Complex Variables

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Preface

These notes present complex analysis from a more structural and conceptual point of view. The main topics include the classical theory of holomorphic functions, Riemann surfaces, sheaves and sheaf cohomology, divisors and line bundles, Serre duality, and the Riemann–Roch theorem.

The material naturally requires more prerequisites than a standard introductory course in complex analysis. The reader should be familiar with basic real analysis and with some elementary tools from homological algebra. In terms of university coursework, I would recommend at least the level of a first course in complex analysis and a first course in abstract algebra. Some familiarity with category theory would also be helpful, although it is not strictly necessary.

Complex analysis is traditionally regarded as a branch of analysis. Nevertheless, its internal structure contains ideas of remarkable depth, and it is closely connected with analysis, topology, algebraic geometry, and number theory. A solid understanding of complex analysis is therefore valuable far beyond the subject itself. However, many undergraduate courses in complex analysis stop almost entirely at applications of the residue theorem to the evaluation of integrals. This was one of the main reasons I decided to write these notes.

The notes are based primarily on the material from the complex analysis course I took during my second year of undergraduate study. I also consulted Rick Miranda's *Algebraic Curves and Riemann Surfaces*, Robin Hartshorne's *Algebraic Geometry*, and my own notes on algebraic geometry.

The original motivation for these notes is quite simple: how should one understand multivalued functions on the complex plane? In attempting to answer this question, one is naturally led to Riemann surfaces and eventually to several tools from algebraic geometry. As a result, the reader will encounter a considerable amount of terminology that is usually associated with algebraic geometry. This should not be intimidating. I believe that, with sufficient patience and attention, the necessary language can be learned naturally as the theory develops.

Finally, GPT-5.6 also contributed to the production of these notes, especially by assisting with repetitive tasks involved in organizing, editing, and compiling the LaTeX source. This substantially reduced the amount of routine work and made it possible to complete the notes within a relatively short period of time.

Contents

Chapter 1

Holomorphic Functions and the Cauchy–Riemann Equations

1.1 Continuity and complex differentiability

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ and let $f : \Omega \to \mathbb{C}$. The function f is *continuous at* $z_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon$$

whenever $z \in \Omega$ and $|z - z_0| < \delta$.

Definition 1.2. Let $\Omega \subseteq \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is *complex differentiable at* $z_0 \in \Omega$ if the limit

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. It is *holomorphic on* Ω if it is complex differentiable at every point of Ω. We write

$$\mathscr{O}(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is holomorphic}\}.$$

A function holomorphic on all of $\mathbb{C}$ is called *entire*.

Proposition 1.3. *If f is complex differentiable at z_0, then f is continuous at z_0.*

Proof. For $z \neq z_0$ we may write

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0).$$

The quotient tends to $f'(z_0)$ as $z \to z_0$, hence is bounded in a neighborhood of z_0. Therefore the product tends to 0, proving continuity. □

1.2 The Cauchy–Riemann equations

Write

$$z = x + \mathrm{i}y, \qquad f(z) = u(x, y) + \mathrm{i}v(x, y),$$

where $u, v : \Omega \to \mathbb{R}$. We may regard f as the real map

$$F(x, y) = (u(x, y), v(x, y)).$$

Theorem 1.4 (Necessary Cauchy–Riemann equations). *Suppose that $f = u + \mathrm{i}v$ is complex differentiable at $z_0 = x_0 + \mathrm{i}y_0$ and that the relevant partial derivatives exist at (x_0, y_0). Then*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

Moreover,

$$f'(z_0) = u_x(z_0) + \mathrm{i}v_x(z_0) = v_y(z_0) - \mathrm{i}u_y(z_0).$$

Proof. Approaching 0 through real values h gives

$$f'(z_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = u_x(z_0) + \mathrm{i}v_x(z_0).$$

Approaching through purely imaginary increments $\mathrm{i}k$ gives

$$\begin{aligned} f'(z_0) &= \lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{\mathrm{i}k} \\ &= -\mathrm{i}\left(u_y(z_0) + \mathrm{i}v_y(z_0)\right) = v_y(z_0) - \mathrm{i}u_y(z_0). \end{aligned}$$

Equality of real and imaginary parts yields the two equations. □

Definition 1.5. The operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)$$

are called the *Wirtinger derivatives.*

Proposition 1.6. *If $f = u + \mathrm{i}v$ is C^1, then the Cauchy–Riemann equations are equivalent to*

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

When they hold,

$$f' = \frac{\partial f}{\partial z}.$$

Proof. A direct calculation gives

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[(u_x - v_y) + \mathrm{i}(v_x + u_y)\right].$$

Thus this derivative vanishes exactly when $u_x = v_y$ and $u_y = -v_x$. Under these equations,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[(u_x + v_y) + \mathrm{i}(v_x - u_y)\right] = u_x + \mathrm{i}v_x = f'.$$

□

Theorem 1.7 (Sufficient Cauchy–Riemann equations). *Let $f = u + \mathrm{i}v$ be defined on an open set Ω. Suppose that u and v are continuously differentiable and satisfy the Cauchy–Riemann equations throughout Ω. Then f is holomorphic on Ω.*

Proof. Fix $z = x + \mathrm{i}y$ and write $h = h_1 + \mathrm{i}h_2$. Real differentiability gives

$$\begin{aligned} u(x + h_1, y + h_2) - u(x, y) &= u_x h_1 + u_y h_2 + o(|h|), \\ v(x + h_1, y + h_2) - v(x, y) &= v_x h_1 + v_y h_2 + o(|h|). \end{aligned}$$

Using $v_x = -u_y$ and $v_y = u_x$, we obtain

$$f(z + h) - f(z) = (u_x + \mathrm{i}v_x)(h_1 + \mathrm{i}h_2) + o(|h|).$$

Hence

$$\frac{f(z + h) - f(z)}{h} = u_x + \mathrm{i}v_x + o(1),$$

so the complex derivative exists. □

1.3 The real differential and conformality

Proposition 1.8. *Let $f = u + \mathrm{i}v$ be holomorphic near z_0. The Jacobian matrix of the associated real map $F = (u, v)$ is*

$$DF(z_0) = \begin{pmatrix} \Re f'(z_0) & -\Im f'(z_0) \\ \Im f'(z_0) & \Re f'(z_0) \end{pmatrix},$$

and

$$\det DF(z_0) = \left|f'(z_0)\right|^2.$$

Proof. The Cauchy–Riemann equations give

$$DF = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}.$$

Since $f' = u_x + \mathrm{i}v_x$, the displayed matrix follows. Its determinant is $u_x^2 + v_x^2 = |f'|^2$. □

Corollary 1.9 (Local conformality). *If f is holomorphic near z_0 and $f'(z_0) \neq 0$, then the real differential of f at z_0 is the composition of a rotation and a positive dilation. In particular, f preserves oriented angles infinitesimally at z_0.*

Proof. Write $f'(z_0) = re^{\mathrm{i}\theta}$ with $r > 0$. Multiplication by this complex number is dilation by r followed by rotation through θ, and the previous proposition identifies this multiplication with the real differential. □

Corollary 1.10 (Holomorphic inverse function theorem). *If f is holomorphic near z_0 and $f'(z_0) \neq 0$, then there are neighborhoods U of z_0 and V of $f(z_0)$ such that $f : U \to V$ is bijective and its inverse is holomorphic. Moreover,*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}.$$

Proof. The preceding proposition shows that the real Jacobian determinant is nonzero at z_0. The real inverse function theorem gives a C^1 local inverse. Differentiating $f(f^{-1}(w)) = w$ and using the chain rule yields the displayed formula, which shows that the inverse is complex differentiable. □

1.4 Radius of convergence

Theorem 1.11 (Cauchy–Hadamard). *For a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

there exists $R \in [0, \infty]$ such that the series converges absolutely for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. Moreover,

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n},$$

with the conventions $1/0 = \infty$ and $1/\infty = 0$.

Proof. Set

$$L = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Suppose first that $0 < L < \infty$. If $|z - z_0| < 1/L$, choose $\varepsilon > 0$ such that

$$(L + \varepsilon)\,|z - z_0| < 1.$$

For all sufficiently large n,

$$|a_n|^{1/n} \leq L + \varepsilon,$$

so

$$|a_n(z - z_0)^n| \leq ((L + \varepsilon)\,|z - z_0|)^n\,.$$

The geometric majorant proves absolute convergence. If $|z - z_0| > 1/L$, choose $\varepsilon > 0$ with

$$(L - \varepsilon)\,|z - z_0| > 1.$$

Infinitely many n satisfy $|a_n|^{1/n} > L - \varepsilon$, so the corresponding terms fail to tend to zero. Thus the series diverges. The cases $L = 0$ and $L = \infty$ follow by the same argument. □

Proposition 1.12 (Uniform convergence on smaller discs)**.** *Let R be the radius of convergence of $\sum a_n(z - z_0)^n$. For every $0 < r < R$, the series converges uniformly and absolutely on the closed disc $|z - z_0| \leq r$.*

Proof. Choose ρ with $r < \rho < R$. Since the series converges absolutely at a point with modulus ρ, the numerical series $\sum |a_n|\,\rho^n$ converges. For $|z - z_0| \leq r$,

$$|a_n(z - z_0)^n| \leq |a_n|\,\rho^n.$$

The Weierstrass M-test gives uniform absolute convergence. □

1.5 Termwise differentiation

Theorem 1.13. *Inside its disc of convergence, a power series defines a holomorphic function and may be differentiated term by term:*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \Longrightarrow \quad f'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}.$$

The differentiated series has the same radius of convergence.

Proof. The Cauchy–Hadamard formula gives

$$\limsup_{n\to\infty} |n a_n|^{1/n} = \limsup_{n\to\infty} n^{1/n}\,|a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}\,,$$

so the two radii agree.

Fix $r < R$. Both the original series and the differentiated series converge uniformly on every smaller closed disc. Let

$$S_N(z) = \sum_{n=0}^{N} a_n(z - z_0)^n.$$

Then $S_N \to f$ uniformly and $S_N' \to g$ uniformly on $|z - z_0| \leq r$, where

$$g(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}.$$

For any line segment from w to z contained in the disc,

$$S_N(z) - S_N(w) = \int_{[w,z]} S_N'(\zeta)\,\mathrm{d}\zeta.$$

Passing to the limit under uniform convergence yields

$$f(z) - f(w) = \int_{[w,z]} g(\zeta)\,\mathrm{d}\zeta.$$

Dividing by $z - w$ and letting $z \to w$ proves $f'(w) = g(w)$. □

Corollary 1.14. *A power series may be differentiated or integrated term by term arbitrarily many times on every compact subset of its disc of convergence.*

Proof. Repeated application of the theorem proves the differentiation statement. For integration, define

$$F(z) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}.$$

The new series has the same radius, and termwise differentiation gives $F' = f$. □

Definition 1.15. A function f on an open set Ω is *analytic at* $z_0 \in \Omega$ if there is a power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

with positive radius of convergence that agrees with f in a neighborhood of z_0. It is *analytic on* Ω if it is analytic at every point.

Proposition 1.16 (Uniqueness of power-series coefficients)**.** *If*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

in a neighborhood of z_0, then $a_n = b_n$ for every n.

Proof. Termwise differentiation gives

$$f^{(n)}(z_0) = n!a_n = n!b_n.$$

Hence $a_n = b_n$. □

1.6 Curves and line integrals

Definition 1.17. A *piecewise C^1 curve* is a continuous map $\gamma : [a, b] \to \mathbb{C}$ that is C^1 on finitely many subintervals. If f is continuous on the image of γ, define

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

The reversed curve is denoted $-\gamma$, and concatenation is denoted $\gamma_1 * \gamma_2$.

Proposition 1.18. *Line integrals are invariant under orientation-preserving piecewise C^1 reparametrization, additive under concatenation, and change sign under reversal.*

Proof. Each statement follows from the real change-of-variables formula applied on the finitely many smooth subintervals. For example, if $\phi : [c, d] \to [a, b]$ is increasing and piecewise C^1, then

$$\int_c^d f(\gamma(\phi(s)))\gamma'(\phi(s))\phi'(s)\,\mathrm{d}s = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

The other assertions are immediate from splitting or reversing the parameter interval. □

Definition 1.19. Let f be continuous on a domain Ω. A function $F \in \mathscr{O}(\Omega)$ is a *primitive* of f if $F' = f$.

Theorem 1.20 (Fundamental theorem for complex line integrals)**.** *If F is a primitive of f on Ω and γ is a piecewise C^1 curve from w_0 to w_1, then*

$$\int_\gamma f(z)\,\mathrm{d}z = F(w_1) - F(w_0).$$

In particular, the integral of f around every closed curve is zero.

Proof. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t)) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t)$$

on every smooth subinterval. Integrating and summing over the pieces gives the result. □

1.7 Goursat's theorem

Theorem 1.21 (Goursat)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let T be a closed triangle whose interior and boundary lie in Ω. If $f \in \mathscr{O}(\Omega)$, then*

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0.$$

Proof. Subdivide T into four congruent triangles by joining the midpoints of its sides. The integrals over the interior edges cancel, so

$$\int_{\partial T} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \int_{\partial T_j} f(z)\,\mathrm{d}z.$$

At least one T_j therefore satisfies

$$\left|\int_{\partial T_j} f(z)\,\mathrm{d}z\right| \geq \frac{1}{4}\left|\int_{\partial T} f(z)\,\mathrm{d}z\right|.$$

Choose such a triangle and call it T_1. Iterating, we obtain nested closed triangles

$$T \supseteq T_1 \supseteq T_2 \supseteq \cdots$$

with

$$\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| \leq 4^n \left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right|,$$

and with $\operatorname{diam}(T_n) = 2^{-n}\operatorname{diam}(T)$. Compactness gives a unique point

$$z_0 \in \bigcap_{n\geq 0} T_n.$$

Since f is complex differentiable at z_0,

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \eta(z)(z-z_0), \qquad \eta(z) \to 0$$

as $z \to z_0$. The constant and linear terms have primitives, hence integrate to zero around ∂T_n. Therefore

$$\left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \leq \sup_{z\in T_n} |\eta(z)|\ \operatorname{diam}(T_n)\ \operatorname{length}(\partial T_n).$$

Both diameter and perimeter are multiplied by 2^{-n}, so

$$4^n \operatorname{diam}(T_n) \operatorname{length}(\partial T_n) = \operatorname{diam}(T) \operatorname{length}(\partial T).$$

Consequently,

$$\left| \int_{\partial T} f(z)\, \mathrm{d}z \right| \le \sup_{z \in T_n} |\eta(z)| \, \operatorname{diam}(T) \operatorname{length}(\partial T),$$

and the right-hand side tends to 0. Hence the integral vanishes. □

Corollary 1.22. *If R is a closed rectangle contained with its interior in Ω and $f \in \mathscr{O}(\Omega)$, then*

$$\int_{\partial R} f(z)\, \mathrm{d}z = 0.$$

Proof. Divide R along a diagonal into two triangles. The diagonal integrals cancel, and Goursat's theorem applies to each triangle. □

1.8 Primitives on convex domains

Theorem 1.23. *Every holomorphic function on a convex domain has a primitive.*

Proof. Let Ω be convex, fix $z_* \in \Omega$, and define

$$F(z) = \int_{[z_*, z]} f(\zeta)\, \mathrm{d}\zeta.$$

For $z, z+h \in \Omega$ with h small, the boundary of the triangle with vertices $z_*, z, z+h$ lies in Ω. By Goursat's theorem,

$$F(z+h) - F(z) = \int_{[z, z+h]} f(\zeta)\, \mathrm{d}\zeta.$$

Parametrizing the segment by $\zeta = z + th$ gives

$$\frac{F(z+h) - F(z)}{h} = \int_0^1 f(z+th)\, \mathrm{d}t.$$

Continuity of f implies that the right-hand side tends to $f(z)$ as $h \to 0$. Thus $F' = f$. □

Corollary 1.24 (Cauchy's theorem on convex domains)**.** *If f is holomorphic on a convex domain Ω, then*

$$\int_\gamma f(z)\, \mathrm{d}z = 0$$

for every closed piecewise C^1 curve γ in Ω.

Proof. By the theorem, f has a primitive. Apply the fundamental theorem for line integrals. □

Chapter 2

The Cauchy Integral Formula and Its Consequences

2.1 The integral formula

Theorem 2.1 (Cauchy integral formula for a disc)**.** *Let $\overline{D(z_0, R)} \subseteq \Omega$ and let $f \in \mathscr{O}(\Omega)$. If C is the positively oriented boundary circle of $D(z_0, R)$, then for every $z \in D(z_0, R)$,*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta.$$

Proof. Fix z in the disc and choose $\varepsilon > 0$ so small that $\overline{D(z, \varepsilon)} \subset D(z_0, R)$. Remove the small disc and connect its boundary to C by a thin corridor. The resulting region can be triangulated, so Goursat's theorem gives

$$\int_C \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta = \int_{|\zeta - z| = \varepsilon} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta,$$

where the small circle is positively oriented. Write

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z}.$$

The first quotient is bounded near z, so its integral over the small circle tends to 0 as $\varepsilon \to 0$. Parametrizing $\zeta = z + \varepsilon e^{\mathrm{i}\theta}$ gives

$$\int_{|\zeta - z| = \varepsilon} \frac{f(z)}{\zeta - z} \,\mathrm{d}\zeta = 2\pi \mathrm{i} f(z).$$

This proves the formula. □

Theorem 2.2 (Cauchy differentiation formula)**.** *Under the same hypotheses, for every $n \geq 0$ and every $z \in D(z_0, R)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \,\mathrm{d}\zeta.$$

Proof. The case $n = 0$ is the Cauchy integral formula. For z in a smaller concentric disc, the denominator is uniformly bounded away from zero for $\zeta \in C$. Therefore the integral may be differentiated with respect to z, yielding

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{(\zeta - z)^{n+1}} = \frac{n+1}{(\zeta - z)^{n+2}}.$$

Induction proves the formula. □

2.2 Holomorphic functions are analytic

Theorem 2.3. *Every holomorphic function is analytic. More precisely, if $\overline{D(z_0, R)} \subseteq \Omega$ and $f \in \mathscr{O}(\Omega)$, then for $|z - z_0| < R$,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = R} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta.$$

In particular,

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Proof. For $|z - z_0| < R$ and $|\zeta - z_0| = R$,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

The geometric series converges uniformly on the circle whenever z ranges in a smaller closed disc. We may therefore integrate term by term in the Cauchy formula, obtaining the expansion. The coefficient identity follows from the differentiation formula. □

Corollary 2.4. *A holomorphic function has derivatives of all orders, and those derivatives are holomorphic.*

Proof. Locally the function is represented by a power series, which may be differentiated term by term arbitrarily many times. □

Theorem 2.5 (Morera)**.** *Let f be continuous on a domain Ω. Suppose that*

$$\int_{\partial T} f(z) \, \mathrm{d}z = 0$$

for every closed triangle T whose interior and boundary lie in Ω. Then f is holomorphic on Ω.

