(\mathscr{O}_X(C)) \in H^1(X, \Omega^1_X).$$

Proof. Choose local equations f_i for C on an affine cover $\{U_i\}$. With local frames $e_i = f_i^{-1}$, the relation $e_i = (f_j/f_i)e_j$ shows that $c_1(\mathscr{O}_X(C))$ is represented by the Cech cocycle

$$d\log(f_j/f_i) = \frac{df_j}{f_j} - \frac{df_i}{f_i}.$$

Use the residue sequence of lemma 11.65. On the same Cech cover, wedging a cocycle for α with the displayed $d\log$ cocycle is the Cech formula for the connecting homomorphism applied to $i^*\alpha$: locally the polar part is $(df_i/f_i) \wedge \alpha_i$, whose residue is $i^*\alpha_i$. Compatibility of the boundary map with Serre traces gives $\mathrm{Tr}_X \circ \delta = \mathrm{Tr}_C$. Consequently, for every $\alpha \in H^1(X, \Omega^1_X)$,

$$\mathrm{Tr}_X(c_1(\mathscr{O}_X(C)) \smile \alpha) = \mathrm{Tr}_C(i^*\alpha).$$

This is precisely the characterization (8.2), and Serre duality gives equality. □

Theorem 11.68 (Cohomological interpretation of intersection)**.** *For divisors D, E on X,*

$$\boxed{(D \cdot E)\, 1_k = \mathrm{Tr}_X(c_1(D) \smile c_1(E)).}$$

In characteristic zero we identify the integer on the left with its image in k; in positive characteristic the identity records the intersection number modulo $\mathrm{char}\, k$*.*

Proof. First let $D = C$ be a smooth curve. By equation (8.2) and proposition 11.67,

$$\mathrm{Tr}_X(c_1(C) \smile c_1(E)) = \mathrm{Tr}_C(i^* c_1(E)).$$

On a smooth projective curve, the trace of the first Chern class of a line bundle is its degree by lemma 11.64. The right side is therefore $\deg_C \mathscr{O}_X(E)|_C = C \cdot E$.

For arbitrary D, choose a very ample divisor H and $m \gg 0$ so that $D + mH$ and mH are very ample. Bertini gives smooth representatives, and the connectedness theorem for effective ample divisors from Chapter 10 makes those representatives connected and hence integral. Bilinearity of c_1, cup product, trace, and intersection reduces the assertion to the smooth case. □

Proposition 11.69 (Cup product as the class of the intersection)**.** *Let $C, D \subset X$ be smooth integral curves with no common component. Then*

$$\boxed{[C] \smile [D] = \sum_{P \in C \cap D} i_P(C, D)[P] \quad \text{in } H^2(X, \Omega^2_X).}$$

For transverse curves this is simply the sum of the classes of their geometric intersection points.

Proof. Serre duality identifies $H^2(X, \Omega^2_X)$ with the dual of $H^0(X, \mathscr{O}_X) = k$, so the trace map is an isomorphism. By proposition 11.67 and theorem 11.68, the trace of the left side is $C \cdot D$. The local-sum formula gives

$$C \cdot D = \sum_P i_P(C, D),$$

while every point class has trace one by proposition 8.63. The two classes therefore have the same trace and hence are equal. □

Example 11.70 (Bezout in cohomological form)**.** Let $h = c_1(\mathscr{O}_{\mathbb{P}^2}(1))$. Since a line has square one,

$$\mathrm{Tr}_{\mathbb{P}^2}(h^2) = 1_k.$$

If C and D have degrees d and e, then

$$c_1(C) = dh, \qquad c_1(D) = eh,$$

and hence

$$\mathrm{Tr}_{\mathbb{P}^2}([C] \smile [D]) = de\, 1_k.$$

When C, D are smooth without a common component, proposition 11.69 refines this trace identity to

$$de\,[P_0] = \sum_{P \in C \cap D} i_P(C, D)[P] \quad \text{in } H^2(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}),$$

where P_0 is any closed point and all point classes have trace one. Over characteristic zero, taking traces recovers the integer Bezout formula. Over positive characteristic, the cohomology class retains its reduction in k, while the algebraic local lengths retain the full integer equality.

Remark 11.71 (Loss of integral information in characteristic p)*.* Suppose $\operatorname{char} k = p > 0$ and let H be the class of a line in $\mathbb{P}^2$. For the Cartier divisor $D = pH$,

$$D \cdot H = p,$$

but

$$c_1(D) = p\, c_1(H) = 0 \quad \text{in } H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}).$$

Accordingly the cohomological trace of $c_1(D)c_1(H)$ is zero. There is no contradiction: the integer p maps to zero in k. This concrete example is why the integral intersection form, rather than Hodge cohomology alone, remains indispensable in positive characteristic.

Example 11.72 (The new cohomology class of an exceptional curve)**.** For $X = \mathrm{Bl}_P\, \mathbb{P}^2$, put

$$h = c_1(H), \qquad e = c_1(E) \in H^1(X, \Omega^1_X).$$

The algebraic intersection table becomes

$$\mathrm{Tr}_X(h^2) = 1_k, \qquad \mathrm{Tr}_X(h \smile e) = 0, \qquad \mathrm{Tr}_X(e^2) = -1_k.$$

Thus the exceptional curve contributes a class orthogonal to pullbacks and with negative square. Over characteristic zero this is literally the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

inside the cup-product pairing. In characteristic p the same calculation in Hodge cohomology records only the reduction of that integral matrix modulo p; the integer-valued intersection pairing of section 11.2 must therefore remain the primary object in the proof of the Hodge index theorem.

Thus the integer pairing constructed algebraically in section 11.2 is also the trace of a cup-product pairing. No general Chow ring is needed for this comparison.

11.6 Frobenius, the Hodge Index, and the Weil Bound

Let $C/\mathbb{F}_q$ be a smooth projective geometrically integral curve of genus g, and put $N = \#C(\mathbb{F}_q)$. Base change to $\overline{\mathbb{F}}_q$ and work on the smooth projective surface

$$S = C \times C.$$

Choose $P \in C(\overline{\mathbb{F}}_q)$ and set

$$F_1 = C \times \{P\}, \qquad F_2 = \{P\} \times C.$$

Let Δ be the diagonal and let Γ be the graph of the q-power Frobenius $F : C \to C$.

Lemma 11.73 (Normal bundle of a graph)**.** *For a morphism $f : C \to C$, the graph embedding $\gamma_f : C \hookrightarrow C \times C$ has normal bundle*

$$N_{\Gamma_f/(C\times C)} \simeq f^*T_C.$$

If a fixed point P of f satisfies $1 - df_P \neq 0$, then Γ_f and Δ meet transversely at (P, P).

Proof. After pulling the tangent bundle of the product back along the graph, the map

$$T_C \oplus f^*T_C \longrightarrow f^*T_C, \qquad (a, b) \longmapsto b - df(a)$$

is surjective and has kernel equal to the image of the differential of γ_f. This identifies the normal bundle with f^*T_C. At a fixed point, the tangent lines of the diagonal and graph are respectively the images of $a \mapsto (a, a)$ and $a \mapsto (a, df_P(a))$; they are distinct exactly when $1 - df_P$ is invertible. □

Proposition 11.74 (The four basic curves on $C \times C$)**.** *Their intersection numbers are*

$$F_1^2 = F_2^2 = 0, \qquad F_1 \cdot F_2 = 1,$$

$$\Delta \cdot F_1 = \Delta \cdot F_2 = 1, \qquad \Delta^2 = 2 - 2g,$$

and

$$\Gamma \cdot F_1 = q, \qquad \Gamma \cdot F_2 = 1, \qquad \Gamma^2 = q(2 - 2g), \qquad \Gamma \cdot \Delta = N.$$

Proof. The fibres move in their respective pencils, giving square zero, and two fibres of opposite projections meet transversely once. The diagonal meets each fibre once. Its normal bundle is T_C, so adjunction gives $\Delta^2 = \deg T_C = 2 - 2g$.