Proof. Fix $z_0 \in \Omega$ and choose a disc D centered at z_0 with $\overline{D} \subseteq \Omega$. Since D is convex, define

$$F(z) = \int_{[z_0, z]} f(\zeta) \, \mathrm{d}\zeta.$$

The same triangular argument used for primitives on convex domains shows that $F' = f$ on D. Thus F is holomorphic, and so $f = F'$ is holomorphic on D. Since z_0 was arbitrary, f is holomorphic on Ω. □

2.3 Cauchy estimates, Liouville, and the fundamental theorem of algebra

Theorem 2.6 (Cauchy estimates)**.** *If f is holomorphic on a neighborhood of $\overline{D(z_0, R)}$ and*

$$M_R = \max_{|z - z_0| = R} |f(z)|,$$

then

$$\left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n}$$

for every $n \geq 0$.

Proof. By the Cauchy differentiation formula,

$$\left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi} \int_{|\zeta-z_0|=R} \frac{|f(\zeta)|}{R^{n+1}} \,|\,\mathrm{d}\zeta| \le \frac{n!}{2\pi} \frac{M_R}{R^{n+1}} (2\pi R) = \frac{n! M_R}{R^n}.$$

□

Corollary 2.7 (Liouville). *Every bounded entire function is constant.*

Proof. Suppose $|f| \le M$ on $\mathbb{C}$. For every $R > 0$, Cauchy's estimate gives

$$|f'(z_0)| \le \frac{M}{R}.$$

Letting $R \to \infty$ yields $f'(z_0) = 0$. Since z_0 is arbitrary, f' vanishes identically, and f is constant on the connected set $\mathbb{C}$. □

Corollary 2.8 (Fundamental theorem of algebra). *Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$. Consequently every degree-n polynomial factors into n linear factors over $\mathbb{C}$, counted with multiplicity.*

Proof. Let $P(z) = a_n z^n + \cdots + a_0$ with $n \ge 1$. If P had no zero, then $1/P$ would be entire. For large $|z|$,

$$|P(z)| \ge |a_n|\,|z|^n - \sum_{k=0}^{n-1} |a_k|\,|z|^k,$$

so $|P(z)| \to \infty$ as $|z| \to \infty$. Hence $1/P$ is bounded outside a large disc. It is also bounded on that disc by continuity and nonvanishing. Liouville's theorem would make $1/P$ constant, a contradiction. Thus P has a root w, and polynomial division gives $P(z) = (z - w)Q(z)$. Induction on the degree yields the full factorization. □

Proposition 2.9. *If f is holomorphic on a connected domain Ω and $f' = 0$, then f is constant.*

Proof. Every point has a disc neighborhood on which the power series of f has all coefficients of positive degree equal to zero. Thus f is locally constant. The level set $\{z \in \Omega : f(z) = f(z_0)\}$ is both open and closed in the connected set Ω, hence equals Ω. □

2.4 Zeros and their orders

Theorem 2.10 (Local factorization at a zero). *Let f be holomorphic on a connected domain Ω, let $z_0 \in \Omega$, and suppose f is not identically zero. If $f(z_0) = 0$, then there is a unique integer $m \ge 1$ and a holomorphic function g near z_0 such that*

$$f(z) = (z - z_0)^m g(z), \qquad g(z_0) \ne 0.$$

The integer m is called the order *or* multiplicity *of the zero.*

Proof. Expand f at z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Since f is not identically zero, some coefficient is nonzero. Let m be the least such index. Then

$$f(z) = (z - z_0)^m \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} = (z - z_0)^m g(z),$$

with $g(z_0) = a_m \ne 0$. Continuity makes g nonvanishing on a smaller neighborhood. If also $f = (z - z_0)^n h$ with $h(z_0) \ne 0$, comparison of the first nonzero Taylor coefficient gives $m = n$. □

Corollary 2.11. *The zeros of a nonzero holomorphic function are isolated.*

Proof. Near each zero z_0, the factor g in the theorem is nonvanishing, so z_0 is the only zero in a sufficiently small disc. □

Theorem 2.12 (Identity principle)**.** *Let Ω be connected and let $f \in \mathscr{O}(\Omega)$. If the zero set of f has a limit point in Ω, then $f \equiv 0$. Equivalently, if two holomorphic functions agree on a set with a limit point in Ω, they agree everywhere.*

Proof. Let

$$E = \{z \in \Omega : f \text{ vanishes on a neighborhood of } z\}\,.$$

The set E is open. Suppose $z_n \to z_0 \in \Omega$ and $f(z_n) = 0$ for all n. If f were not identically zero near z_0, the local factorization theorem would make z_0 an isolated zero, contradicting the existence of the sequence. Hence $z_0 \in E$, so E is also closed. It is nonempty by the assumed limit point. Connectedness gives $E = \Omega$. □

2.5 Locally uniform convergence

Definition 2.13. A sequence f_n on a domain Ω converges *locally uniformly* to f if every point of Ω has a neighborhood on which $f_n \to f$ uniformly.

Lemma 2.14. *For functions on a domain Ω, the following are equivalent:*

(i) *$f_n \to f$ locally uniformly;*

(ii) *$f_n \to f$ uniformly on every compact subset of Ω.*

Proof. Assume local uniform convergence and let $K \subseteq \Omega$ be compact. Choose finitely many neighborhoods $U_1, \ldots, U_m$ covering K on each of which convergence is uniform. Taking the maximum of the finitely many corresponding indices gives uniform convergence on K.

Conversely, fix $z_0 \in \Omega$ and choose $r > 0$ with $\overline{D(z_0, r)} \subseteq \Omega$. Uniform convergence on this compact disc implies uniform convergence on the smaller neighborhood $D(z_0, r)$. □

Theorem 2.15 (Weierstrass convergence theorem)**.** *If $f_n \in \mathscr{O}(\Omega)$ and $f_n \to f$ locally uniformly, then $f \in \mathscr{O}(\Omega)$. Moreover, for every $k \geq 1$,*

$$f_n^{(k)} \longrightarrow f^{(k)}$$

locally uniformly on Ω.

Proof. Let T be a triangle compactly contained in Ω. Uniform convergence on ∂T gives

$$\int_{\partial T} f(z)\,\mathrm{d}z = \lim_{n\to\infty} \int_{\partial T} f_n(z)\,\mathrm{d}z = 0.$$

Morera's theorem implies that f is holomorphic.

Fix a compact set $K \subseteq \Omega$. Choose $r > 0$ such that every point of K lies in a closed disc of radius r contained in some compact set $K' \subseteq \Omega$. For $z \in K$, Cauchy's formula on $|\zeta - z| = r$ yields

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \leq \frac{k!}{r^k} \sup_{\zeta \in K'} |f_n(\zeta) - f(\zeta)|\,.$$

The right-hand side tends to zero uniformly in $z \in K$. □

Theorem 2.16 (Hurwitz)**.** *Let f_n be holomorphic and nowhere zero on a connected domain Ω, and suppose $f_n \to f$ locally uniformly. Then either f is nowhere zero or $f \equiv 0$.*

Proof. Suppose f is not identically zero and has a zero at z_0. Choose a small closed disc $\overline{D} \subseteq \Omega$ centered at z_0 such that f has no zero on ∂D. Set

$$m = \min_{\partial D} |f| > 0.$$

For all sufficiently large n,

$$|f_n - f| < m \leq |f|$$

on ∂D. Rouché's theorem, proved in Chapter 3.3, implies that f_n and f have the same number of zeros in D, counted with multiplicity. Since f has at least one zero and f_n has none, this is impossible. Hence f has no zeros. □

Remark 2.17. The proof of Hurwitz uses Rouché's theorem, which appears later. The theorem is placed here because conceptually it belongs to locally uniform convergence; the argument may be read after Chapter 3.3.

2.6 Laurent expansions

Theorem 2.18 (Laurent expansion). *Let*

$$A = \{z \in \mathbb{C} : r < |z - z_0| < R\}, \qquad 0 \leq r < R \leq \infty,$$

and let $f \in \mathscr{O}(A)$. Then there is a unique doubly infinite series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

converging absolutely and locally uniformly on A. For every circle C_ρ with $r < \rho < R$,

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{C_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, \mathrm{d}\zeta.$$

Proof. Fix $z \in A$ and choose radii $r < \rho_1 < |z - z_0| < \rho_2 < R$. Apply the Cauchy formula to the annular region bounded by the two circles, obtaining

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{C_{\rho_2}} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{C_{\rho_1}} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta.$$

On the outer circle,

$$\frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}},$$

while on the inner circle,

$$\frac{1}{\zeta - z} = -\sum_{m=0}^{\infty} \frac{(\zeta - z_0)^m}{(z - z_0)^{m+1}}.$$

Both geometric series converge uniformly on the relevant circles, so termwise integration yields the Laurent expansion. The coefficient formula follows directly. Independence of ρ is a consequence of Cauchy's theorem on the annulus between two such circles.

For uniqueness, suppose a locally uniformly convergent Laurent series represents zero. Multiplying by $(z - z_0)^{-n-1}$ and integrating around a circle extracts the coefficient a_n, so every coefficient vanishes. □

Definition 2.19. The sum

$$\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

is the *principal part* of the Laurent expansion at z_0. The coefficient a_{-1} is the *residue* of f at z_0, denoted

$$\mathrm{Res}(f; z_0) = a_{-1}.$$

2.7 Classification of isolated singularities

Definition 2.20. A point z_0 is an *isolated singularity* of f if f is holomorphic on a punctured disc

$$0 < |z - z_0| < R.$$

It is *removable* if f extends holomorphically to z_0; it is a *pole* if $|f(z)| \to \infty$ as $z \to z_0$; otherwise it is an *essential singularity.*

Theorem 2.21 (Riemann removable singularity theorem)**.** *Let f be holomorphic on $0 < |z - z_0| < R$. If f is bounded near z_0, then z_0 is removable.*

Proof. Define

$$g(z) = \begin{cases} (z - z_0)^2 f(z), & z \neq z_0, \\ 0, & z = z_0. \end{cases}$$

Boundedness of f shows that g is complex differentiable at z_0 with $g'(z_0) = 0$, and it is holomorphic elsewhere. Hence g is holomorphic on the full disc. Since $g(z_0) = g'(z_0) = 0$, the local factorization theorem gives

$$g(z) = (z - z_0)^2 h(z)$$

with h holomorphic. On the punctured disc, $h = f$, so h is the required extension. □

Theorem 2.22 (Laurent classification)**.** *Let f have an isolated singularity at z_0 with Laurent expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Then:

(i) *z_0 is removable if and only if $a_n = 0$ for every $n < 0$;*

(ii) *z_0 is a pole if and only if only finitely many negative-power coefficients are nonzero;*

(iii) *z_0 is essential if and only if infinitely many negative-power coefficients are nonzero.*

Proof. If all negative coefficients vanish, the nonnegative part is a convergent power series at z_0, giving a holomorphic extension. Conversely, a holomorphic extension has an ordinary Taylor series, so uniqueness of Laurent coefficients forces the principal part to vanish.

If the principal part terminates at $a_{-m}(z - z_0)^{-m}$ with $a_{-m} \neq 0$, then

$$f(z) = (z - z_0)^{-m} g(z), \qquad g(z_0) = a_{-m} \neq 0,$$

so $|f(z)| \to \infty$. Conversely, if f has a pole, then $1/f$ extends holomorphically with a zero of some finite order m, which gives the same representation. The essential case is the remaining possibility. □

Proposition 2.23 (Order and residue at a pole)**.** *If f has a pole of order m at z_0, then*

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

for a holomorphic function g with $g(z_0) \neq 0$, and

$$\operatorname{Res}(f; z_0) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left((z - z_0)^m f(z)\right)\Big|_{z=z_0}.$$

Proof. The representation follows from the Laurent classification. Since $(z-z_0)^m f(z) = g(z)$, the coefficient of $(z-z_0)^{m-1}$ in the Taylor series of g is precisely the coefficient of $(z-z_0)^{-1}$ in f. That coefficient is $g^{(m-1)}(z_0)/(m-1)!$. □

Theorem 2.24 (Casorati–Weierstrass). *If z_0 is an essential singularity of f, then the image of every punctured neighborhood of z_0 is dense in $\mathbb{C}$.*

Proof. Suppose the image of some punctured disc omits an open disc $D(w,\varepsilon)$. Then

$$g(z) = \frac{1}{f(z)-w}$$

is holomorphic and bounded on that punctured disc. By the removable singularity theorem, g extends holomorphically to z_0.

If $g(z_0) \neq 0$, then $1/g + w$ gives a holomorphic extension of f, making the singularity removable. If $g(z_0) = 0$, then g has a zero of finite order, so $1/g$ has a pole and hence f has a pole. Both alternatives contradict essentiality. □

Definition 2.25. A function f on a domain Ω is *meromorphic* if it is holomorphic away from a discrete subset of Ω and has a pole at each point of that subset.

2.8 The residue theorem

Theorem 2.26 (Residue theorem for a simple closed contour). *Let γ be a positively oriented simple closed piecewise C^1 curve, and suppose f is holomorphic on a neighborhood of γ and its interior except for finitely many isolated singularities $z_1, \ldots, z_N$ inside γ. Then*

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{N} \operatorname{Res}(f; z_k).$$

Proof. Choose pairwise disjoint small circles C_k around the singularities, all contained inside γ. Remove the corresponding discs. Triangulating the remaining region and applying Goursat's theorem gives

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{N} \int_{C_k} f(z)\,\mathrm{d}z,$$

where each C_k is positively oriented. Integrating the Laurent expansion of f at z_k term by term shows that every term except the coefficient of $(z-z_k)^{-1}$ integrates to zero, while

$$\int_{C_k} \frac{\mathrm{d}z}{z-z_k} = 2\pi\mathrm{i}.$$

Thus the kth integral equals $2\pi\mathrm{i}\operatorname{Res}(f; z_k)$. □

Definition 2.27. If f is meromorphic near infinity, define its *residue at infinity* by

$$\operatorname{Res}(f;\infty) = -\operatorname{Res}\left(\frac{1}{w^2} f\left(\frac{1}{w}\right); 0\right).$$

Proposition 2.28. *If f is meromorphic on the Riemann sphere, then the sum of all residues, including the residue at infinity, is zero.*

Proof. Choose a large circle containing all finite poles. The residue theorem gives

$$\frac{1}{2\pi \mathrm{i}} \int_{|z|=R} f(z)\, \mathrm{d}z = \sum_{a \in \mathbb{C}} \operatorname{Res}(f; a).$$

Set $z = 1/w$. The circle becomes a small negatively oriented circle around 0, and

$$f(z)\, \mathrm{d}z = -\frac{1}{w^2} f\left(\frac{1}{w}\right) \mathrm{d}w.$$

By the definition of the residue at infinity, the left-hand side equals $-\operatorname{Res}(f; \infty)$. Rearranging proves the claim. ☐

2.9 Jordan's lemma and a classical integral

Lemma 2.29 (Jordan)**.** *Let $a > 0$, let C_R be the upper semicircle $z = Re^{\mathrm{i}\theta}$, $0 \le \theta \le \pi$, and let g be continuous on C_R. If*

$$M_R = \sup_{z \in C_R} |g(z)| \longrightarrow 0$$

as $R \to \infty$, then

$$\int_{C_R} g(z) e^{\mathrm{i}az}\, \mathrm{d}z \longrightarrow 0.$$

Proof. Since $\left| e^{\mathrm{i}aRe^{\mathrm{i}\theta}} \right| = e^{-aR\sin\theta}$,

$$\left| \int_{C_R} g(z) e^{\mathrm{i}az}\, \mathrm{d}z \right| \le M_R R \int_0^\pi e^{-aR\sin\theta}\, \mathrm{d}\theta.$$

By symmetry and the inequality $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi/2$,

$$\int_0^\pi e^{-aR\sin\theta}\, \mathrm{d}\theta \le 2 \int_0^{\pi/2} e^{-2aR\theta/\pi}\, \mathrm{d}\theta \le \frac{\pi}{aR}.$$

Hence the integral is bounded by $\pi M_R / a$, which tends to zero. ☐

Example 2.30. We compute

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, \mathrm{d}x = \pi.$$

Proof. Integrate $e^{\mathrm{i}z}/z$ over the upper semicircle of radius R, but indent the contour by a small upper semicircle of radius ε around 0. There are no poles inside the indented contour, so the total integral is zero. Jordan's lemma shows that the large arc tends to zero. On the small arc, oriented clockwise,

$$\int_{\text{small arc}} \frac{e^{\mathrm{i}z}}{z}\, \mathrm{d}z \longrightarrow -\mathrm{i}\pi.$$

Therefore

$$\text{p. v.} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}x}}{x}\, \mathrm{d}x = \mathrm{i}\pi.$$

Taking imaginary parts gives the stated integral. Since $\sin x / x$ is continuous at 0 after defining its value there to be 1, no principal value is needed for the imaginary part. ☐

Chapter 3

Topological Properties

3.1 Homotopy invariance

Definition 3.1. Two closed curves $\gamma_0, \gamma_1 : [0,1] \to \Omega$ are *homotopic in* Ω if there is a continuous map

$$H : [0,1] \times [0,1] \to \Omega$$

such that

$$H(0,t) = \gamma_0(t), \qquad H(1,t) = \gamma_1(t), \qquad H(s,0) = H(s,1).$$

For the integral theorem below, we call the homotopy *piecewise* C^1 if the square admits a finite rectangular subdivision on each piece of which H is C^1.

Theorem 3.2 (Homotopy invariance of holomorphic integrals)**.** *Let $f \in \mathscr{O}(\Omega)$ and let γ_0, γ_1 be piecewise C^1 closed curves joined by a piecewise C^1 homotopy in Ω. Then*

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z.$$

Proof. The compact image of the homotopy is covered by finitely many discs on which f has primitives. By uniform continuity of H, subdivide the square $[0,1]^2$ into sufficiently small rectangles so that the image of each rectangle lies in one such disc. The integral of f around the image of the boundary of each small rectangle is zero because f has a primitive there. Summing over all rectangles cancels the integrals along interior edges. The remaining boundary terms are precisely the integrals along γ_1 and $-\gamma_0$, proving equality. □

Corollary 3.3. *Every holomorphic function on a simply connected domain has a primitive.*

Proof. Fix $z_* \in \Omega$ and define

$$F(z) = \int_\gamma f(\zeta)\,\mathrm{d}\zeta,$$

where γ is any path from z_* to z. If γ_1 and γ_2 are two such paths, then $\gamma_1 * (-\gamma_2)$ is a closed loop. In a simply connected domain it is null-homotopic, so homotopy invariance gives equality of the two path integrals. Thus F is well-defined. A local line-segment argument shows $F' = f$. □

Proposition 3.4 (Primitive criterion)**.** *For a continuous function f on a domain Ω, the following are equivalent:*

(i) *f has a primitive on Ω;*

(ii) *$\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed piecewise C^1 curve γ in Ω;*

(iii) *the integral of f between two points is independent of the path.*

Proof. The implication (i)⇒(ii) follows from the fundamental theorem. If (ii) holds and γ_1, γ_2 have the same endpoints, then $\gamma_1 * (-\gamma_2)$ is closed, so their integrals agree; hence (ii)⇒(iii). Assuming (iii), fix z_* and define $F(z)$ by integrating from z_* to z. Path independence makes F well-defined, and a local line-segment argument gives $F' = f$. Thus (iii)⇒(i). □

3.2 Winding numbers

Definition 3.5. Let γ be a closed piecewise C^1 curve and let $a \notin \gamma([0,1])$. The *winding number* or *index* of γ about a is

$$\operatorname{Ind}(\gamma, a) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{d}z}{z-a}.$$

Proposition 3.6. *The winding number is an integer.*

Proof. Write $\gamma : [0,1] \to \mathbb{C} \setminus \{a\}$ and define

$$I(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a}\, \mathrm{d}s$$

piecewise on the smooth subintervals. Consider

$$G(t) = \frac{e^{I(t)}}{\gamma(t) - a}.$$

On each smooth subinterval,

$$G'(t) = 0,$$

so G is constant. Since $\gamma(1) = \gamma(0)$, we get $e^{I(1)} = 1$. Hence $I(1) = 2\pi \mathrm{i} n$ for some $n \in \mathbb{Z}$, proving the assertion. □

Proposition 3.7. *The function $a \mapsto \operatorname{Ind}(\gamma, a)$ is locally constant on $\mathbb{C} \setminus \gamma([0,1])$ and vanishes on the unbounded component.*

Proof. For a away from the compact image of γ, the integral depends continuously on a. Since it takes values in the discrete set $\mathbb{Z}$, it is locally constant. For $|a|$ sufficiently large,

$$\frac{1}{z-a} = -\frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$$

uniformly on the curve. Every term has a primitive as a function of z, so the integral is zero. □

Proposition 3.8 (Homotopy invariance of winding number)**.** *If two closed curves are homotopic in $\mathbb{C} \setminus \{a\}$, then they have the same winding number about a.*

Proof. Apply homotopy invariance of holomorphic integrals to the function $1/(z-a)$. □

3.3 Holomorphic logarithms and roots

Theorem 3.9 (Criterion for a holomorphic logarithm)**.** *Let f be holomorphic and nowhere zero on a domain Ω. The following are equivalent:*

(i) *there exists $g \in \mathscr{O}(\Omega)$ such that $e^g = f$;*

(ii) *f'/f has a primitive on Ω;*

(iii) *for every closed curve γ in Ω,*

$$\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = 0.$$

Proof. If $e^g = f$, then differentiation gives $g' = f'/f$, proving (i)⇒(ii). The primitive criterion gives (ii)⇔(iii).