The first projection identifies Γ with C, while the second restricts to Frobenius of degree q. This gives the intersections with F_1, F_2. By lemma 11.73, the normal bundle of the graph is F^*T_C; hence $\Gamma^2 = q(2 - 2g)$. Finally $\Gamma \cap \Delta$ is the fixed-point scheme of Frobenius. Its geometric points are exactly $C(\mathbb{F}_q)$, and it is reduced because $dF = 0$, so the differential of $F - \mathrm{id}$ is $-\mathrm{id}$. Thus the intersection number is N. □

The divisor $H = F_1 + F_2$ is ample. Indeed,

$$\mathscr{O}_S(H) \simeq p_1^*\mathscr{O}_C(P) \otimes p_2^*\mathscr{O}_C(P),$$

and sufficiently high powers of the two factors define projective embeddings; their external tensor product defines the Segre embedding of the product. Define

$$D = \Gamma - F_1 - qF_2, \qquad E = \Delta - F_1 - F_2.$$

The orthogonality is immediate from the bidegrees:

$$D \cdot H = E \cdot H = 0,$$

while the squares are

$$\begin{aligned} D^2 &= \Gamma^2 + (F_1 + qF_2)^2 - 2\Gamma \cdot (F_1 + qF_2) \\ &= q(2 - 2g) + 2q - 2(q + q) = -2gq, \end{aligned}$$

and

$$\begin{aligned} E^2 &= \Delta^2 + (F_1 + F_2)^2 - 2\Delta \cdot (F_1 + F_2) \\ &= (2 - 2g) + 2 - 4 = -2g. \end{aligned}$$

Finally, expanding the mixed product gives

$$\begin{aligned} D \cdot E &= (\Gamma - F_1 - qF_2) \cdot (\Delta - F_1 - F_2) \\ &= N - (q+1). \end{aligned}$$

Thus

$$D^2 = -2gq, \qquad E^2 = -2g, \qquad D \cdot E = N - (q+1).$$

The same table works simultaneously over every finite extension. Let Γ_n be the graph of F^n and put

$$D_n = \Gamma_n - F_1 - q^n F_2.$$

Since F^n has degree q^n and differential zero,

$$\Gamma_n \cdot F_1 = q^n, \quad \Gamma_n \cdot F_2 = 1, \quad \Gamma_n^2 = q^n(2-2g), \quad \Gamma_n \cdot \Delta = N_n.$$

Consequently

$$D_n \cdot H = 0, \qquad D_n^2 = -2gq^n, \qquad D_n \cdot E = N_n - (q^n + 1).$$

Thus the extension-field estimate is the same surface calculation with the iterated Frobenius graph.

Theorem 11.75 (Weil bound)**.** *For a smooth projective geometrically integral curve of genus g over $\mathbb{F}_q$,*

$$\boxed{|\#C(\mathbb{F}_q) - (q+1)| \leq 2g\sqrt{q}.}$$

More generally, if $N_n = \#C(\mathbb{F}_{q^n})$, then

$$\boxed{|N_n - (q^n + 1)| \leq 2gq^{n/2}} \qquad (n \geq 1).$$

Proof. The Hodge index theorem makes $-(\,\cdot\,)$ a positive definite pairing on $H^\perp$. Cauchy–Schwarz therefore gives

$$(D \cdot E)^2 \leq D^2 E^2 = 4g^2 q.$$

Substitution gives the first inequality. Apply the same argument to D_n and E to obtain the second. □