Assume (ii), and let $G' = f'/f$. Then

$$\left(fe^{-G}\right)' = e^{-G}(f' - fG') = 0.$$

Thus $fe^{-G} = c \neq 0$ is constant. Choose any complex number c_0 with $e^{c_0} = c$. Then $g = G + c_0$ satisfies $e^g = f$. □

Corollary 3.10. *Every nowhere-zero holomorphic function on a simply connected domain admits a holomorphic logarithm.*

Proof. The function f'/f is holomorphic and therefore has a primitive on a simply connected domain. □

Corollary 3.11. *If f is nowhere zero and holomorphic on a simply connected domain, then for every $n \geq 1$ there exists a holomorphic function h such that $h^n = f$.*

Proof. Choose a holomorphic logarithm g of f and set $h = e^{g/n}$. □

Example 3.12 (Why $\log z$ does not exist globally on $\mathbb{C}^\times$)**.** There is no holomorphic function g on $\mathbb{C}^\times$ such that $e^{g(z)} = z$.

Proof. If such g existed, then $g'(z) = 1/z$. Integrating around the unit circle would give

$$0 = \int_{|z|=1} g'(z)\,\mathrm{d}z = \int_{|z|=1} \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i},$$

a contradiction. □

3.4 The argument principle

Theorem 3.13 (Argument principle)**.** *Let γ be a positively oriented simple closed contour, and let f be meromorphic on a neighborhood of γ and its interior, with no zeros or poles on γ. Then*

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = N - P,$$

where N and P are the numbers of zeros and poles inside γ, counted with multiplicity and order.

Proof. At a zero or pole a, write

$$f(z) = (z-a)^m h(z),$$

where h is holomorphic and nonvanishing, and where $m > 0$ for a zero and $m < 0$ for a pole. Then

$$\frac{f'}{f} = \frac{m}{z-a} + \frac{h'}{h},$$

so

$$\operatorname{Res}\left(\frac{f'}{f}; a\right) = m.$$

The residue theorem now sums these integers over all zeros and poles. □

Corollary 3.14. *Under the same hypotheses,*

$$\operatorname{Ind}(f \circ \gamma, 0) = N - P.$$

Proof. Parametrizing $f \circ \gamma$ and applying the chain rule gives

$$\int_{f\circ\gamma} \frac{\mathrm{d}w}{w} = \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z.$$

Now use the definition of winding number and the argument principle. □

3.5 Rouché's theorem

Theorem 3.15 (Rouché)**.** *Let γ be a positively oriented simple closed contour, and suppose f and g are holomorphic on a neighborhood of γ and its interior. If*

$$|g(z)| < |f(z)| \qquad (z \in \gamma),$$

then f and $f+g$ have the same number of zeros inside γ, counted with multiplicity.

Proof. For $0 \le t \le 1$, set $F_t = f + tg$. On γ,

$$|tg| < |f|\,,$$

so F_t never vanishes there. The integer

$$N(t) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{F_t'(z)}{F_t(z)}\,\mathrm{d}z$$

counts the zeros of F_t inside γ. The integrand depends continuously on t uniformly along the compact curve, so $N(t)$ is continuous. Since it is integer-valued, it is constant. Thus $N(0) = N(1)$. □

3.6 The local mapping theorem and open mapping theorem

Theorem 3.16 (Local mapping theorem)**.** *Let f be nonconstant and holomorphic near z_0, and let m be the order of the zero of $f(z) - f(z_0)$ at z_0. Then there exist neighborhoods U of z_0 and V of $f(z_0)$ such that every $w \in V \setminus \{f(z_0)\}$ has exactly m preimages in U, counted with multiplicity. For sufficiently small V, these preimages are distinct.*

Proof. Choose $r > 0$ so small that z_0 is the only zero of $f - f(z_0)$ in $\overline{D(z_0, r)}$ and f' has no zero there except possibly at z_0. Set

$$\delta = \min_{|z - z_0| = r} |f(z) - f(z_0)| > 0.$$

If $0 < |w - f(z_0)| < \delta$, then on the boundary circle

$$|w - f(z_0)| < |f(z) - f(z_0)|\,.$$

Rouché's theorem shows that $f(z) - w$ and $f(z) - f(z_0)$ have the same number m of zeros in the disc, counted with multiplicity. Any zero distinct from z_0 is simple because f' does not vanish there, so for $w \ne f(z_0)$ the m preimages are distinct. □

Corollary 3.17 (Open mapping theorem)**.** *A nonconstant holomorphic function maps open sets to open sets.*

Proof. Let U be open and $z_0 \in U$. Apply the local mapping theorem inside a small disc contained in U. It shows that a neighborhood of $f(z_0)$ is contained in $f(U)$. Hence $f(U)$ is open. □

3.7 Maximum and minimum principles

Theorem 3.18 (Maximum modulus principle). *Let f be holomorphic on a domain Ω. If $|f|$ has a local maximum at an interior point, then f is constant.*

Proof. If f were nonconstant, the open mapping theorem would imply that the image of every neighborhood of the point is open. An open neighborhood of $f(z_0)$ contains points of modulus strictly larger than $|f(z_0)|$, contradicting local maximality. □

Corollary 3.19. *Let Ω be bounded, let f be continuous on $\overline{\Omega}$ and holomorphic on Ω. Then*

$$\max_{\overline{\Omega}} |f| = \max_{\partial\Omega} |f|$$

unless f is constant.

Proof. Compactness gives a point where $|f|$ attains its maximum. If that point lies in the interior and f is nonconstant, the maximum modulus principle gives a contradiction. Hence a maximum occurs on the boundary. □

Corollary 3.20 (Minimum modulus principle). *If f is holomorphic and nowhere zero on a domain Ω, and if $|f|$ has a local minimum at an interior point, then f is constant.*

Proof. Apply the maximum modulus principle to $1/f$. □

3.8 Schwarz's lemma

Theorem 3.21 (Schwarz lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and suppose $f(0) = 0$. Then*

$$|f(z)| \leq |z| \qquad (z \in \mathbb{D}),$$

and

$$\left|f'(0)\right| \leq 1.$$

If equality holds at some nonzero point, or if $|f'(0)| = 1$, then

$$f(z) = e^{\mathrm{i}\theta} z$$

for some real θ.

Proof. Define

$$g(z) = \begin{cases} f(z)/z, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

The removable singularity theorem makes g holomorphic on $\mathbb{D}$. For $0 < r < 1$, the maximum modulus principle on $|z| \leq r$ gives

$$|g(z)| \leq \frac{1}{r} \qquad (|z| \leq r),$$

because $|f| < 1$ on $|z| = r$. Letting $r \to 1$ gives $|g| \leq 1$, hence $|f|(z) \leq |z|$ and $|f'(0)| \leq 1$.

If equality occurs, then $|g|$ attains its maximum 1 in the interior, so g is constant by the maximum modulus principle. Its constant value has modulus 1, yielding the stated form. □

Corollary 3.22 (Automorphisms of the unit disc). *Every biholomorphic self-map of $\mathbb{D}$ has the form*

$$\varphi(z) = e^{\mathrm{i}\theta} \frac{z - a}{1 - \bar{a}z}$$

for some $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$.

Proof. Let $a = \varphi^{-1}(0)$ and define

$$\psi_a(z) = \frac{z-a}{1-\bar{a}z}.$$

A direct calculation shows that ψ_a is an automorphism of $\mathbb{D}$ sending a to 0. Then

$$F = \varphi \circ \psi_a^{-1}$$

is an automorphism fixing 0. Schwarz's lemma applied to F and F^{-1} gives

$$|F(z)| = |z|.$$

The equality case yields $F(z) = e^{\mathrm{i}\theta} z$, and rearranging gives the formula. □

3.9 Harmonicity and harmonic conjugates

Definition 3.23. A twice continuously differentiable real-valued function u on a domain $\Omega \subseteq \mathbb{R}^2$ is *harmonic* if

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Proposition 3.24. *If $f = u + \mathrm{i}v$ is holomorphic, then both u and v are harmonic.*

Proof. Holomorphic functions are smooth because they are locally represented by power series. Differentiating the Cauchy–Riemann equations gives

$$u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy},$$

so $\Delta u = 0$. Similarly,

$$v_{xx} = -u_{yx}, \qquad v_{yy} = u_{xy},$$

so $\Delta v = 0$. □

Theorem 3.25 (Existence of harmonic conjugates)**.** *Let Ω be simply connected and let u be harmonic on Ω. Then there exists a harmonic function v such that $u + \mathrm{i}v$ is holomorphic. The function v is unique up to an additive real constant.*

Proof. Set

$$g = u_x - \mathrm{i}u_y.$$

Since u is harmonic,

$$\frac{\partial g}{\partial \bar{z}} = \frac{1}{2}(u_{xx} + u_{yy}) = 0,$$

so g is holomorphic. On the simply connected domain Ω, choose a primitive F of g. Writing $F = U + \mathrm{i}V$, we have

$$F' = U_x + \mathrm{i}V_x = u_x - \mathrm{i}u_y.$$

Thus $U_x = u_x$ and, by the Cauchy–Riemann equations for F, $U_y = -V_x = u_y$. Therefore $U - u$ has zero gradient and is constant. After subtracting that constant from F, we obtain $F = u + \mathrm{i}v$. If v_1, v_2 are two conjugates, then $\mathrm{i}(v_1 - v_2)$ is holomorphic with zero real part, hence constant; the difference is a real constant. □

3.10 Mean value and maximum principles

Theorem 3.26 (Mean value property)**.** *If u is harmonic on a neighborhood of $\overline{D(z_0, R)}$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{\mathrm{i}\theta})\, \mathrm{d}\theta.$$

Proof. The disc is simply connected, so u has a harmonic conjugate v there. Let $f = u + \mathrm{i}v$. Cauchy's integral formula at z_0 gives

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{\mathrm{i}\theta})\, \mathrm{d}\theta.$$

Taking real parts proves the formula. □

Theorem 3.27 (Maximum principle for harmonic functions)**.** *Let u be harmonic on a connected domain Ω. If u attains a local maximum or minimum at an interior point, then u is constant.*

Proof. Suppose u has a local maximum at z_0. Choose a small disc centered at z_0 on which $u(z) \leq u(z_0)$. The mean value property says that the average of the nonnegative function $u(z_0) - u$ on every sufficiently small boundary circle is zero. Therefore $u = u(z_0)$ throughout a neighborhood of z_0.

Now set $g = u_x - \mathrm{i}u_y$. Harmonicity gives $\partial g/\partial \bar{z} = 0$, so g is holomorphic on Ω. Since u is constant near z_0, the function g vanishes on a nonempty open set. The identity principle implies $g \equiv 0$ on the connected domain Ω. Hence $u_x = u_y = 0$ everywhere, so u is constant. The minimum case follows by applying the argument to $-u$. □

Corollary 3.28. *Let Ω be bounded, let u be continuous on $\overline{\Omega}$, and harmonic on Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \qquad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u,$$

unless u is constant.

Proof. Compactness gives points of maximum and minimum. If either lies in the interior and u is nonconstant, the harmonic maximum principle gives a contradiction. □

3.11 Regular points and natural boundaries

Definition 3.29. Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

have disc of convergence $D(z_0, R)$. A point $\zeta \in \partial D(z_0, R)$ is a *regular boundary point* if there exists a neighborhood U of ζ and a holomorphic function g on U such that

$$g = f \qquad \text{on } U \cap D(z_0, R).$$

Otherwise ζ is a *singular boundary point.* If every boundary point is singular, the circle $\partial D(z_0, R)$ is called a *natural boundary* of f.

Example 3.30. The geometric series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

has radius of convergence 1. Its only singular boundary point is $z = 1$; every other point of the unit circle is regular because $1/(1-z)$ is holomorphic there.

Proposition 3.31 (A lacunary series with a natural boundary)**.** *The function*

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

has the unit circle as a natural boundary.

Proof. The Cauchy–Hadamard formula gives radius of convergence 1. Let ζ be a root of unity of order 2^m. Then $\zeta^{2^n} = 1$ for all $n \geq m$, and for $0 < r < 1$,

$$f(r\zeta) = \sum_{n=0}^{m-1} r^{2^n} \zeta^{2^n} + \sum_{n=m}^{\infty} r^{2^n}.$$

The second sum tends to $+\infty$ as $r \to 1^-$. Indeed, for every $N > m$, its first $N - m$ terms tend to 1, so the liminf is at least $N - m$, and N is arbitrary. Hence f cannot extend holomorphically across such a point.

The roots of unity whose order is a power of 2 are dense in the unit circle. If f extended across any boundary point, it would extend to a neighborhood meeting an arc of the unit circle and would be bounded on a smaller compact neighborhood there. That arc contains a dyadic root of unity, contradicting the radial blow-up just proved. Thus every boundary point is singular. □

Chapter 4

Riemann Surfaces

The purpose of this chapter is to replace the fixed complex plane by a space which is locally indistinguishable from an open subset of $\mathbb{C}$. The familiar theorems of one-variable complex analysis then become local statements on such a space, while topology governs whether local objects can be assembled globally. This is the first step toward sheaves, divisors, cohomology, Serre duality, and the Riemann–Roch theorem.

Throughout this chapter, all topological spaces are assumed Hausdorff unless explicitly stated otherwise. A *domain* means a connected open set.

4.1 Complex atlases

Definition 4.1. Let X be a topological space. A *complex chart* on X is a pair (U, φ), where $U \subseteq X$ is open and

$$\varphi : U \longrightarrow \varphi(U) \subseteq \mathbb{C}$$

is a homeomorphism onto an open subset of $\mathbb{C}$. Two charts (U, φ) and (V, ψ) are *holomorphically compatible* if either $U \cap V = \varnothing$, or the transition map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is biholomorphic.

Definition 4.2. A *complex atlas* on X is a collection of pairwise compatible complex charts whose domains cover X. Two atlases are equivalent if their union is again a complex atlas. A complex atlas is *maximal* if it contains every chart compatible with all of its charts.

Proposition 4.3 (Maximal atlas)**.** *Every complex atlas $\mathcal{A}$ is contained in a unique maximal complex atlas.*

Proof. Let $\overline{\mathcal{A}}$ be the collection of all complex charts compatible with every chart in $\mathcal{A}$. Clearly $\mathcal{A} \subseteq \overline{\mathcal{A}}$. We first show that any two charts $(U, \varphi), (V, \psi) \in \overline{\mathcal{A}}$ are compatible.

Fix $p \in U \cap V$. Since $\mathcal{A}$ covers X, there is a chart $(W, \theta) \in \mathcal{A}$ with $p \in W$. On a neighborhood of $\varphi(p)$, the transition map between φ and ψ factors as

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1}).$$

Both factors are holomorphic because each of the two new charts is compatible with (W, θ). Thus $\psi \circ \varphi^{-1}$ is holomorphic near every point of its domain. Interchanging the two charts shows that its inverse is holomorphic. Hence the charts are compatible, so $\overline{\mathcal{A}}$ is an atlas.

By construction, every chart compatible with $\overline{\mathcal{A}}$ is compatible with $\mathcal{A}$, hence already belongs to $\overline{\mathcal{A}}$. Thus it is maximal. If $\mathcal{B}$ is another maximal atlas containing $\mathcal{A}$, every chart in $\mathcal{B}$ is compatible with $\mathcal{A}$, so $\mathcal{B} \subseteq \overline{\mathcal{A}}$. Maximality gives equality. $\square$

Definition 4.4. A *Riemann surface* is a Hausdorff, second-countable topological space equipped with a maximal complex atlas. Unless stated otherwise, a Riemann surface will be assumed connected.

Remark 4.5. Some authors include connectedness in the definition, while others do not. The Hausdorff condition guarantees uniqueness of limits. Second countability excludes pathologically large examples and implies, among other things, that every open cover admits a countable refinement. Both assumptions are standard in the theory of manifolds.

Proposition 4.6. *Every Riemann surface is locally path connected. Consequently, a connected Riemann surface is path connected.*

Proof. Each point has a chart neighborhood homeomorphic to an open subset of $\mathbb{C}$, and every open subset of $\mathbb{C}$ has a basis of open discs. Hence every point has a basis of path-connected neighborhoods, so the surface is locally path connected.

In any locally path-connected space, path components are open: if p belongs to a path component C, every sufficiently small path-connected neighborhood of p is contained in C. The complement of a path component is a union of the other path components and is therefore also open. Thus every path component is both open and closed. If the space is connected, it has only one path component. □

4.2 Holomorphic maps

Definition 4.7. Let X and Y be Riemann surfaces and let $F : X \to Y$ be continuous. The map F is *holomorphic at* $p \in X$ if there are charts (U, φ) at p and (V, ψ) at $F(p)$, with $F(U) \subseteq V$, such that

$$\psi \circ F \circ \varphi^{-1}$$

is holomorphic near $\varphi(p)$. It is holomorphic if it is holomorphic at every point. A bijective holomorphic map with holomorphic inverse is a *biholomorphism.*

Proposition 4.8 (Coordinate independence)**.** *Holomorphicity of a map between Riemann surfaces does not depend on the chosen charts.*

Proof. Suppose $(U, \varphi), (V, \psi)$ are charts in which the coordinate representation

$$g = \psi \circ F \circ \varphi^{-1}$$

is holomorphic. Let $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ be other charts at p and $F(p)$. After restricting all chart domains, the new coordinate representation is

$$\widetilde{\psi} \circ F \circ \widetilde{\varphi}^{-1} = (\widetilde{\psi} \circ \psi^{-1}) \circ g \circ (\varphi \circ \widetilde{\varphi}^{-1}).$$

The outer maps are holomorphic transition maps, so the composition is holomorphic. □

Proposition 4.9 (Chain rule)**.** *The composition of holomorphic maps between Riemann surfaces is holomorphic. Moreover, in local coordinates its derivative satisfies the ordinary complex chain rule.*

Proof. Let $F : X \to Y$ and $G : Y \to Z$ be holomorphic. Choose charts φ, ψ, θ around $p, F(p), G(F(p))$. In coordinates,

$$\theta \circ (G \circ F) \circ \varphi^{-1} = (\theta \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}).$$

The right-hand side is a composition of holomorphic functions between plane domains, hence is holomorphic and satisfies the usual chain rule. □

Definition 4.10. Let $F : X \to Y$ be holomorphic and let $p \in X$. In charts $z = \varphi(q)$ and $w = \psi(y)$, the complex derivative of the coordinate representation at $z(p)$ will be denoted by $F'_{\psi,\varphi}(p)$. Its numerical value depends on the coordinates, but the condition

$$F'_{\psi,\varphi}(p) = 0$$

does not.

Proposition 4.11. *The vanishing or nonvanishing of the derivative of a holomorphic map at a point is independent of local coordinates.*

Proof. Under a change of coordinates, the chain rule gives

$$(\widetilde{\psi} \circ F \circ \widetilde{\varphi}^{-1})' = (\widetilde{\psi} \circ \psi^{-1})'(\psi \circ F \circ \varphi^{-1})'(\varphi \circ \widetilde{\varphi}^{-1})'.$$

The derivatives of the transition maps are nonzero because the transition maps are biholomorphic. Therefore the middle derivative vanishes in one pair of coordinates exactly when it vanishes in every pair. □

Proposition 4.12 (Local biholomorphism criterion)**.** *Let $F : X \to Y$ be holomorphic. If the derivative of F at p is nonzero, then F restricts to a biholomorphism from a neighborhood of p onto a neighborhood of $F(p)$.*

Proof. Choose local coordinates around p and $F(p)$. The coordinate representation has nonzero derivative at the corresponding point. The holomorphic inverse function theorem from Chapter 1 gives a biholomorphic inverse on sufficiently small coordinate neighborhoods. Transporting this inverse through the charts proves the assertion. □

Theorem 4.13 (Identity principle)**.** *Let X be connected and let $f, g \in \mathscr{O}(X)$. If the set*

$$\{p \in X : f(p) = g(p)\}$$

has an accumulation point in X, then $f = g$ on X. In particular, a holomorphic function which is constant on a nonempty open set is constant everywhere.

Proof. Set $h = f - g$. In a chart around an accumulation point, the ordinary identity theorem implies that h vanishes on a nonempty open subset of X. Let A be the set of points at which the germ of h is zero. By definition, A is open and nonempty.

We show that its complement is open. If $q \notin A$ and $h(q) \neq 0$, continuity gives a neighborhood disjoint from A. If $h(q) = 0$, then in a local coordinate the zero of h at q has finite order, because the local representative is not identically zero. Thus h is nonzero on a punctured neighborhood of q, and that neighborhood contains no point of A. Hence $X \setminus A$ is open. Connectedness gives $A = X$, so $h = 0$ everywhere. □

4.3 Basic examples

4.3.1 Plane domains and the Riemann sphere

Every open subset of $\mathbb{C}$ is a Riemann surface with the identity chart on each connected component. The first genuinely global example is obtained by adding a point at infinity.

Definition 4.14. The *Riemann sphere* is the one-point compactification

$$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}.$$

It has two chart domains

$$U_0 = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}, \qquad U_\infty = \mathbb{P}^1(\mathbb{C}) \setminus \{0\},$$

with coordinates

$$z : U_0 \longrightarrow \mathbb{C}, \qquad \zeta : U_\infty \longrightarrow \mathbb{C},$$

given by

$$z(a) = a, \qquad \zeta(a) = \frac{1}{a} \quad (a \in \mathbb{C}^\times), \qquad \zeta(\infty) = 0.$$

Proposition 4.15. *The two charts above define a Riemann surface structure on* $\mathbb{P}^1(\mathbb{C})$*, and* $\mathbb{P}^1(\mathbb{C})$ *is compact.*

Proof. On the overlap $U_0 \cap U_\infty = \mathbb{C}^\times$, the transition map is

$$\zeta \circ z^{-1}(w) = \frac{1}{w},$$

which is biholomorphic on $\mathbb{C}^\times$. Thus the charts form a complex atlas. The one-point compactification of the locally compact Hausdorff second-countable space $\mathbb{C}$ is Hausdorff and second countable, so this atlas defines a Riemann surface.

For compactness, let $\mathcal{U}$ be an open cover of $\mathbb{P}^1(\mathbb{C})$. Choose $U \in \mathcal{U}$ containing ∞. By the definition of the one-point compactification, $\mathbb{P}^1(\mathbb{C}) \setminus U$ is a compact subset of $\mathbb{C}$. It is covered by finitely many members of $\mathcal{U}$; adding U gives a finite subcover of $\mathbb{P}^1(\mathbb{C})$. $\square$

Example 4.16 (Mobius transformations)**.** For $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, the formula

$$T(z) = \frac{az + b}{cz + d}$$

extends in the usual way to a biholomorphism of $\mathbb{P}^1(\mathbb{C})$. Its inverse is represented by the inverse matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

Matrices differing by a nonzero scalar define the same transformation, so these automorphisms form the projective linear group $\mathrm{PGL}_2(\mathbb{C})$.

4.3.2 Complex tori

Definition 4.17. A *lattice* in $\mathbb{C}$ is a subgroup

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$$

where ω_1, ω_2 are linearly independent over $\mathbb{R}$. The quotient topological space $\mathbb{C}/\Lambda$ is called a *complex torus.*

Lemma 4.18. *A lattice* $\Lambda \subseteq \mathbb{C}$ *is discrete. More precisely, there exists* $r > 0$ *such that*

$$D(0, 2r) \cap \Lambda = \{0\}.$$

Proof. The real-linear map

$$T : \mathbb{R}^2 \longrightarrow \mathbb{C}, \qquad T(x, y) = x\omega_1 + y\omega_2,$$

is an isomorphism of finite-dimensional real normed spaces. Hence there is a constant $c > 0$ such that

$$|T(x, y)| \geq c\sqrt{x^2 + y^2}.$$

For every nonzero $(m, n) \in \mathbb{Z}^2$, we therefore have

$$|m\omega_1 + n\omega_2| \geq c.$$

Taking $r < c/2$ gives the assertion. $\square$

Theorem 4.19. *Let $\Lambda \subseteq \mathbb{C}$ be a lattice and let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the quotient map. There is a unique Riemann surface structure on $\mathbb{C}/\Lambda$ for which π is locally biholomorphic. With this structure, $\mathbb{C}/\Lambda$ is compact.*

Proof. Choose $r > 0$ as in the preceding lemma. If $a \in \mathbb{C}$, then the translates

$$D(a,r) + \lambda, \qquad \lambda \in \Lambda,$$

are pairwise disjoint. Therefore π restricts to a homeomorphism

$$\pi|_{D(a,r)} : D(a,r) \longrightarrow \pi(D(a,r)).$$

Use its inverse as a chart on $\pi(D(a,r))$. If two such chart domains overlap, the corresponding coordinate maps differ on each connected component of the overlap by a translation $z \mapsto z + \lambda$, hence the transition map is holomorphic. These charts define a complex atlas.

The quotient is Hausdorff because the translation action of the discrete group Λ is properly discontinuous: two points not in the same orbit have sufficiently small discs whose lattice translates remain disjoint. It is second countable because the images under π of a countable basis of discs in $\mathbb{C}$ form a countable basis of the quotient. By construction, π is locally biholomorphic. Uniqueness follows because any complex structure with this property must contain precisely these local inverse charts.

Let

$$P = \{s\omega_1 + t\omega_2 : 0 \leq s, t \leq 1\}.$$

The set P is compact and $\pi(P) = \mathbb{C}/\Lambda$. Since a continuous image of a compact space is compact, the quotient is compact. □

4.3.3 Regular plane curves

Theorem 4.20 (Holomorphic implicit function theorem in two variables)**.** *Let F be holomorphic on a neighborhood of $(a,b) \in \mathbb{C}^2$, suppose $F(a,b) = 0$, and assume*

$$\frac{\partial F}{\partial w}(a,b) \neq 0.$$

Then there exist neighborhoods U of a and V of b, and a unique holomorphic function $g : U \to V$, such that

$$F(z,w) = 0 \text{ in } U \times V \quad \iff \quad w = g(z).$$

Proof. Consider the holomorphic map

$$\Phi(z,w) = (z, F(z,w)).$$

Its complex Jacobian determinant at (a,b) equals $F_w(a,b)$, which is nonzero. The real Jacobian determinant is therefore $|F_w(a,b)|^2 \neq 0$. By the real inverse function theorem, after shrinking neighborhoods, Φ is a real C^1-diffeomorphism. Its inverse is holomorphic: differentiating $\Phi \circ \Phi^{-1} = \mathrm{id}$ shows that the differential of the inverse is complex linear, and the Cauchy–Riemann criterion applies. Write

$$\Phi^{-1}(z,\eta) = (z, G(z,\eta)).$$

Then $g(z) = G(z,0)$ is holomorphic and parametrizes the zero set. Uniqueness follows from injectivity of Φ on the chosen neighborhood. □

Corollary 4.21. *Let F be holomorphic on a domain in $\mathbb{C}^2$, and suppose that at every point of the zero set*

$$X = \{(z,w) : F(z,w) = 0\}$$

at least one of F_z, F_w is nonzero. Then X, with the subspace topology, carries a natural Riemann surface structure.

Proof. At a point where $F_w \neq 0$, projection onto the z-coordinate identifies a neighborhood in X with an open subset of $\mathbb{C}$ by the implicit function theorem. At a point where $F_z \neq 0$, projection onto the w-coordinate does the same. On overlaps, the transition maps are restrictions of holomorphic coordinate functions and are therefore holomorphic. The subspace is Hausdorff and second countable because it is a subspace of $\mathbb{C}^2$. □

Example 4.22. If $\lambda \in \mathbb{C} \setminus \{0,1\}$, the affine curve

$$y^2 = x(x-1)(x-\lambda)$$

is nonsingular: a common zero of the defining equation and both partial derivatives would force $y = 0$ and a repeated root of $x(x-1)(x-\lambda)$, which does not occur. Thus its affine zero set is a Riemann surface. Its compactification will later give a basic example of a genus-one curve.

4.4 Holomorphic and meromorphic functions

Definition 4.23. A *holomorphic function* on an open set $U \subseteq X$ is a holomorphic map $f : U \to \mathbb{C}$. A function is *meromorphic* on U if it is locally a quotient g/h of holomorphic functions with h not identically zero on any connected component.

Proposition 4.24 (Order is coordinate independent)**.** *Let f be a nonzero meromorphic function near $p \in X$. If z and ζ are two local coordinates centered at p, then the order of the corresponding meromorphic functions at 0 is the same. Thus $\mathrm{ord}_p(f) \in \mathbb{Z}$ is well defined.*

Proof. The transition map has the form

$$\zeta = \alpha(z) = zu(z), \qquad u(0) \neq 0,$$

because $\alpha(0) = 0$ and $\alpha'(0) \neq 0$. If in the z-coordinate

$$f(z) = z^m v(z), \qquad v(0) \neq 0,$$

with $m \in \mathbb{Z}$, then $z = \zeta\, q(\zeta)$ for a holomorphic nonvanishing function q. Hence

$$f = \zeta^m q(\zeta)^m v(z(\zeta)),$$

and the factor multiplying ζ^m is holomorphic and nonvanishing. Therefore the order remains m. □

Theorem 4.25 (Meromorphic functions as sphere-valued maps)**.** *Let X be a Riemann surface. Meromorphic functions on X are naturally equivalent to holomorphic maps*

$$F : X \longrightarrow \mathbb{P}^1(\mathbb{C})$$

which are not identically ∞ on any connected component.

Proof. Let f be meromorphic. Away from its poles, regard it as a map to the affine chart $\mathbb{C} \subseteq \mathbb{P}^1(\mathbb{C})$. Near a pole p, the reciprocal $1/f$ is holomorphic and vanishes at p. In the coordinate $\zeta = 1/w$ near $\infty \in \mathbb{P}^1(\mathbb{C})$, the map is represented by $1/f$, so extending $f(p) = \infty$ makes the map holomorphic at p.

Conversely, let $F : X \to \mathbb{P}^1(\mathbb{C})$ be holomorphic and not identically ∞. On the inverse image of the affine chart, it is an ordinary holomorphic function. If $F(p) = \infty$, then in the coordinate $\zeta = 1/w$ near infinity, the function $g = \zeta \circ F$ is holomorphic and satisfies $g(p) = 0$. Since F is not identically infinity, g is not identically zero, and locally $F = 1/g$. Thus F is meromorphic. The two constructions are inverse to each other. □

Corollary 4.26. *The poles and zeros of a nonzero meromorphic function on a Riemann surface are isolated.*

Proof. In a local coordinate, a meromorphic function is an ordinary meromorphic function of one complex variable. The factorization theorem for zeros and poles shows that each zero or pole is isolated unless the function vanishes identically. A nonzero meromorphic function cannot vanish identically on a nonempty open set without vanishing identically on the connected surface. □

4.5 The local form of a holomorphic map

Theorem 4.27 (Local normal form)**.** *Let $F : X \to Y$ be a nonconstant holomorphic map of connected Riemann surfaces and let $p \in X$. There exist local coordinates z centered at p and w centered at $F(p)$ such that*

$$w \circ F = z^e$$

for a unique integer $e \geq 1$.

Proof. Choose arbitrary centered coordinates z at p and w at $F(p)$. The coordinate representation

$$g = w \circ F \circ z^{-1}$$

is holomorphic, satisfies $g(0) = 0$, and is not identically zero. Let $e \geq 1$ be the order of its zero at 0. Then

$$g(\xi) = \xi^e h(\xi), \qquad h(0) \neq 0.$$

After shrinking the coordinate disc, h has a holomorphic logarithm and hence a holomorphic e-th root u, with $u^e = h$. Define a new source coordinate

$$\widetilde{z} = \xi u(\xi).$$

Its derivative at 0 is $u(0) \neq 0$, so it is a valid local coordinate by the inverse function theorem. In this coordinate,

$$g(\xi) = \widetilde{z}^e.$$

To prove uniqueness, observe that e is the order of the zero of any coordinate representation of $F - F(p)$, and the preceding coordinate-independence argument shows that this order is invariant under changes of source and target coordinates. □

Definition 4.28. The integer $e = e_p(F)$ in the local normal form is the *local degree* or *ramification index* of F at p. The point p is a *ramification point* if $e_p(F) > 1$, and $F(p)$ is then a *branch value.*

Proposition 4.29. *For a nonconstant holomorphic map $F : X \to Y$, the following are equivalent at a point $p \in X$:*

(i) $e_p(F) = 1$;

(ii) *the derivative of F at p is nonzero;*

(iii) *F is locally biholomorphic at p.*

Proof. In the local normal form, F is represented by $z \mapsto z^e$. Its derivative at 0 is nonzero exactly when $e = 1$. If $e = 1$, the map is locally the identity in suitable coordinates and hence locally biholomorphic. Conversely, a local biholomorphism has nonzero derivative by the chain rule applied to its inverse. □

Corollary 4.30 (Open mapping theorem on Riemann surfaces)**.** *A nonconstant holomorphic map between connected Riemann surfaces is open.*

Proof. In suitable local coordinates, the map is $z \mapsto z^e$. This map sends every sufficiently small disc centered at the origin onto a disc centered at the origin. Therefore the original map sends a neighborhood of each point to a neighborhood of its image, and hence is open. □

Corollary 4.31. *Every fiber of a nonconstant holomorphic map between connected Riemann surfaces is discrete.*

Proof. Near a point p in the fiber over y, the local normal form is $z \mapsto z^{e_p}$. In a sufficiently small source neighborhood, the equation $z^{e_p} = 0$ has only the solution $z = 0$. Thus every point of the fiber is isolated in the fiber. □

Proposition 4.32. *The ramification points of a nonconstant holomorphic map are discrete.*

Proof. In local coordinates the ramification points are precisely the zeros of the derivative of the coordinate representation. Since the map is nonconstant, that derivative is not identically zero on any coordinate neighborhood. Its zeros are therefore isolated. Hence the ramification set is locally discrete, and thus discrete. □

4.6 Germs and analytic continuation

Definition 4.33. Let X be a Riemann surface and $p \in X$. Two holomorphic functions $f \in \mathscr{O}(U)$ and $g \in \mathscr{O}(V)$, with $p \in U \cap V$, define the same *germ at p* if they agree on some neighborhood of p. The germ of f at p is denoted $[f]_p$, and the set of all such germs is denoted $\mathscr{O}_{X,p}$.

Definition 4.34. Let $\gamma : [0,1] \to X$ be a path. An *analytic continuation* of a germ $\mathfrak{f}_0 \in \mathscr{O}_{X,\gamma(0)}$ along γ consists of a subdivision

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

and holomorphic functions $f_j \in \mathscr{O}(U_j)$, where $\gamma([t_{j-1}, t_j]) \subseteq U_j$, such that $[f_1]_{\gamma(0)} = \mathfrak{f}_0$ and consecutive functions agree near $\gamma(t_j)$. The resulting germ $[f_n]_{\gamma(1)}$ is called the *terminal germ.*

Proposition 4.35 (Uniqueness along a fixed path)**.** *If a germ admits two analytic continuations along the same path, then the two continuations have the same germ at every parameter value, in particular the same terminal germ.*

Proof. Refine the two subdivisions to a common subdivision. Let $A \subseteq [0,1]$ be the set of parameters t for which the two continuations determine the same germ at $\gamma(t)$. The set A is nonempty because it contains 0. It is open: equality of two germs means equality on a neighborhood, and continuity of γ keeps nearby path points in that neighborhood. It is closed: if $t_n \in A$ and $t_n \to t$, choose representatives of the two continuations on a common connected neighborhood of $\gamma(t)$ valid for path parameters near t. They agree near $\gamma(t_n)$ for large n, hence agree on the whole connected overlap by the identity theorem; in particular their germs at $\gamma(t)$ agree. Thus $A = [0,1]$. □

Lemma 4.36 (Homotopy invariance of analytic continuation)**.** *Let $H : [0,1]^2 \to X$ be a homotopy of paths with fixed endpoints:*

$$H(s,0) = p, \qquad H(s,1) = q.$$

Suppose a germ at p can be analytically continued along every path $t \mapsto H(s,t)$. Then the terminal germ at q is independent of s.