Corollary 11.76 (Equality in the Weil bound)**.** *Assume $g > 0$. Equality*

$$|N_n - (q^n + 1)| = 2gq^{n/2}$$

holds if and only if the numerical classes D_n and E are proportional in $N^1(C \times C)_{\mathbb{R}}$. In that case

$$D_n \equiv \pm q^{n/2} E.$$

Proof. Equality in the Weil bound is equality in Cauchy–Schwarz for the positive definite form $-(\,\cdot\,)$ on $H^\perp$. It occurs exactly when D_n and E are linearly dependent. Comparing $D_n^2 = -2gq^n$ with $E^2 = -2g$ determines the absolute value of the proportionality factor. □

The equality case is therefore geometrically rigid: an extremal point count forces the Frobenius correspondence, after its two fibre components are removed, to lie on the same numerical line as the diagonal correspondence.

Corollary 11.77 (Points over sufficiently large extensions)**.** *For all sufficiently large n,*

$$C(\mathbb{F}_{q^n}) \neq \varnothing.$$

More explicitly, for $g \geq 1$ it is enough that

$$q^{n/2} > g + \sqrt{g^2 - 1}.$$

Proof. Put $x = q^{n/2}$. The extended Weil bound gives

$$N_n \geq x^2 + 1 - 2gx.$$

The right side is positive for $x > g + \sqrt{g^2 - 1}$. For $g = 0$ the exact value is $q^n + 1$. □

This apparently modest consequence is the arithmetic input needed in lemma 11.81: it rules out a common divisor greater than one for the degrees of all closed points. Thus the same Hodge-index inequality that controls the error term also supplies the degree-one divisor used to prove rationality of the zeta function.

Example 11.78 (Genus zero and genus one)**.** For $g = 0$, the two orthogonal classes in the proof have square zero and hence are numerically trivial; the bound becomes the exact identity

$$\#C(\mathbb{F}_{q^n}) = q^n + 1.$$

For $g = 1$, put

$$a = q + 1 - \#C(\mathbb{F}_q).$$

The bound reads $|a| \leq 2\sqrt{q}$. In the next section this same integer is the middle coefficient of the numerator of the zeta function. Thus the surface intersection inequality already predicts the size of both reciprocal roots of that quadratic polynomial.

11.7 Zeta Functions and the Arithmetic of Curves

Continue with $C/\mathbb{F}_q$ as in the preceding section. For a closed point $x \in |C|$, put $\deg x = [\kappa(x) : \mathbb{F}_q]$.

We shall use Riemann–Roch over $\mathbb{F}_q$, whereas Chapter 10 stated it over an algebraically closed field. This causes no gap. For every line bundle L on C, flat cohomology base change gives

$$H^i(C, L) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q \simeq H^i(C_{\overline{\mathbb{F}}_q}, L_{\overline{\mathbb{F}}_q}).$$

The degree of L and the genus are unchanged by this extension. Applying the algebraically closed version to the base change and comparing dimensions proves the same Riemann–Roch formula over $\mathbb{F}_q$.

Definition 11.79 (Zeta function)**.**

$$\boxed{Z(C,t) = \exp\left(\sum_{n\geq 1} N_n \frac{t^n}{n}\right) \in 1 + t\mathbb{Q}[[t]].}$$

Proposition 11.80 (Euler product)**.**

$$\boxed{Z(C,t) = \prod_{x\in|C|} (1 - t^{\deg x})^{-1}.}$$

Equivalently, if A_m is the number of effective divisors of degree m, then

$$Z(C,t) = \sum_{m\geq 0} A_m t^m.$$

Proof. A closed point of degree d contributes d geometric points fixed by F^n exactly when $d \mid n$. Hence

$$N_n = \sum_{d|n} dB_d,$$

where B_d is the number of closed points of degree d. Taking logarithms,

$$\sum_{n\geq 1} N_n \frac{t^n}{n} = \sum_{d\geq 1} B_d \sum_{r\geq 1} \frac{t^{dr}}{r} = -\sum_{x\in|C|} \log(1 - t^{\deg x}).$$

Exponentiation gives the product. Expanding each factor as a geometric series chooses an arbitrary nonnegative multiplicity for every closed point, which is exactly an effective divisor. □

The infinite product is legitimate in the t-adic topology. Its coefficient of t^m involves only closed points of degree at most m. There are finitely many of those, because a degree-d point contributes a Frobenius orbit to the finite set $C(\mathbb{F}_{q^d})$. Thus every coefficient is determined by a finite product.