Proof. Fix $s_0 \in [0,1]$. Choose an analytic continuation along $\gamma_{s_0}(t) = H(s_0, t)$, represented by finitely many holomorphic functions on open sets $U_1, \dots, U_n$ along a subdivision. Shrinking the parameter intervals slightly if necessary, compactness of each path segment and continuity of H imply that for all s sufficiently close to s_0, the corresponding segment of γ_s remains in the same U_j. The same functions therefore give an analytic continuation along γ_s, with the same terminal representative near q. By uniqueness along a fixed path, this agrees with any analytic continuation along γ_s. Hence the terminal germ is locally constant as a function of s. Since $[0,1]$ is connected, it is constant. □

Theorem 4.37 (Monodromy theorem)**.** *Let X be a simply connected Riemann surface, let $p_0 \in X$, and let $\mathfrak{f}_0 \in \mathscr{O}_{X,p_0}$. Suppose $\mathfrak{f}_0$ admits analytic continuation along every path beginning at p_0. Then there exists a unique holomorphic function $F \in \mathscr{O}(X)$ whose germ at p_0 is $\mathfrak{f}_0$ and whose germ at any point is obtained by analytic continuation from $\mathfrak{f}_0$.*

Proof. For $q \in X$, choose a path from p_0 to q and define $F(q)$ to be the value at q of the terminal germ. Since X is simply connected, any two such paths are homotopic relative to their endpoints. The homotopy invariance lemma shows that the terminal germ, and hence its value, is independent of the chosen path.

To see that F is holomorphic, fix $q \in X$ and choose one continuation ending in a representative $f \in \mathscr{O}(U)$ near q. For any q' in a sufficiently small connected neighborhood $V \subseteq U$ of q, extend the chosen path to q by a path inside V from q to q'. The resulting terminal germ is represented by f, so $F = f$ on V. Hence F is holomorphic. Uniqueness follows from the identity theorem, or directly from the prescribed germs. □

Example 4.38 (The logarithm)**.** A germ of $\log z$ at $1 \in \mathbb{C}^\times$ can be analytically continued along every path in $\mathbb{C}^\times$, but continuation once around the unit circle changes its value by $2\pi\mathrm{i}$. Thus the continuation depends on the homotopy class of the path and does not define a single-valued holomorphic function on $\mathbb{C}^\times$. On any simply connected domain avoiding 0, the monodromy theorem produces a holomorphic branch of the logarithm.

Example 4.39 (Roots)**.** A germ of $z^{1/n}$ at 1 can be analytically continued along every path in $\mathbb{C}^\times$. Going once around the origin multiplies the terminal value by $e^{2\pi\mathrm{i}/n}$. This is the simplest finite monodromy phenomenon.

4.7 Covering maps

Definition 4.40. A continuous surjection $\pi : E \to B$ is a *covering map* if every point $b \in B$ has an open neighborhood U such that

$$\pi^{-1}(U) = \coprod_{\alpha \in A} V_\alpha,$$

where each V_α is open in E and $\pi|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Such a neighborhood is *evenly covered*. If E and B are Riemann surfaces and the local homeomorphisms are biholomorphic, π is a *holomorphic covering map*.

Theorem 4.41 (Path lifting)**.** *Let $\pi : E \to B$ be a covering map, let $\gamma : [0,1] \to B$ be a path, and choose $e_0 \in E$ with $\pi(e_0) = \gamma(0)$. Then there exists a unique path $\widetilde{\gamma} : [0,1] \to E$ satisfying*

$$\widetilde{\gamma}(0) = e_0, \qquad \pi \circ \widetilde{\gamma} = \gamma.$$

Proof. The compact set $\gamma([0,1])$ is covered by evenly covered neighborhoods. By the Lebesgue number lemma, there is a subdivision

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that each path segment $\gamma([t_{j-1}, t_j])$ lies in one evenly covered neighborhood U_j. Starting with e_0, choose the unique sheet V_1 over U_1 containing e_0, and define

$$\widetilde{\gamma}(t) = (\pi|_{V_1})^{-1}(\gamma(t))$$

on the first interval. Inductively, the endpoint already constructed lies in a unique sheet over U_j, which determines the lift on the next interval. This constructs a continuous lift.

For uniqueness, two lifts agreeing at a time t_0 must, on a small interval mapped into an evenly covered neighborhood, lie in the same sheet and equal the same local inverse of π. Hence the set of times at which they agree is nonempty, open, and closed in $[0,1]$, and therefore is all of $[0,1]$. ☐

Theorem 4.42 (Homotopy lifting)**.** *Let $\pi : E \to B$ be a covering map and let $H : [0,1]^2 \to B$ be continuous. Suppose a lift $\widetilde{H}(0,t)$ of the path $H(0,t)$ has been chosen. Then there is a unique continuous lift $\widetilde{H} : [0,1]^2 \to E$ extending it.*

Proof. Cover the compact square by finitely many small rectangles whose images under H lie in evenly covered neighborhoods. Choose a rectangular grid refining this cover. Starting from the lifted left edge, lift successively across adjacent rectangles using the inverse on the unique sheet containing the already lifted common edge. On overlaps, uniqueness of path lifting guarantees that the local constructions agree. Thus they patch to a continuous lift on the whole square. Uniqueness follows by applying the same open-and-closed argument to the set of points where two lifts agree, or rectangle by rectangle from uniqueness of local inverses. ☐

Proposition 4.43. *The exponential map*

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$$

is a holomorphic covering map. Its deck transformations are exactly

$$z \longmapsto z + 2\pi\mathrm{i}k, \qquad k \in \mathbb{Z}.$$

Proof. Fix $w_0 \in \mathbb{C}^\times$. Choose a small simply connected disc U around w_0 avoiding 0, and choose a holomorphic branch L of the logarithm on U. Then

$$\exp^{-1}(U) = \coprod_{k \in \mathbb{Z}} \big(L(U) + 2\pi\mathrm{i}k\big),$$

and exp restricts biholomorphically from each component onto U, with inverse $L + 2\pi\mathrm{i}k$. Thus exp is a holomorphic covering.

A deck transformation T satisfies $e^{T(z)} = e^z$, so $T(z) - z \in 2\pi\mathrm{i}\mathbb{Z}$. The left-hand side is continuous and $\mathbb{C}$ is connected, hence it is constant. Therefore $T(z) = z + 2\pi\mathrm{i}k$. Conversely, every such translation is a deck transformation. ☐

Proposition 4.44. *For a lattice* Λ*, the quotient map*

$$\pi : \mathbb{C} \longrightarrow \mathbb{C}/\Lambda$$

is a holomorphic covering map. Its deck transformations are precisely the translations $z \mapsto z + \lambda$*,* $\lambda \in \Lambda$*.*

Proof. The local charts used to define the complex structure on $\mathbb{C}/\Lambda$ show directly that sufficiently small quotient neighborhoods are evenly covered and that each sheet maps biholomorphically onto the base neighborhood. Hence π is a holomorphic covering.

Every lattice translation is plainly a deck transformation. Conversely, if T is a deck transformation, then $T(z) - z \in \Lambda$ for all z. Since Λ is discrete and $T(z) - z$ is continuous on connected $\mathbb{C}$, it is constant, so T is translation by an element of Λ. □

Remark 4.45. The exponential covering turns the multivalued logarithm on $\mathbb{C}^\times$ into the ordinary global coordinate on its covering surface $\mathbb{C}$. Similarly, elliptic functions are precisely meromorphic functions on $\mathbb{C}$ invariant under the deck group Λ, and hence descend to meromorphic functions on the torus $\mathbb{C}/\Lambda$.

4.8 Compact Riemann surfaces

Theorem 4.46. *Every holomorphic function on a compact connected Riemann surface is constant.*

Proof. Let $f : X \to \mathbb{C}$ be holomorphic. Since X is compact, $|f|$ attains a maximum at some point p. In a coordinate neighborhood of p, the ordinary maximum modulus principle implies that f is constant near p. The set on which f equals this constant is both open and closed by the identity theorem. Since X is connected, f is constant on all of X. □

Theorem 4.47. *Let* $F : X \to Y$ *be a nonconstant holomorphic map, where* X *is compact and* Y *is connected. Then* F *is surjective and every fiber is finite.*

Proof. By the open mapping theorem, $F(X)$ is open in Y. It is also compact, hence closed because Y is Hausdorff. Since Y is connected and $F(X) \neq \varnothing$, we have $F(X) = Y$.

Every fiber is discrete by the local normal form. A discrete closed subset of a compact space is finite: otherwise it would have an accumulation point by compactness, contradicting discreteness. Thus all fibers are finite. □

Theorem 4.48 (Degree of a holomorphic map)**.** *Let* $F : X \to Y$ *be a nonconstant holomorphic map between compact connected Riemann surfaces. For* $y \in Y$*, define*

$$d_y(F) = \sum_{p \in F^{-1}(y)} e_p(F).$$

Then $d_y(F)$ *is independent of* y*. Its common value is the* degree $\deg F$ *of* F*.*

Proof. Fix $y_0 \in Y$, and write

$$F^{-1}(y_0) = \{p_1, \ldots, p_r\}, \qquad e_i = e_{p_i}(F).$$

Choose pairwise disjoint coordinate discs U_i around the p_i, and a coordinate disc V around y_0, so that in suitable coordinates the restriction on U_i is $z \mapsto z^{e_i}$. After shrinking V, every $y \in V \setminus \{y_0\}$ has exactly e_i preimages in U_i, counted with multiplicity; this follows directly from the equation $z^{e_i} = w(y)$.

It remains to exclude additional preimages outside the U_i. The compact set

$$K = X \setminus \bigcup_{i=1}^{r} U_i'$$

obtained by taking slightly smaller coordinate discs $U_i' \Subset U_i$ does not meet the fiber of y_0. Hence $y_0 \notin F(K)$. Since $F(K)$ is compact and therefore closed, we may shrink V so that $V \cap F(K) = \varnothing$. Thus every preimage of a point of V lies in one of the U_i, and

$$d_y(F) = e_1 + \cdots + e_r = d_{y_0}(F)$$

for all $y \in V$. Therefore $d_y(F)$ is locally constant on connected Y, and hence constant. □

Corollary 4.49. *Let f be a nonconstant meromorphic function on a compact connected Riemann surface X. The number of zeros of f, counted with multiplicity, equals the number of poles, counted with multiplicity. Both numbers equal the degree of the map $f : X \to \mathbb{P}^1(\mathbb{C})$.*

Proof. Under the sphere-valued interpretation, the zeros of f form the fiber over 0, and the local degree at a zero is its order. The poles form the fiber over ∞, and the local degree at a pole is its pole order. The degree theorem says that the weighted cardinalities of these two fibers are equal. □

Corollary 4.50. *A holomorphic map of degree one between compact connected Riemann surfaces is a biholomorphism.*

Proof. If $\deg F = 1$, every fiber consists of a single point of local degree one. Hence F is bijective and locally biholomorphic everywhere. The local holomorphic inverses patch to a global holomorphic inverse. □

Proposition 4.51. *Let $F : X \to Y$ be a nonconstant holomorphic map between compact connected Riemann surfaces. Away from the finitely many branch values, every fiber consists of exactly $\deg F$ distinct points, and the restriction*

$$F : X \setminus F^{-1}(B) \longrightarrow Y \setminus B$$

is a holomorphic covering map, where B is the set of branch values.

Proof. The ramification set is discrete and closed in compact X, hence finite; therefore its image B is finite. If $y \notin B$, all points in the fiber have local degree one, so the degree formula implies that the fiber has exactly $\deg F$ distinct points. Around each point of such a fiber, F is biholomorphic. Compactness allows the target neighborhood to be chosen small enough that no additional sheets appear, exactly as in the proof of the degree theorem. Thus the complement of the branch locus is evenly covered. □

4.8.1 Rational functions on the sphere

Theorem 4.52. *Every meromorphic function on $\mathbb{P}^1(\mathbb{C})$ is a rational function.*

Proof. Let f be meromorphic on $\mathbb{P}^1(\mathbb{C})$. Its poles form a discrete subset of the compact space $\mathbb{P}^1(\mathbb{C})$, hence are finite. List the finite poles as $a_1, \ldots, a_r$. For each a_j, let $P_j(z)$ be the principal part of the Laurent expansion of f at a_j. Then

$$g(z) = f(z) - \sum_{j=1}^{r} P_j(z)$$

is entire on $\mathbb{C}$. Since f is meromorphic at infinity and each P_j is holomorphic at infinity, the function g is meromorphic at infinity. Thus for some $N \geq 0$, the function $g(z)/z^N$ is bounded for sufficiently large z. Cauchy's estimates applied on circles of radius R show that $g^{(n)}(0) = 0$ for every $n > N$: indeed,

$$\left|g^{(n)}(0)\right| \leq \frac{n!}{R^n} \max_{|z|=R} |g(z)| \leq Cn!R^{N-n},$$

and the right-hand side tends to zero as $R \to \infty$. Hence g is a polynomial. Since every principal part P_j is rational, so is f. ☐

Proposition 4.53. *Let*

$$F(z) = \frac{P(z)}{Q(z)}$$

be a nonconstant rational map $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$*, where* $P, Q \in \mathbb{C}[z]$ *are coprime. Then*

$$\deg F = \max\{\deg P, \deg Q\}.$$

Proof. Set $d = \max\{\deg P, \deg Q\}$. Choose a finite value $w \in \mathbb{C}$ such that the leading terms of $P - wQ$ do not cancel. Then $P - wQ$ has degree d. Its roots, counted with multiplicity, are exactly the finite points of the fiber $F^{-1}(w)$. Indeed, coprimality ensures that a root of $P - wQ$ is not a root of Q, and the multiplicity of that root equals the local degree of F there. By the fundamental theorem of algebra, the sum of these multiplicities is d. The degree theorem therefore gives $\deg F = d$. ☐

Corollary 4.54. *Every holomorphic map* $F : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ *is represented by a rational function. Moreover,*

$$\mathrm{Aut}(\mathbb{P}^1(\mathbb{C})) \cong \mathrm{PGL}_2(\mathbb{C}).$$

Proof. A holomorphic map to $\mathbb{P}^1(\mathbb{C})$ is a meromorphic function, so the theorem shows that it is rational. If it is an automorphism, it has degree one because the fiber of every point consists of one point of local degree one. Writing it as P/Q in lowest terms, the proposition gives

$$\max\{\deg P, \deg Q\} = 1.$$

Thus

$$F(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$. Conversely, every such Mobius transformation is a biholomorphism of the sphere. Two matrices define the same transformation exactly when they differ by a nonzero scalar, yielding $\mathrm{PGL}_2(\mathbb{C})$. ☐

4.9 The structure sheaf: a preview

The passage from plane domains to Riemann surfaces forces us to keep track of holomorphic functions on every open subset at once. This naturally leads to the language of sheaves.

Definition 4.55. For a Riemann surface X, define

$$\mathscr{O}_X(U) = \mathscr{O}(U)$$

for every open set $U \subseteq X$, with the usual restriction maps. This assignment is called the *structure sheaf* of X.

Proposition 4.56 (Locality and gluing). *Let $U = \bigcup_{i \in I} U_i$ be an open cover of an open subset of a Riemann surface.*

(i) *If $f, g \in \mathscr{O}_X(U)$ and $f|_{U_i} = g|_{U_i}$ for every i, then $f = g$.*

(ii) *If $f_i \in \mathscr{O}_X(U_i)$ and*

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for all i, j, then there exists a unique $f \in \mathscr{O}_X(U)$ with $f|_{U_i} = f_i$.

Proof. The first assertion is immediate because the U_i cover U. For the second, define $f(p) = f_i(p)$ whenever $p \in U_i$. The compatibility condition makes this definition independent of i. Near every point, f agrees with one of the holomorphic functions f_i, so it is holomorphic. Uniqueness follows from the first assertion. $\square$

Remark 4.57. The local logarithms on $\mathbb{C}^\times$ do not contradict the gluing property: on overlaps, different branches need not agree. Their differences are locally constant multiples of $2\pi \mathrm{i}$. Sheaf cohomology will measure precisely the obstruction to modifying local branches so that they do agree.

Chapter 5

Sheaves on Riemann Surfaces

The preceding chapter replaced plane domains by spaces which are locally modelled on open subsets of $\mathbb{C}$. This change makes the local-to-global problem unavoidable. A holomorphic function, a branch of a logarithm, or a meromorphic function may be defined on many small open sets, yet the local objects need not combine to form a global object. Sheaves provide the natural language for organizing such local data.

The abstract definitions in the first half of this chapter apply to an arbitrary topological space. Beginning with Section 5.3, we specialize to a Riemann surface X. No theory of general complex analytic spaces is needed: the only analytic input is the notion of a holomorphic or meromorphic function on an open subset of a Riemann surface.

5.1 Presheaves and sheaves

5.1.1 Presheaves and restriction maps

Definition 5.1. Let X be a topological space. A *presheaf of sets* $\mathscr{F}$ on X consists of the following data:

(i) for every open subset $U \subseteq X$, a set $\mathscr{F}(U)$;

(ii) for every inclusion $U \subseteq V$ of open subsets, a restriction map

$$\rho_{V,U} : \mathscr{F}(V) \longrightarrow \mathscr{F}(U),$$

such that

$$\rho_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$$

and

$$\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V} \qquad (U \subseteq V \subseteq W).$$

The elements of $\mathscr{F}(U)$ are called *sections of* $\mathscr{F}$ *over* U. If $s \in \mathscr{F}(V)$, we usually write $s|_U$ for $\rho_{V,U}(s)$.

Let $\mathrm{Open}(X)$ denote the category whose objects are the open subsets of X and whose morphisms are inclusions. Equivalently, a presheaf of sets is a contravariant functor

$$\mathscr{F} : \mathrm{Open}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

Replacing **Set** by **Ab**, **Ring**, or another category gives presheaves with additional algebraic structure.

Example 5.2 (Functions)**.** For a topological space X, the assignment

$$U \longmapsto C^0(U, \mathbb{C})$$

with the usual restriction maps is a presheaf of rings. If X is a Riemann surface, the assignment

$$U \longmapsto \mathscr{O}_X(U) := \{f : U \to \mathbb{C} : f \text{ is holomorphic}\}$$

is also a presheaf of rings.

5.1.2 The sheaf axiom

Definition 5.3. A presheaf $\mathscr{F}$ on X is a *sheaf* if, for every open subset $U \subseteq X$ and every open cover $U = \bigcup_{i \in I} U_i$, the following conditions hold:

(i) *Locality.* If $s, t \in \mathscr{F}(U)$ satisfy

$$s|_{U_i} = t|_{U_i} \qquad \text{for every } i,$$

then $s = t$.

(ii) *Gluing.* If sections $s_i \in \mathscr{F}(U_i)$ satisfy

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \qquad \text{for every } i, j,$$

then there is a section $s \in \mathscr{F}(U)$ such that

$$s|_{U_i} = s_i \qquad \text{for every } i.$$

The section obtained by gluing is unique by locality.

Remark 5.4. For a sheaf of sets, $\mathscr{F}(\varnothing)$ is a singleton. For a sheaf of abelian groups, rings, or modules, it is the zero object.

Proposition 5.5. *The presheaf $\mathscr{O}_X$ of holomorphic functions on a Riemann surface is a sheaf of $\mathbb{C}$-algebras.*

Proof. Let $U = \bigcup_i U_i$. If two holomorphic functions on U agree on every U_i, they agree pointwise on U, proving locality. Suppose now that $f_i \in \mathscr{O}_X(U_i)$ agree on all overlaps. Define

$$f(p) = f_i(p) \qquad (p \in U_i).$$

The compatibility condition makes this definition independent of the chosen index. Every point of U has a neighborhood on which f equals one of the holomorphic functions f_i; hence f is holomorphic. The algebraic operations are defined pointwise and commute with restriction. □

Example 5.6 (A presheaf which is not a sheaf)**.** Let A be a set and assign A to every nonempty open subset of X, with identity restriction maps. If $U = U_1 \sqcup U_2$ is disconnected, one may choose different elements of A on U_1 and U_2, but these sections cannot be glued in the constant presheaf. Its sheafification will be the sheaf of locally constant A-valued functions.