The next lemma removes a hidden rational-point assumption from the elementary proof of rationality.

Lemma 11.81 (Existence of a degree-one divisor)**.** *The degree map* $\deg : \mathrm{Div}(C) \to \mathbb{Z}$ *is surjective. Thus there exists a divisor A of degree one, although A need not be effective.*

Proof. Let d be the greatest common divisor of the degrees of all closed points. If $d > 1$, then $C(\mathbb{F}_{q^n})$ is empty whenever $d \nmid n$. But the extended Weil bound gives

$$N_n \geq q^n + 1 - 2gq^{n/2} > 0$$

for every sufficiently large n. There are arbitrarily large n not divisible by d, a contradiction. Hence $d = 1$, and an integral linear combination of closed points has degree one. □

Lemma 11.82 (Finiteness of the degree-zero class group)**.** *The group* $\mathrm{Pic}^0(C)$ *of degree-zero divisor classes defined over* $\mathbb{F}_q$ *is finite.*

Proof. Fix a degree-one divisor A and an integer $m \geq 2g$. For any degree-zero divisor D, Riemann–Roch gives

$$\ell(D + mA) \geq m + 1 - g > 0.$$

Thus every degree-zero class is represented, after adding mA, by an effective divisor of degree m. There are only finitely many such divisors: only closed points of degree at most m can occur, and each $C(\mathbb{F}_{q^r})$ is finite. □

Write $h_C = \#\,\mathrm{Pic}^0(C)$.

Theorem 11.83 (Rationality of the zeta function)**.** *There is a polynomial* $P_C(t) \in \mathbb{Z}[t]$ *such that*

$$\boxed{Z(C,t) = \frac{P_C(t)}{(1-t)(1-qt)},}$$

with

$$P_C(0) = 1, \qquad \deg P_C = 2g.$$

Proof. Tensoring with $\mathscr{O}_C(-nA)$ identifies the degree-n divisor classes with $\mathrm{Pic}^0(C)$, so there are h_C of them. If $n > 2g-2$, Riemann–Roch gives $\ell(D) = n+1-g$ for every degree-n class. Its effective representatives are the nonzero sections of $\mathscr{O}_C(D)$ modulo $\mathbb{F}_q^\times$, hence number

$$\frac{q^{n+1-g}-1}{q-1}.$$

Therefore, for $n > 2g-2$,

$$A_n = h_C\frac{q^{n+1-g}-1}{q-1}.$$

The coefficients of $Z(C,t)$ consequently have a tail which is a linear combination of 1 and q^n. Multiplication by $(1-t)(1-qt)$ leaves a polynomial of degree at most $2g$. Its coefficients are integers because $Z(C,t) = \sum A_n t^n$ has integer coefficients. The functional equation proved next shows that the degree is exactly $2g$ and the leading coefficient is q^g. □

Theorem 11.84 (Functional equation)**.**

$$\boxed{P_C(t) = q^g t^{2g} P_C\left(\frac{1}{qt}\right).}$$

Equivalently,

$$Z\left(C, \frac{1}{qt}\right) = q^{1-g}t^{2-2g}Z(C,t).$$

Proof. If $g=0$, Riemann–Roch makes every degree-zero divisor class effective and hence trivial; it also gives

$$A_n = \frac{q^{n+1}-1}{q-1} \qquad (n \geq 0),$$

so $Z(C,t) = ((1-t)(1-qt))^{-1}$ and both forms of the functional equation are immediate. Assume henceforth that $g \geq 1$.

For a divisor D of degree n, Riemann–Roch gives

$$\ell(D) = n+1-g+\ell(K_C-D).$$

The involution $[D] \mapsto [K_C - D]$ pairs degree-n classes with degree-$(2g-2-n)$ classes. Put $A_n = 0$ for $n < 0$. Summing $(q^{\ell(D)}-1)/(q-1)$ over the h_C classes of degree n gives the explicit coefficient relation

$$(q-1)A_n = q^{n+1-g}\big((q-1)A_{2g-2-n} + h_C\big) - h_C \qquad (n \in \mathbb{Z}).$$

Moreover the tail computed in theorem 11.83 gives

$$Z(C,t) = \sum_{n=0}^{2g-2} A_n t^n + \frac{h_C}{q-1}\left(\frac{q^{1-g}(qt)^{2g-1}}{1-qt} - \frac{t^{2g-1}}{1-t}\right).$$

Substitute the coefficient relation into the finite sum and reindex it by $n \mapsto 2g-2-n$. The two displayed geometric tails are exchanged by $t \mapsto 1/(qt)$, and hence

$$Z\left(C, \frac{1}{qt}\right) = q^{1-g}t^{2-2g}Z(C,t).$$

After multiplying by the two denominator factors, this is

$$P_C(t) = q^g t^{2g} P_C(1/(qt)).$$

Since $P_C(0) = 1$, the right side has leading term $q^g t^{2g}$, proving at the same time that the degree is $2g$. □

Corollary 11.85 (Palindromic coefficients)**.** *Write*

$$P_C(t) = a_0 + a_1 t + \cdots + a_{2g} t^{2g}.$$

Then $a_0 = 1$, $a_{2g} = q^g$, and

$$\boxed{a_{2g-i} = q^{g-i} a_i \qquad (0 \le i \le g).}$$

In particular the functional equation is an explicit symmetry of the integer coefficients, not merely a symmetry of the zeros.

Proof. Substitute the coefficient expansion into $P_C(t) = q^g t^{2g} P_C(1/(qt))$ and compare the coefficient of t^{2g-i}. □

Factor over $\mathbb{C}$:

$$P_C(t) = \prod_{i=1}^{2g} (1 - \alpha_i t).$$

Taking logarithms in the rational expression for Z gives

$$\boxed{N_n = q^n + 1 - \sum_{i=1}^{2g} \alpha_i^n.} \tag{11.3}$$

Lemma 11.86 (Power sums)**.** *If complex numbers $\beta_1, \ldots, \beta_r$ satisfy $|\sum_i \beta_i^n| \le M R^n$ for every $n \ge 1$, then $|\beta_i| \le R$ for every i.*

Proof. The generating function of the power sums is

$$\sum_{n \ge 1} \left(\sum_i \beta_i^n \right) z^n = \sum_i \frac{\beta_i z}{1 - \beta_i z}.$$

The bound makes the left side holomorphic for $|z| < R^{-1}$. Equal values among the β_i add their positive multiplicities, so none of the poles on the right cancels. Hence every pole $z = \beta_i^{-1}$ lies outside that disc. □

Theorem 11.87 (Riemann hypothesis for curves over finite fields)**.** *Every reciprocal root of P_C satisfies*

$$\boxed{|\alpha_i| = \sqrt{q}.}$$

Equivalently, every nontrivial zero of $Z(C, t)$ has $|t| = q^{-1/2}$; under $t = q^{-s}$, it lies on $\Re(s) = 1/2$.