5.1.3 The equalizer form of the sheaf axiom

For presheaves of abelian groups, locality and gluing can be combined into one exactness statement.

Proposition 5.7. *A presheaf $\mathscr{F}$ of abelian groups is a sheaf if and only if, for every open cover $U = \bigcup_{i \in I} U_i$, the sequence*

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\rho} \prod_{i \in I} \mathscr{F}(U_i) \xrightarrow{\delta} \prod_{(i,j) \in I \times I} \mathscr{F}(U_i \cap U_j)$$

is exact, where

$$\rho(s) = (s|_{U_i})_i$$

and

$$\delta((s_i)_i) = \left(s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}\right)_{i,j}.$$

Proof. The injectivity of ρ is exactly the locality axiom. The equality $\ker \delta = \operatorname{im} \rho$ says that a family of sections lies in the image of a global section precisely when its members agree on every overlap. This is exactly the gluing axiom together with the uniqueness furnished by locality. $\square$

Remark 5.8. Proposition 5.7 is the first indication of the Čech complex: compatibility on pairwise intersections is expressed as the vanishing of a coboundary.

5.2 Germs and stalks

5.2.1 The stalk of a presheaf

Definition 5.9. Let $\mathscr{F}$ be a presheaf on X and let $x \in X$. The *stalk* of $\mathscr{F}$ at x is the direct limit

$$\mathscr{F}_x := \varinjlim_{x \in U} \mathscr{F}(U),$$

where U ranges over the open neighborhoods of x, ordered by reverse inclusion.

Concretely, an element of $\mathscr{F}_x$ is represented by a pair (U, s) with $x \in U$ and $s \in \mathscr{F}(U)$. Two pairs (U, s) and (V, t) represent the same element if there is a neighborhood $W \subseteq U \cap V$ of x on which

$$s|_W = t|_W.$$

The equivalence class of (U, s) is denoted by s_x and is called the *germ of s at x*.

Proposition 5.10 (Sections are determined by their germs)**.** *Let $\mathscr{F}$ be a sheaf on X, let $U \subseteq X$ be open, and let $s, t \in \mathscr{F}(U)$. If*

$$s_x = t_x \qquad \text{for every } x \in U,$$

then $s = t$.

Proof. For each $x \in U$, equality of the germs gives a neighborhood $U_x \subseteq U$ on which s and t agree. The sets U_x cover U, so locality gives $s = t$. $\square$

5.2.2 Morphisms and induced maps on stalks

Definition 5.11. A *morphism of presheaves* $\varphi : \mathscr{F} \to \mathscr{G}$ is a family of maps

$$\varphi_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$$

which commutes with every restriction map. If the presheaves have algebraic structure, the maps are required to preserve that structure.

For each $x \in X$, a morphism induces a map

$$\varphi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x, \qquad \varphi_x(s_x) = \big(\varphi_U(s)\big)_x.$$

It is well-defined because φ commutes with restrictions.

Proposition 5.12 (Stalk criterion for isomorphisms)**.** *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of sets. Then φ is an isomorphism if and only if*

$$\varphi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x$$

is a bijection for every $x \in X$.

Proof. An isomorphism plainly induces bijections on stalks. Conversely, suppose all stalk maps are bijective. Let $t \in \mathscr{G}(U)$. For every $x \in U$, surjectivity of φ_x gives a neighborhood $U_x \subseteq U$ and a section $s_x \in \mathscr{F}(U_x)$ whose image has the same germ as t at x. After shrinking U_x, we may assume

$$\varphi(s_x) = t|_{U_x}.$$

On an overlap $U_x \cap U_y$, the germs of s_x and s_y have the same image at every point. Injectivity of the stalk maps and the preceding proposition imply that the two sections agree on the overlap. Hence the s_x glue to a section $s \in \mathscr{F}(U)$ with $\varphi(s) = t$, proving surjectivity on sections. If $s, t \in \mathscr{F}(U)$ have the same image, then their germs have the same image at every point; stalkwise injectivity implies $s_x = t_x$, so $s = t$. Thus every φ_U is bijective and the sectionwise inverses form a morphism of sheaves. □

5.3 Basic analytic sheaves

Throughout this section, X is a Riemann surface.

5.3.1 Holomorphic, invertible, and meromorphic functions

The structure sheaf $\mathscr{O}_X$ has already appeared. Its group of units on an open subset U is

$$\mathscr{O}_X^\times(U) := \{f \in \mathscr{O}_X(U) : f(p) \neq 0 \text{ for every } p \in U\}.$$

With pointwise multiplication, $\mathscr{O}_X^\times$ is a sheaf of abelian groups.

Let $\mathscr{M}_X(U)$ be the set of meromorphic functions on U, with the usual restriction maps. If U is disconnected, a meromorphic function is understood componentwise and is not allowed to be identically ∞ on a component. The locality of meromorphicity immediately implies that $\mathscr{M}_X$ is a sheaf of fields in the local sense: each stalk is a field, although $\mathscr{M}_X(U)$ need not be a field when U is disconnected.

5.3.2 The local ring of holomorphic germs

Proposition 5.13. *For every $p \in X$, the stalk $\mathscr{O}_{X,p}$ is a local $\mathbb{C}$-algebra. Its unique maximal ideal is*

$$\mathfrak{m}_p = \{f_p \in \mathscr{O}_{X,p} : f(p) = 0\},$$

and evaluation induces an isomorphism

$$\mathscr{O}_{X,p}/\mathfrak{m}_p \cong \mathbb{C}.$$

Proof. Evaluation at p defines a surjective homomorphism

$$\mathrm{ev}_p : \mathscr{O}_{X,p} \longrightarrow \mathbb{C}, \qquad f_p \longmapsto f(p),$$

whose kernel is $\mathfrak{m}_p$. Thus $\mathfrak{m}_p$ is maximal and the quotient is $\mathbb{C}$. If $f_p \notin \mathfrak{m}_p$, then some representative f satisfies $f(p) \neq 0$. After shrinking its domain, f is nowhere zero, so $1/f$ is holomorphic and f_p is a unit. Conversely, a germ vanishing at p cannot be a unit. Hence the nonunits are exactly $\mathfrak{m}_p$, which is therefore the unique maximal ideal. □

Proposition 5.14. *Choose a local coordinate z centered at p. Composition with the coordinate identifies $\mathscr{O}_{X,p}$ with the ring $\mathbb{C}\{z\}$ of convergent power-series germs at 0. In particular, every nonzero germ has a unique expression*

$$f_p = z^n u, \qquad n \geq 0, \qquad u \in \mathscr{O}_{X,p}^{\times}.$$

Proof. A holomorphic germ at p becomes, in the coordinate z, a holomorphic germ at 0, and Taylor's theorem identifies the latter with a convergent power series. This correspondence respects sums and products and has an inverse obtained by evaluating the convergent series in the coordinate.

For a nonzero germ, let n be the first index with nonzero Taylor coefficient. Factoring out z^n leaves a convergent power series with nonzero constant term, hence a unit. The integer n is unique because it is the order of the zero, and then the unit is uniquely determined. □

Corollary 5.15. *For every $p \in X$, there is a natural isomorphism*

$$\mathscr{M}_{X,p} \cong \mathrm{Frac}(\mathscr{O}_{X,p}).$$

Every nonzero meromorphic germ can be written uniquely as

$$f_p = z^n u, \qquad n \in \mathbb{Z}, \qquad u \in \mathscr{O}_{X,p}^{\times}.$$

Proof. A quotient of two holomorphic germs with nonzero denominator defines a meromorphic germ. Conversely, by the local description of a meromorphic function, every meromorphic germ is such a quotient. This proves the first assertion. Applying Proposition 5.14 to a numerator and denominator and cancelling powers of z gives the displayed form. Its exponent is unique because the order of a product is the sum of the orders, and a unit has order zero. □

Remark 5.16. The integer n in Corollary 5.15 is $\mathrm{ord}_p(f)$. This local invariant will later be used to define divisors.

5.3.3 Locally constant and skyscraper sheaves

Definition 5.17. Let A be a set, group, or ring, equipped with the discrete topology. The *constant sheaf* $\underline{A}_X$ is defined by

$$\underline{A}_X(U) = \{s : U \to A : s \text{ is locally constant}\}.$$

Proposition 5.18. *The sheaf $\underline{A}_X$ is the sheafification of the constant presheaf with value A. Moreover,*

$$(\underline{A}_X)_p \cong A$$

for every $p \in X$.

Proof. A section of the sheafification is locally represented by a section of the constant presheaf, so it is precisely a locally constant function. Conversely, a locally constant function is locally represented by constant sections. The stalk at p is determined by the value of a locally constant function near p, and every element of A occurs as the germ of a constant function. Thus evaluation gives the stated isomorphism. □

Definition 5.19. Let $p \in X$ and let A be an abelian group. The *skyscraper sheaf at p with value A*, denoted A_p, is defined by

$$A_p(U) = \begin{cases} A, & p \in U, \\ 0, & p \notin U. \end{cases}$$

The restriction maps are the identity whenever both open sets contain p, and are the zero map otherwise.

Proposition 5.20. *The assignment A_p is a sheaf, and its stalks are*

$$(A_p)_q \cong \begin{cases} A, & q = p, \\ 0, & q \neq p. \end{cases}$$

Proof. The sheaf axioms follow directly from the fact that an open cover of an open set containing p has at least one member containing p, and compatibility forces every member containing p to carry the same element of A. At p, all sufficiently small neighborhoods have section group A with identity restrictions, so the stalk is A. If $q \neq p$, the Hausdorff property gives a neighborhood of q not containing p; hence every germ at q is zero. □

5.4 Sheafification

5.4.1 The universal property

Definition 5.21. Let $\mathscr{P}$ be a presheaf on X. A *sheafification* of $\mathscr{P}$ is a sheaf $\mathscr{P}^+$ together with a morphism

$$\eta : \mathscr{P} \longrightarrow \mathscr{P}^+$$

such that, for every sheaf $\mathscr{G}$ and every morphism $\alpha : \mathscr{P} \to \mathscr{G}$, there is a unique morphism $\widetilde{\alpha} : \mathscr{P}^+ \to \mathscr{G}$ satisfying

$$\alpha = \widetilde{\alpha} \circ \eta.$$

Thus sheafification is left adjoint to the inclusion of sheaves into presheaves.

5.4.2 Construction by locally representable germs

For an open subset $U \subseteq X$, let $\mathscr{P}^+(U)$ be the set of maps

$$\sigma : U \longrightarrow \bigsqcup_{x \in U} \mathscr{P}_x$$

such that $\sigma(x) \in \mathscr{P}_x$ for every x, and such that σ is locally represented by sections of $\mathscr{P}$: for every $x \in U$, there are a neighborhood $V \subseteq U$ of x and a section $s \in \mathscr{P}(V)$ satisfying

$$\sigma(y) = s_y \qquad (y \in V).$$

Restrictions are obtained by restricting the domain of σ.

Theorem 5.22 (Existence of sheafification)**.** *The presheaf $\mathscr{P}^+$ is a sheaf. The maps*

$$\eta_U : \mathscr{P}(U) \longrightarrow \mathscr{P}^+(U), \qquad s \longmapsto (x \mapsto s_x),$$

define a sheafification of $\mathscr{P}$. Moreover,

$$\eta_x : \mathscr{P}_x \xrightarrow{\sim} (\mathscr{P}^+)_x$$

is an isomorphism for every $x \in X$.

Proof. Since elements of $\mathscr{P}^+(U)$ are functions, locality is immediate. Compatible locally representable functions glue as functions, and the glued function is locally representable because this property can be checked near each point. Hence $\mathscr{P}^+$ is a sheaf.

The maps η_U commute with restrictions. To prove stalkwise surjectivity, let a germ in $(\mathscr{P}^+)_x$ be represented by $\sigma \in \mathscr{P}^+(U)$. Near x, the function σ is represented by some $s \in \mathscr{P}(V)$, so the germ of σ equals $\eta_x(s_x)$. If two germs $s_x, t_x \in \mathscr{P}_x$ have the same image, then the corresponding locally representable functions agree on a neighborhood of x; evaluating at x gives $s_x = t_x$. Thus η_x is bijective.

Finally, let $\alpha : \mathscr{P} \to \mathscr{G}$ be a morphism to a sheaf. If $\sigma \in \mathscr{P}^+(U)$ is locally represented by sections $s_i \in \mathscr{P}(U_i)$, define $\widetilde{\alpha}(\sigma)$ by gluing the sections $\alpha(s_i)$. If two local representatives determine the same germs, they agree after shrinking near every point, and therefore their images under α also agree locally. The construction is consequently well-defined. It is a morphism of sheaves and satisfies $\alpha = \widetilde{\alpha} \circ \eta$. Uniqueness follows because every section of $\mathscr{P}^+$ is locally in the image of η. □

Corollary 5.23. *A sheafification is unique up to a unique isomorphism compatible with the maps from the original presheaf.*

Proof. Suppose $(\mathscr{P}^+, \eta)$ and $(\mathscr{Q}, \theta)$ are two sheafifications. The universal property of $\mathscr{P}^+$ gives a unique map $u : \mathscr{P}^+ \to \mathscr{Q}$ with $u \circ \eta = \theta$, and the universal property of $\mathscr{Q}$ gives a unique map $v : \mathscr{Q} \to \mathscr{P}^+$ with $v \circ \theta = \eta$. Both $v \circ u$ and $\mathrm{id}_{\mathscr{P}^+}$ compose with η to give η, so uniqueness implies $v \circ u = \mathrm{id}$. Similarly $u \circ v = \mathrm{id}$. The same universal property proves uniqueness of the isomorphism. □

5.5 Sheaves of abelian groups and exactness

5.5.1 Kernels, images, and cokernels

A presheaf of abelian groups is a presheaf whose section sets are abelian groups and whose restriction maps are homomorphisms. It is an abelian sheaf if its underlying presheaf of sets is a sheaf.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups. Its kernel is computed sectionwise:

$$(\ker \varphi)(U) = \ker(\varphi_U).$$

The presheaf image

$$U \longmapsto \operatorname{im}(\varphi_U)$$

need not be a sheaf; its sheafification is the *image sheaf* $\operatorname{im} \varphi$. Similarly, the *cokernel sheaf* is the sheafification of the presheaf

$$U \longmapsto \operatorname{coker}(\varphi_U).$$

Proposition 5.24. *For every $x \in X$, there are natural isomorphisms*

$$(\ker \varphi)_x \cong \ker(\varphi_x),$$

$$(\operatorname{im} \varphi)_x \cong \operatorname{im}(\varphi_x),$$

and

$$(\operatorname{coker} \varphi)_x \cong \operatorname{coker}(\varphi_x).$$

Proof. For the kernel, a germ lies in $(\ker \varphi)_x$ precisely when it has a representative s whose image vanishes on a neighborhood of x. This is equivalent to $\varphi_x(s_x) = 0$, proving the first isomorphism.

The stalk of the presheaf image consists of germs which are locally images of sections of $\mathscr{F}$; these are exactly the elements of $\operatorname{im}(\varphi_x)$. Sheafification does not change stalks, so the second isomorphism follows.

Finally, direct limits over neighborhoods are filtered, and filtered direct limits of abelian groups preserve cokernels. Therefore

$$\begin{aligned}
(\operatorname{coker}_{\mathrm{pre}} \varphi)_x &= \varinjlim_{x \in U} \operatorname{coker}(\varphi_U) \\
&\cong \operatorname{coker}\left(\varinjlim_{x \in U} \mathscr{F}(U) \longrightarrow \varinjlim_{x \in U} \mathscr{G}(U) \right) \\
&= \operatorname{coker}(\varphi_x).
\end{aligned}$$

Sheafification again leaves the stalk unchanged. □

5.5.2 Stalkwise exactness

Definition 5.25. A sequence of abelian sheaves

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$$

is *exact at $\mathscr{G}$* if

$$\operatorname{im}(\alpha) = \ker(\beta)$$

as subsheaves of $\mathscr{G}$.

Theorem 5.26 (Stalkwise criterion for exactness)**.** *The sequence*

$$\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$$

is exact at $\mathscr{G}$ if and only if

$$\mathscr{F}_x \xrightarrow{\alpha_x} \mathscr{G}_x \xrightarrow{\beta_x} \mathscr{H}_x$$

is exact at $\mathscr{G}_x$ for every $x \in X$.

Proof. By Proposition 5.24,

$$(\operatorname{im}\alpha)_x \cong \operatorname{im}(\alpha_x) \qquad \text{and} \qquad (\ker\beta)_x \cong \ker(\beta_x).$$

Thus exactness of the sheaf sequence implies exactness of every stalk sequence. Conversely, if all stalk sequences are exact, then the natural inclusion

$$\operatorname{im}(\alpha) \longrightarrow \ker(\beta)$$

is an isomorphism on every stalk. Proposition 5.12 implies that it is an isomorphism of sheaves. ☐

Corollary 5.27. *A morphism of abelian sheaves is a monomorphism, epimorphism, or isomorphism if and only if its maps on all stalks are injective, surjective, or bijective, respectively.*

Proof. A morphism is a monomorphism precisely when its kernel is zero, and this can be checked on stalks by Proposition 5.24. It is an epimorphism precisely when its cokernel is zero, which is likewise a stalkwise condition. The isomorphism statement is Proposition 5.12. ☐

Remark 5.28 (Local lifting). A surjective morphism of sheaves need not be surjective on sections over every open set. Instead, if $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective and $s \in \mathscr{G}(U)$, then every point $x \in U$ has a neighborhood $V \subseteq U$ on which $s|_V$ lifts to a section of $\mathscr{F}(V)$. This is exactly what surjectivity of the stalk maps means.

5.5.3 The global section functor

For an open subset $U \subseteq X$, write

$$\Gamma(U, \mathscr{F}) := \mathscr{F}(U).$$

In particular, $\Gamma(X, \mathscr{F})$ is the group of global sections.

Proposition 5.29. *The functor $\Gamma(U, -)$ is left exact. Thus a short exact sequence*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \mathscr{F}'(U) \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{F}''(U),$$

but the final map need not be surjective.

Proof. Kernels of sheaf morphisms are computed sectionwise, so taking sections preserves the kernel of every morphism. Equivalently, injectivity and exactness at the middle term remain valid after taking sections. Surjectivity may fail because a section of $\mathscr{F}''$ may have local liftings which cannot be chosen compatibly on overlaps. ☐

5.6 The exponential sequence and the logarithm

We now return to the motivating local-to-global problem. The exponential sequence is exact as a sequence of sheaves because logarithms exist locally, but it need not remain exact after taking global sections.

Lemma 5.30 (Local holomorphic logarithms)**.** *Let g be holomorphic near a point p of a Riemann surface and suppose $g(p) \neq 0$. There is a neighborhood U of p and a holomorphic function $h \in \mathscr{O}_X(U)$ such that*

$$\exp(h) = g|_U.$$

Proof. Choose a coordinate neighborhood U of p which is mapped biholomorphically onto a disc and shrink it so that g has no zeros on U. In the coordinate, the holomorphic function g'/g has a primitive H on the disc. Then

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(g\mathrm{e}^{-H}\right) = 0,$$

so $g\mathrm{e}^{-H} = c$ for some nonzero constant c. Choose a complex number a with $\mathrm{e}^a = c$. Then $h = H + a$, transported back through the coordinate, satisfies $\mathrm{e}^h = g$. $\square$

Theorem 5.31 (The exponential sequence)**.** *For every Riemann surface X, the sequence of sheaves of abelian groups*

$$0 \longrightarrow 2\pi\mathrm{i}\,\underline{\mathbb{Z}}_X \longrightarrow \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \longrightarrow 1$$

is exact. Here $\mathscr{O}_X$ is written additively, $\mathscr{O}_X^\times$ multiplicatively, and the final 1 denotes the trivial multiplicative sheaf.

Proof. The first map sends a locally constant function with values in $2\pi\mathrm{i}\mathbb{Z}$ to the same function regarded as holomorphic. It is injective. A holomorphic function f satisfies $\mathrm{e}^f = 1$ precisely when its values lie in $2\pi\mathrm{i}\mathbb{Z}$. Since this set is discrete and f is continuous, such an f is locally constant. Hence the kernel of exp is $2\pi\mathrm{i}\underline{\mathbb{Z}}_X$. Finally, Lemma 5.30 says that every germ of a nowhere-vanishing holomorphic function is locally an exponential. Thus $\exp : \mathscr{O}_X \to \mathscr{O}_X^\times$ is surjective on all stalks, and therefore is an epimorphism of sheaves by Corollary 5.27. $\square$

Proposition 5.32. *The coordinate function*

$$z \in \mathscr{O}_{\mathbb{C}^\times}^\times(\mathbb{C}^\times)$$

does not have a global holomorphic logarithm. Consequently, applying global sections to the exponential sequence need not preserve surjectivity.

Proof. Suppose $h \in \mathscr{O}(\mathbb{C}^\times)$ satisfied $\mathrm{e}^{h(z)} = z$. Differentiating gives

$$h'(z) = \frac{1}{z}.$$

Integrating around the positively oriented unit circle γ, we obtain

$$0 = \int_\gamma h'(z)\,\mathrm{d}z = \int_\gamma \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i},$$

a contradiction. Therefore the global section z of $\mathscr{O}_{\mathbb{C}^\times}^\times$ has no preimage under the global exponential map. $\square$

Example 5.33 (Two local branches)**.** Set

$$U = \mathbb{C}^\times \setminus \mathbb{R}_{\leq 0}, \qquad V = \mathbb{C}^\times \setminus \mathbb{R}_{\geq 0}.$$

Choose branches

$$\mathrm{Log}_U\, z = \log|z| + \mathrm{i}\arg_U(z), \qquad -\pi < \arg_U(z) < \pi,$$

and

$$\mathrm{Log}_V\, z = \log|z| + \mathrm{i}\arg_V(z), \qquad 0 < \arg_V(z) < 2\pi.$$

The overlap $U \cap V$ has an upper and a lower connected component. On one component the branches agree, while on the other they differ by $2\pi\mathrm{i}$. Thus they are valid local liftings of z through the exponential map, but they do not satisfy the compatibility condition required by the sheaf axiom.

5.7 Ringed Space

5.7.1 Basic Definitions

Definition 5.34. A **ringed space** is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. The sheaf $\mathcal{O}_X$ is called the **structure sheaf**.

A ringed space is called a **locally ringed space** if every stalk $\mathcal{O}_{X,x}$ is a local ring. In this case we denote by $\mathfrak{m}_x$ the maximal ideal of $\mathcal{O}_{X,x}$.

Definition 5.35. Let $(X, \mathcal{O}_X)$ be a locally ringed space and let $x \in X$. The **residue field** of X at x is

$$k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x.$$

Example 5.36 (Holomorphic Functions on a Complex Manifold)**.** Let X be a complex manifold. We can naturally equip X with the sheaf of holomorphic functions $\mathcal{O}_X$. For an open set $U \subseteq X$, $\mathcal{O}_X(U)$ is the ring of holomorphic functions on U.

Let us examine the local ring structure at a point $p \in X$. The stalk $\mathcal{O}_{X,p}$ consists of germs of holomorphic functions at p. Let

$$\mathfrak{m}_p = \{f_p \in \mathcal{O}_{X,p} \mid f(p) = 0\}.$$

If $f_p \notin \mathfrak{m}_p$, then $f(p) \neq 0$. Hence f is non-zero on a sufficiently small neighbourhood of p, and $1/f$ is again holomorphic there. Thus f_p is invertible. Therefore $\mathfrak{m}_p$ is the unique maximal ideal and $\mathcal{O}_{X,p}$ is a local ring.

The evaluation map

$$\mathcal{O}_{X,p} \longrightarrow \mathbb{C}, \qquad f_p \longmapsto f(p)$$

is surjective and has kernel $\mathfrak{m}_p$. Hence

$$k(p) \cong \mathbb{C}.$$

Geometrically, passing to the residue field keeps only the value of a function at p and forgets higher order infinitesimal information.

Definition 5.37. A morphism of ringed spaces

$$(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

consists of a continuous map $f : X \to Y$ and a morphism of sheaves of rings

$$f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X.$$

Equivalently, for each open set $V \subseteq Y$, there is a ring homomorphism

$$f^{\#}(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}V)$$

compatible with restrictions.

If X and Y are locally ringed spaces, then $(f, f^{\#})$ is a morphism of locally ringed spaces if for every $x \in X$ the induced homomorphism on stalks

$$f_x^{\#} : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is local, i.e.

$$(f_x^{\#})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}.$$

5.7.2 Open Immersions

Definition 5.38. Let $(X, \mathcal{O}_X)$ be a ringed space and let $U \subseteq X$ be open. The **open subspace** associated to U is

$$(U, \mathcal{O}_X|_U).$$

A morphism $j : Y \to X$ of locally ringed spaces is an **open immersion** if j induces a homeomorphism from Y onto an open subset $U \subseteq X$ and the natural morphism

$$\mathcal{O}_X|_U \longrightarrow j_*\mathcal{O}_Y|_U$$

is an isomorphism. After identifying Y with U, this simply says that

$$\mathcal{O}_Y \cong \mathcal{O}_X|_U.$$

Remark 5.39. Algebraically, open immersions correspond to localization. At the stalk level, if $y \in Y$ maps to $x = j(y) \in X$, then

$$\mathcal{O}_{X,x} \xrightarrow{\sim} \mathcal{O}_{Y,y}.$$

Thus restricting to an open neighbourhood does not change the local ring of a point.

5.7.3 Closed Immersions

Closed immersions are more subtle. Topologically, the image is a closed subset; algebraically, functions on the closed subspace are obtained from functions on the ambient space by quotienting out functions vanishing on the subspace.

Definition 5.40. A morphism $i : Y \to X$ of locally ringed spaces is a **closed immersion** if:

1. i induces a homeomorphism from Y onto a closed subset $Z \subseteq X$;
2. the sheaf morphism
$$i^\# : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$$
is surjective.

The kernel

$$\mathcal{I} = \ker(i^\#)$$

is called the **ideal sheaf** of the closed immersion. Then

$$i_*\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{I}.$$

After identifying Y with its image Z, one may write

$$\mathcal{O}_Y \cong i^{-1}(\mathcal{O}_X/\mathcal{I}).$$

Remark 5.41. The surjectivity of $i^\#$ says that every function on Y locally lifts to a function on X. At the stalk level, for $y \in Y$ and $x = i(y)$,

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}$$

is surjective, and

$$\mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x}/\mathcal{I}_x.$$

This is the local manifestation of the idea that a closed subspace is cut out by equations.

Example 5.42 (Complex Manifolds Revisited)**.** Let X be a complex manifold with the sheaf $\mathcal{O}_X$ of holomorphic functions.

- An open immersion $U \hookrightarrow X$ is simply an open submanifold. The stalks remain unchanged.
- A closed immersion $Y \hookrightarrow X$ of complex analytic spaces is locally described by a sheaf of ideals $\mathcal{I}_Y \subseteq \mathcal{O}_X$, and the structure sheaf is

  $$\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y.$$

  Modding out by $\mathcal{I}_Y$ means that we identify two ambient functions if their restrictions to Y agree.

Proposition 5.43. *Open immersions and closed immersions are monomorphisms in the category of locally ringed spaces, and hence also in the category of schemes.*

Proof. Let $i : Y \to X$ be either an open immersion or a closed immersion, and let $g, h : T \to Y$ be two morphisms such that

$$i \circ g = i \circ h.$$

Since i is a homeomorphism onto its image, the equality after composing with i implies that g and h have the same underlying continuous map.

It remains to compare the morphisms on structure sheaves. For an open immersion this is immediate after identifying Y with an open subspace of X, because $\mathcal{O}_Y = \mathcal{O}_X|_Y$.

For a closed immersion, the map

$$i^\# : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$$

is surjective. Pulling back to T, the equality $i \circ g = i \circ h$ gives equality of the two maps from the pullback of $\mathcal{O}_X$. Since $\mathcal{O}_Y$ is a quotient of $i^{-1}\mathcal{O}_X$, the induced maps to $\mathcal{O}_T$ are equal. Hence $g = h$. $\square$

Chapter 6

Coherent Sheaves and Sheaf Cohomology

The previous chapter introduced sheaves as a language for local holomorphic objects and ended with a basic failure of global lifting: every nowhere vanishing holomorphic function has a logarithm locally, but not necessarily globally. We now develop the machinery which measures such failures.

There are two complementary themes. First, the analytic objects which arise in geometry are rarely arbitrary sheaves of modules. They satisfy a local finiteness condition called *coherence*. On a Riemann surface this condition is particularly transparent because every local ring of holomorphic germs is a discrete valuation ring. Second, the global section functor is left exact but not exact. Its right derived functors are the sheaf cohomology groups.

Throughout the chapter, X denotes a Riemann surface unless a general topological space is explicitly allowed. We write $\mathscr{O}_X$ for the structure sheaf and $\mathscr{M}_X$ for the sheaf of meromorphic functions.

6.1 Coherent analytic sheaves

6.1.1 Modules over the structure sheaf

Definition 6.1. Let $(X, \mathscr{O}_X)$ be a ringed space. A *sheaf of $\mathscr{O}_X$-modules* is a sheaf $\mathscr{F}$ of abelian groups such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module for every open set U, and all restriction maps are compatible with scalar multiplication.

The most basic examples on a Riemann surface are $\mathscr{O}_X$ itself, finite direct sums $\mathscr{O}_X^{\oplus r}$, and the sheaf $\mathscr{M}_X$. A holomorphic vector bundle of rank r has a sheaf of holomorphic sections which is locally isomorphic to $\mathscr{O}_X^{\oplus r}$.

Definition 6.2. An $\mathscr{O}_X$-module $\mathscr{E}$ is *locally free of rank r* if every point has an open neighbourhood U such that

$$\mathscr{E}|_U \cong \mathscr{O}_U^{\oplus r}.$$

A locally free sheaf of rank one is called *invertible*.

Definition 6.3. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module.

(i) It is *of finite type* if every point has a neighbourhood U for which there is a surjection

$$\mathscr{O}_U^{\oplus n} \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

with $n < \infty$.

(ii) It is *locally finitely presented* if every point has a neighbourhood U for which there is an exact sequence

$$\mathscr{O}_U^{\oplus m} \longrightarrow \mathscr{O}_U^{\oplus n} \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

with $m, n < \infty$.

(iii) It is *coherent* if it is of finite type and, for every open set U and every morphism

$$\varphi : \mathscr{O}_U^{\oplus n} \longrightarrow \mathscr{F}|_U,$$

the kernel $\ker \varphi$ is of finite type.

Thus coherence says that both generators and relations can be chosen locally and finitely. It is the sheaf-theoretic analogue of finite presentation over a noetherian ring.

6.1.2 The local analytic ring

Proposition 6.4. *Let $p \in X$, and choose a local coordinate z centred at p. Then*

$$\mathscr{O}_{X,p} \cong \mathbb{C}\{z\},$$

the ring of convergent power series at the origin. Every nonzero element has a unique expression

$$f = z^n u, \qquad n \geq 0, \qquad u \in \mathscr{O}_{X,p}^\times.$$

Consequently, $\mathscr{O}_{X,p}$ is a discrete valuation ring and, in particular, a noetherian principal ideal domain.

Proof. A germ of a holomorphic function at p is represented in the coordinate z by a convergent power series, which gives the first isomorphism. If the germ f is nonzero, its power series has a first nonzero coefficient. If this occurs in degree n, then

$$f = z^n u$$

where $u(0) \neq 0$, so u is a unit. The integer n is uniquely determined as the order of vanishing. Every nonzero ideal therefore contains an element of minimal order, and such an element generates the ideal. The maximal ideal is generated by z, so the ring is a discrete valuation ring. □

Lemma 6.5. *Suppose that a sheaf of rings $\mathscr{A}$ is coherent as an $\mathscr{A}$-module. Then the kernel of every morphism between finite free $\mathscr{A}$-modules is of finite type.*

Proof. By the definition of coherence, the assertion holds for a morphism $\mathscr{A}^{\oplus m} \to \mathscr{A}$. We argue by induction on the rank n of the target. Let

$$u : \mathscr{A}^{\oplus m} \longrightarrow \mathscr{A}^{\oplus n}$$

and suppose $n > 1$. Compose u with projection onto the last factor and let $\mathscr{K}$ be its kernel. Then $\mathscr{K}$ is of finite type, so locally there is a surjection

$$\mathscr{A}^{\oplus r} \longrightarrow \mathscr{K}.$$

Compose this map with the remaining $n-1$ components of u. By the induction hypothesis, the kernel of the resulting map $\mathscr{A}^{\oplus r} \to \mathscr{A}^{\oplus (n-1)}$ is of finite type. Its image in $\mathscr{K}$ is precisely $\ker u$. Since a quotient of a finite type sheaf is of finite type, $\ker u$ is of finite type. □

Theorem 6.6. *For every Riemann surface X, the structure sheaf $\mathscr{O}_X$ is a coherent sheaf of rings.*

Proof. It is enough to verify the kernel condition locally. Let U be a coordinate neighbourhood and consider a morphism

$$\varphi : \mathscr{O}_U^{\oplus n} \longrightarrow \mathscr{O}_U.$$

It is given by a row of holomorphic functions $(f_1, \ldots, f_n)$. Fix $p \in U$. If every germ $(f_i)_p$ is zero, then after shrinking around p all f_i vanish and the kernel is the finite free sheaf $\mathscr{O}_U^{\oplus n}$.

Assume that at least one germ is nonzero. Choose j for which $\operatorname{ord}_p(f_j)$ is minimal. Proposition 6.4 implies that $(f_j)_p$ divides every $(f_i)_p$. After shrinking U, there are holomorphic functions h_i such that

$$f_i = f_j h_i, \qquad h_j = 1.$$

Multiplication by the nonzero holomorphic function f_j is injective on holomorphic functions, so $\ker \varphi$ is the kernel of

$$(h_1, \ldots, h_n) : \mathscr{O}_U^{\oplus n} \longrightarrow \mathscr{O}_U.$$

Because $h_j = 1$, this kernel is freely generated by

$$e_i - h_i e_j, \qquad i \neq j.$$

It is therefore locally free of rank $n - 1$. Hence the kernel of every map from a finite free sheaf to $\mathscr{O}_X$ is of finite type, proving coherence. □

Proposition 6.7. *Let $(X, \mathscr{A})$ be a ringed space whose structure sheaf is coherent. An $\mathscr{A}$-module is coherent if and only if it is locally finitely presented.*

Proof. Suppose first that $\mathscr{F}$ is coherent. Locally choose a surjection $\mathscr{A}^{\oplus n} \to \mathscr{F}$. Its kernel is of finite type, so after shrinking there is a surjection $\mathscr{A}^{\oplus m}$ onto that kernel. This gives a finite presentation.

Conversely, suppose locally that

$$\mathscr{A}^{\oplus m} \xrightarrow{A} \mathscr{A}^{\oplus n} \longrightarrow \mathscr{F} \longrightarrow 0$$

is exact. Then $\mathscr{F}$ is of finite type. Let $\alpha : \mathscr{A}^{\oplus r} \to \mathscr{F}$ be a morphism. After shrinking, the images of the standard basis vectors lift through $\mathscr{A}^{\oplus n} \to \mathscr{F}$, giving a map $\widetilde{\alpha} : \mathscr{A}^{\oplus r} \to \mathscr{A}^{\oplus n}$. Define

$$\beta : \mathscr{A}^{\oplus r} \oplus \mathscr{A}^{\oplus m} \longrightarrow \mathscr{A}^{\oplus n}, \qquad \beta(v, w) = \widetilde{\alpha}(v) - A(w).$$

By Lemma 6.5, $\ker \beta$ is of finite type. Projection onto the first factor maps $\ker \beta$ surjectively onto $\ker \alpha$. Therefore $\ker \alpha$ is of finite type, and $\mathscr{F}$ is coherent. □

Corollary 6.8. *Every locally free $\mathscr{O}_X$-module of finite rank on a Riemann surface is coherent.*

Proof. The assertion is local, and a finite free sheaf has the finite presentation

$$0 \longrightarrow 0 \longrightarrow \mathscr{O}_U^{\oplus r} \xrightarrow{\mathrm{id}} \mathscr{O}_U^{\oplus r} \longrightarrow 0.$$

Apply Proposition 6.7 and Theorem 6.6. □

Proposition 6.9. *The coherent $\mathscr{O}_X$-modules on a Riemann surface form an abelian subcategory of the category of all $\mathscr{O}_X$-modules. In particular, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of coherent sheaves, then*

$$\ker \varphi, \qquad \operatorname{im} \varphi, \qquad \operatorname{coker} \varphi$$

are coherent.

Proof. The assertion is local. We first note that every finite type subsheaf $\mathscr{K} \subseteq \mathscr{O}_U^{\oplus n}$ is coherent. Indeed, choose a surjection $\mathscr{O}_U^{\oplus r} \to \mathscr{K}$. Its kernel is also the kernel of the composite map $\mathscr{O}_U^{\oplus r} \to \mathscr{O}_U^{\oplus n}$, hence is of finite type by Lemma 6.5. Thus $\mathscr{K}$ is locally finitely presented and therefore coherent.

Choose finite presentations

$$\mathscr{O}_U^{\oplus a} \xrightarrow{A} \mathscr{O}_U^{\oplus b} \longrightarrow \mathscr{F}|_U \longrightarrow 0, \qquad \mathscr{O}_U^{\oplus c} \xrightarrow{B} \mathscr{O}_U^{\oplus d} \longrightarrow \mathscr{G}|_U \longrightarrow 0.$$

After shrinking, the morphism φ lifts to a morphism $\widetilde{\varphi} : \mathscr{O}_U^{\oplus b} \to \mathscr{O}_U^{\oplus d}$. Define

$$T : \mathscr{O}_U^{\oplus b} \oplus \mathscr{O}_U^{\oplus c} \longrightarrow \mathscr{O}_U^{\oplus d}, \qquad T(v, w) = \widetilde{\varphi}(v) - B(w).$$

By Lemma 6.5, $\ker T$ is of finite type. Its image under projection to $\mathscr{O}_U^{\oplus b}$ is the inverse image $\mathscr{L}$ of $\ker \varphi$ before passing to the quotient defining $\mathscr{F}$. Hence $\mathscr{L}$ is a finite type subsheaf of a finite free sheaf and is coherent by the preceding paragraph. Since

$$\ker \varphi \cong \mathscr{L} / \operatorname{im} A,$$

it is coherent.