Proof. Equations (11.3) and the extended Weil bound give

$$\left| \sum_i \alpha_i^n \right| \le 2g q^{n/2}$$

for all n. The power-sum lemma gives $|\alpha_i| \le \sqrt{q}$. The functional equation makes the multiset of reciprocal roots invariant under $\alpha \mapsto q/\alpha$. Applying the same upper bound to the paired root gives $q/|\alpha_i| \le \sqrt{q}$, hence the reverse inequality. □

Example 11.88 (The projective line)**.** For $C = \mathbb{P}^1$, one has $N_n = q^n + 1$, and therefore

$$\begin{aligned} Z(\mathbb{P}^1, t) &= \exp\left(\sum_{n\geq 1}(q^n+1)\frac{t^n}{n}\right) \\ &= \exp(-\log(1-qt) - \log(1-t)) \\ &= \boxed{\frac{1}{(1-t)(1-qt)}}. \end{aligned}$$

Thus $P_{\mathbb{P}^1}(t) = 1$, exactly as the genus-zero case of the functional equation requires.

Example 11.89 (A genus-one curve)**.** Let $C/\mathbb{F}_q$ have genus one and put

$$a = q + 1 - \#C(\mathbb{F}_q).$$

The numerator has degree two, constant term one, and leading coefficient q. Comparing the coefficient of t in $Z = P_C/((1-t)(1-qt))$ gives

$$\boxed{P_C(t) = 1 - at + qt^2.}$$

If $P_C(t) = (1-\alpha t)(1-\beta t)$, then

$$\alpha + \beta = a, \qquad \alpha\beta = q, \qquad |\alpha| = |\beta| = \sqrt{q}.$$

Consequently

$$\#C(\mathbb{F}_{q^n}) = q^n + 1 - \alpha^n - \beta^n.$$

For an elliptic curve this is the familiar Frobenius trace formula; the Hasse bound $|a| \leq 2\sqrt{q}$ is the $n = 1$ Weil bound.

Corollary 11.90 (Class number)**.**

$$\boxed{h_C = P_C(1),}$$

and

$$(\sqrt{q}-1)^{2g} \leq h_C \leq (\sqrt{q}+1)^{2g}.$$

Proof. From the explicit tail for A_n, comparison of the residue at $t = 1$ in $Z = P_C/((1-t)(1-qt))$ gives $P_C(1) = h_C$. The bounds follow from $P_C(1) = \prod_i(1-\alpha_i)$ and $|\alpha_i| = \sqrt{q}$. □

Corollary 11.91 (Prime divisors of the function field)**.** *Let*

$$B_n = \#\{x \in |C| : \deg x = n\}.$$

Then

$$\boxed{B_n \sim \frac{q^n}{n}.}$$

More precisely, $B_n = q^n/n + O(q^{n/2})$, with the implied constant depending on C and q.

Proof. From $N_n = \sum_{d|n} dB_d$, Mobius inversion gives

$$nB_n = \sum_{d|n} \mu(d) N_{n/d}.$$

The $d = 1$ term is $q^n + 1 + O(q^{n/2})$ by Weil. Every proper divisor contribution has exponent at most $n/2$. More precisely,

$$\sum_{\substack{d|n \\ d>1}} N_{n/d} \le \sum_{m=1}^{\lfloor n/2 \rfloor} (q^m + 1 + 2gq^{m/2}) = O(q^{n/2}).$$

Hence

$$nB_n = q^n + O(q^{n/2}).$$

After division by n this gives the stronger error $O(q^{n/2}/n)$, and in particular the stated estimate. Dividing by q^n/n proves the asymptotic. $\square$

The chapter began by replacing an ideal with a divisor. It ends with closed points behaving as prime divisors:

$$\begin{aligned} &\text{blow-up} \longrightarrow \text{intersection form} \longrightarrow \text{Hodge index} \\ &\qquad \longrightarrow \text{Weil bound} \longrightarrow \text{zeta function and prime divisors.} \end{aligned}$$

In this way the birational geometry of surfaces and the arithmetic of curves meet inside the same intersection pairing.