The image of φ is a quotient of the finite type sheaf $\mathscr{F}$, so it is of finite type. As a finite type subsheaf of the coherent sheaf $\mathscr{G}$, the same presentation argument shows that it is coherent. Finally,

$$\operatorname{coker} \varphi = \mathscr{G} / \operatorname{im} \varphi$$

is locally finitely presented and hence coherent. Thus coherent sheaves are closed under kernels, images, and cokernels. □

6.1.3 The local structure of a coherent sheaf

Theorem 6.10 (Local structure theorem)**.** *Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module and let $p \in X$. There is a coordinate neighbourhood U of p, a coordinate z with $z(p) = 0$, and integers $r, s \geq 0$ and $n_1, \dots, n_s \geq 1$ such that*

$$\mathscr{F}|_U \cong \mathscr{O}_U^{\oplus r} \oplus \bigoplus_{j=1}^{s} \mathscr{O}_U / (z^{n_j}).$$

Proof. By Proposition 6.7, after shrinking around p there is a finite presentation

$$\mathscr{O}_U^{\oplus m} \xrightarrow{A} \mathscr{O}_U^{\oplus n} \longrightarrow \mathscr{F}|_U \longrightarrow 0.$$

At the stalk p, the matrix A_p has entries in the principal ideal domain $\mathscr{O}_{X,p} \cong \mathbb{C}\{z\}$. The Smith normal form gives invertible matrices P and Q over $\mathscr{O}_{X,p}$ for which

$$P A_p Q = \operatorname{diag}(z^{n_1}, \dots, z^{n_s}, 0, \dots, 0),$$

after absorbing units into P and Q. The entries of P, Q and their inverses are germs of holomorphic functions. After shrinking U, they are represented by holomorphic matrices whose determinants are nowhere zero. They therefore define automorphisms of the finite free sheaves on U. Replacing A by the equivalent diagonal matrix does not change its cokernel, and the displayed decomposition follows by taking the cokernel one diagonal entry at a time. □

Definition 6.11. A coherent sheaf $\mathscr{F}$ is *torsion-free* if every stalk $\mathscr{F}_p$ is torsion-free over the domain $\mathscr{O}_{X,p}$. Its *torsion subsheaf* consists locally of sections annihilated by a nonzero holomorphic function.

Corollary 6.12. *Every torsion-free coherent sheaf on a Riemann surface is locally free.*

Proof. In the decomposition of Theorem 6.10, each summand $\mathscr{O}_U/(z^{n_j})$ is torsion. Torsion-freeness forces all such summands to be absent, leaving $\mathscr{F}|_U \cong \mathscr{O}_U^{\oplus r}$. □

Thus, on a nonsingular complex curve, torsion-free coherent sheaves are exactly the sheaves of holomorphic sections of vector bundles. The remaining coherent sheaves differ from vector bundles only by torsion concentrated at isolated points.

Proposition 6.13. *Let $p \in X$, and let $\mathbb{C}_p$ be the skyscraper sheaf with stalk $\mathbb{C}$ at p and zero stalk elsewhere. Then $\mathbb{C}_p$ is a coherent $\mathscr{O}_X$-module. More generally, for every $n \geq 1$, the skyscraper module with stalk $\mathscr{O}_{X,p}/\mathfrak{m}_p^n$ at p is coherent.*

Proof. Choose a coordinate z centred at p. On a sufficiently small coordinate neighbourhood U, evaluation at p gives an exact sequence

$$0 \longrightarrow z\mathscr{O}_U \longrightarrow \mathscr{O}_U \longrightarrow \mathbb{C}_p|_U \longrightarrow 0.$$

The first two terms are locally free of rank one, so the cokernel is coherent by Proposition 6.9. Away from p, the sheaf is zero and hence coherent. Replacing z by z^n gives

$$0 \longrightarrow z^n\mathscr{O}_U \longrightarrow \mathscr{O}_U \longrightarrow (\mathscr{O}_{X,p}/\mathfrak{m}_p^n)_p|_U \longrightarrow 0,$$

which proves the general statement. □

6.2 Derived-functor cohomology

The geometric sheaves of interest are coherent, but injective resolutions are usually not coherent. Cohomology is therefore defined in the larger category of all sheaves of abelian groups. For an $\mathscr{O}_X$-module we simply forget the module structure while taking the resolution; the resulting cohomology groups retain their natural complex vector-space structure.

6.2.1 Enough injective sheaves

Proposition 6.14 (Enough injectives)**.** *For every topological space Y, the category of sheaves of abelian groups on Y has enough injective objects.*

Proof. Let $\mathscr{F}$ be a sheaf of abelian groups. For every $y \in Y$, choose an embedding of abelian groups

$$\mathscr{F}_y \hookrightarrow I_y$$

with I_y injective. Let $\mathscr{S}_y(I_y)$ be the skyscraper sheaf with value I_y at y. There is a natural adjunction

$$\mathrm{Hom}(\mathscr{G}, \mathscr{S}_y(I_y)) \cong \mathrm{Hom}(\mathscr{G}_y, I_y).$$

Since the stalk functor is exact and I_y is injective, $\mathscr{S}_y(I_y)$ is an injective sheaf.

Set

$$\mathscr{I} = \prod_{y \in Y} \mathscr{S}_y(I_y).$$

A product of injective sheaves is injective: a map into the product is the same as a family of maps into its factors, and each factorwise map extends across any monomorphism. The chosen stalk embeddings define a morphism

$$\mathscr{F} \longrightarrow \mathscr{I}.$$

At the stalk y, its composition with the projection to I_y is the chosen injection. Hence the morphism is injective on every stalk and therefore is a monomorphism of sheaves. □

The same construction, using injective $\mathscr{O}_{X,p}$-modules, proves that the category of all $\mathscr{O}_X$-modules also has enough injectives.

Definition 6.15. Let Y be a topological space and let $\mathscr{F}$ be a sheaf of abelian groups. Choose an injective resolution

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \longrightarrow \mathscr{I}^1 \longrightarrow \cdots .$$

The *sheaf cohomology groups* are

$$H^q(Y, \mathscr{F}) := H^q\big(\Gamma(Y, \mathscr{I}^\bullet)\big).$$

Equivalently,

$$H^q(Y, \mathscr{F}) = R^q\Gamma(Y, \mathscr{F}).$$

Standard comparison maps between injective resolutions are unique up to homotopy, so the groups are independent of the chosen resolution and are functorial in $\mathscr{F}$. In degree zero,

$$H^0(Y, \mathscr{F}) = \Gamma(Y, \mathscr{F}).$$

6.2.2 The long exact sequence

Lemma 6.16 (Horseshoe lemma)**.** *Let*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be a short exact sequence of sheaves of abelian groups. There are injective resolutions of the three sheaves which fit into a short exact sequence of complexes

$$0 \longrightarrow \mathscr{I}'^\bullet \longrightarrow \mathscr{I}^\bullet \longrightarrow \mathscr{I}''^\bullet \longrightarrow 0.$$

Proof. Choose injective resolutions of $\mathscr{F}'$ and $\mathscr{F}''$. In degree zero, injectivity allows the inclusion $\mathscr{F}' \to \mathscr{I}'^0$ to be extended across $\mathscr{F}$, and the quotient map to $\mathscr{F}''$ maps into $\mathscr{I}''^0$. Taking $\mathscr{I}^0 = \mathscr{I}'^0 \oplus \mathscr{I}''^0$, one obtains a monomorphism $\mathscr{F} \to \mathscr{I}^0$ compatible with the given short exact sequence. Pass to cokernels and repeat the same construction inductively. At each stage the middle injective is the direct sum of the two outer injectives, and the resulting differentials are chosen to make the diagrams commute. This produces the required short exact sequence of complexes. □

Theorem 6.17 (Long exact sequence)**.** *A short exact sequence*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

induces a natural long exact sequence

$$\begin{aligned} 0 \longrightarrow & H^0(Y, \mathscr{F}') \longrightarrow H^0(Y, \mathscr{F}) \longrightarrow H^0(Y, \mathscr{F}'') \xrightarrow{\delta} H^1(Y, \mathscr{F}') \\ \longrightarrow & H^1(Y, \mathscr{F}) \longrightarrow H^1(Y, \mathscr{F}'') \xrightarrow{\delta} H^2(Y, \mathscr{F}') \longrightarrow \cdots . \end{aligned}$$

Proof. Apply the global section functor to the short exact sequence of injective complexes supplied by Lemma 6.16. Because the sequence is split in every degree, it remains a short exact sequence of complexes of abelian groups. The standard cohomology sequence associated with a short exact sequence of complexes gives the displayed long exact sequence. Its construction is natural with respect to morphisms of short exact sequences. □

The connecting homomorphism is the formal device which records the obstruction to lifting a global section. Exactness gives

$$\delta(s) = 0 \quad \Longleftrightarrow \quad s \in H^0(Y, \mathscr{F}'') \text{ lifts to } H^0(Y, \mathscr{F}).$$

6.3 Flasque sheaves and acyclic resolutions

Definition 6.18. A sheaf $\mathscr{F}$ on a topological space Y is *flasque* if every restriction map

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V), \qquad V \subseteq U,$$

is surjective.

Lemma 6.19. *Every injective sheaf of abelian groups is flasque.*

Proof. For an open set $U \subseteq Y$, let $\mathbb{Z}_U$ denote the extension by zero to Y of the constant sheaf $\underline{\mathbb{Z}}$ on U. It represents sections in the sense that

$$\mathrm{Hom}(\mathbb{Z}_U, \mathscr{F}) \cong \mathscr{F}(U).$$

If $V \subseteq U$, there is a monomorphism $\mathbb{Z}_V \hookrightarrow \mathbb{Z}_U$. For an injective sheaf $\mathscr{I}$, applying $\mathrm{Hom}(-, \mathscr{I})$ gives a surjection

$$\mathscr{I}(U) \cong \mathrm{Hom}(\mathbb{Z}_U, \mathscr{I}) \longrightarrow \mathrm{Hom}(\mathbb{Z}_V, \mathscr{I}) \cong \mathscr{I}(V).$$

Thus every restriction map is surjective. $\square$

Lemma 6.20 (Extension lemma)**.** *Let*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

be exact. If $\mathscr{F}$ is flasque, then

$$\mathscr{G}(Y) \longrightarrow \mathscr{H}(Y)$$

is surjective.

Proof. Let $t \in \mathscr{H}(Y)$. Consider pairs (U, s) where $U \subseteq Y$ is open and $s \in \mathscr{G}(U)$ maps to $t|_U$, ordered by extension. Zorn's lemma gives a maximal pair (U, s). If $U \neq Y$, choose $y \notin U$ and an open neighbourhood V on which t has a lift $s_V \in \mathscr{G}(V)$. On $U \cap V$, the difference $s_V - s$ comes from a section $a \in \mathscr{F}(U \cap V)$. Since $\mathscr{F}$ is flasque, a extends to a section $\widetilde{a} \in \mathscr{F}(V)$. Replacing s_V by $s_V - \widetilde{a}$, the two lifts agree on the overlap and glue to a lift on $U \cup V$, contradicting maximality. Hence $U = Y$. $\square$

Theorem 6.21. *If $\mathscr{F}$ is flasque, then*

$$H^q(Y, \mathscr{F}) = 0$$

for every $q > 0$.

Proof. Embed $\mathscr{F}$ into an injective sheaf $\mathscr{I}$ and let $\mathscr{G} = \mathscr{I}/\mathscr{F}$. By Lemma 6.19, $\mathscr{I}$ is flasque. The extension lemma shows that

$$0 \longrightarrow \mathscr{F}(Y) \longrightarrow \mathscr{I}(Y) \longrightarrow \mathscr{G}(Y) \longrightarrow 0$$

is exact, and the long exact sequence gives $H^1(Y, \mathscr{F}) = 0$.

The quotient $\mathscr{G}$ is flasque. Indeed, over any open set U, the extension lemma makes $\mathscr{I}(U) \to \mathscr{G}(U)$ surjective. A section of $\mathscr{G}(V)$ therefore lifts to $\mathscr{I}(V)$ and extends to $\mathscr{I}(U)$, whose image extends the original section. The same argument may now be repeated with $\mathscr{G}$. Dimension shifting in the long exact sequence yields

$$H^{q+1}(Y, \mathscr{F}) \cong H^q(Y, \mathscr{G}) = 0$$

inductively for every $q > 0$. $\square$

Proposition 6.22. *If*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{R}^0 \longrightarrow \mathscr{R}^1 \longrightarrow \cdots$$

is a resolution by flasque sheaves, then

$$H^q(Y, \mathscr{F}) \cong H^q\big(\Gamma(Y, \mathscr{R}^\bullet)\big).$$

Proof. By Theorem 6.21, every $\mathscr{R}^j$ is acyclic for the global section functor. The comparison theorem for resolutions gives a map from the flasque resolution to an injective resolution, unique up to homotopy. Applying global sections, the induced map is a quasi-isomorphism because both resolutions are acyclic resolutions of the same sheaf. Their cohomology groups are therefore naturally isomorphic. □

6.4 Čech cohomology

6.4.1 The Čech complex

Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of a topological space Y, and fix an ordering of I. Write

$$U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

Definition 6.23. For a sheaf of abelian groups $\mathscr{F}$, the group of Čech p-cochains is

$$C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0 \cdots i_p}).$$

The differential $d : C^p \to C^{p+1}$ is

$$(d\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}}\big|_{U_{i_0 \cdots i_{p+1}}}.$$

The cohomology of this complex is denoted

$$\check{H}^p(\mathfrak{U}, \mathscr{F}).$$

Proposition 6.24. *The Čech differential satisfies $d^2 = 0$.*

Proof. In $(d^2\alpha)_{i_0 \cdots i_{p+2}}$, deleting i_a and then i_b gives the same restricted cochain as deleting i_b and then i_a. The two occurrences have opposite signs, because the position of the second deleted index changes by one when the first deletion is reversed. Thus all terms cancel in pairs. □

Proposition 6.25. *For every open cover $\mathfrak{U}$,*

$$\check{H}^0(\mathfrak{U}, \mathscr{F}) \cong \Gamma(Y, \mathscr{F}).$$

Proof. A zero-cochain is a family $s_i \in \mathscr{F}(U_i)$. It is a cocycle exactly when the sections agree on every overlap. The sheaf gluing axiom identifies such compatible families with unique global sections. □

For every nonempty intersection $U_{i_0 \cdots i_p}$, let $j_{i_0 \cdots i_p} : U_{i_0 \cdots i_p} \hookrightarrow Y$ be the inclusion and set

$$\mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \cdots < i_p} j_{i_0 \cdots i_p, *}\big(\mathscr{F}|_{U_{i_0 \cdots i_p}}\big).$$

Then

$$\Gamma\big(Y, \mathscr{C}^p(\mathfrak{U}, \mathscr{F})\big) = C^p(\mathfrak{U}, \mathscr{F}),$$

and the alternating restriction maps define a complex of sheaves.

Lemma 6.26. *The augmented Čech complex of sheaves*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U},\mathscr{F}) \longrightarrow \cdots$$

is exact.

Proof. Exactness may be checked on stalks. Fix $y \in Y$ and choose i with $y \in U_i$. On the stalk complex, define a homotopy by inserting the fixed index i into a cochain, with the sign dictated by the ordering. After restricting to a sufficiently small neighbourhood contained in U_i, a direct computation gives

$$dh + hd = \mathrm{id}\,.$$

Thus the stalk complex is contractible and hence exact. □

Proposition 6.27. *If $\mathscr{F}$ is flasque, then*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) = 0$$

for every $p > 0$.

Proof. Each term $\mathscr{C}^p(\mathfrak{U},\mathscr{F})$ is a product of direct images of restrictions of $\mathscr{F}$ to finite intersections. Restrictions of a flasque sheaf are flasque, direct images under open inclusions preserve flasqueness, and products of flasque sheaves are flasque. Hence Lemma 6.26 gives a flasque resolution of $\mathscr{F}$. By Proposition 6.22, its complex of global sections computes $H^p(Y,\mathscr{F})$, which vanishes in positive degrees by Theorem 6.21. This complex of global sections is the Čech complex. □

Proposition 6.28. *For every open cover $\mathfrak{U}$ there is a natural comparison morphism*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) \longrightarrow H^p(Y,\mathscr{F}).$$

Proof. The augmented Čech complex and an injective resolution both resolve $\mathscr{F}$. The comparison theorem for resolutions gives a morphism from the former to the latter, unique up to homotopy. Applying global sections and passing to cohomology gives the stated natural map. □

Definition 6.29. A finite open cover $\mathfrak{U} = \{U_i\}$ is *Leray for $\mathscr{F}$* if for every nonempty finite intersection $U_{i_0\cdots i_p}$ one has

$$H^q(U_{i_0\cdots i_p},\mathscr{F}) = 0 \qquad (q > 0).$$

Theorem 6.30 (Leray theorem)**.** *If a finite cover $\mathfrak{U}$ is Leray for $\mathscr{F}$, then the comparison map is an isomorphism:*

$$\check{H}^p(\mathfrak{U},\mathscr{F}) \xrightarrow{\sim} H^p(Y,\mathscr{F})$$

for every $p \geq 0$.

Proof. Take an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$ and form the first-quadrant double complex

$$C^{p,q} = C^p(\mathfrak{U},\mathscr{I}^q).$$

Compute the cohomology of its total complex in two ways. First take horizontal cohomology. Every injective sheaf is flasque, so Proposition 6.27 shows that the horizontal cohomology vanishes for $p > 0$ and equals $\Gamma(Y,\mathscr{I}^q)$ for $p = 0$. The total cohomology is therefore $H^*(Y,\mathscr{F})$.

Now take vertical cohomology. Restricting the injective resolution to each intersection gives a flasque resolution there, because restriction preserves exactness and flasqueness. Hence the vertical cohomology in bidegree (p,q) is

$$\prod_{i_0<\cdots<i_p} H^q(U_{i_0\cdots i_p},\mathscr{F}).$$

The Leray assumption makes this zero for $q > 0$, while the row $q = 0$ is exactly the Čech complex $C^\bullet(\mathfrak{U},\mathscr{F})$. Thus the same total cohomology is $\check{H}^*(\mathfrak{U},\mathscr{F})$. The two identifications agree with the comparison morphism, proving the theorem. □

6.5 The logarithm as a cohomological obstruction

Recall the exponential sequence on a Riemann surface:

$$0 \longrightarrow 2\pi\mathrm{i}\,\underline{\mathbb{Z}}_X \longrightarrow \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \longrightarrow 1.$$

It is exact as a sequence of sheaves because a nowhere vanishing holomorphic function has a local holomorphic logarithm.

Proposition 6.31. *The long exact sequence associated with the exponential sequence contains*

$$\mathscr{O}_X(X) \xrightarrow{\exp} \mathscr{O}_X^\times(X) \xrightarrow{\delta} H^1(X, 2\pi\mathrm{i}\,\underline{\mathbb{Z}}_X) \longrightarrow H^1(X, \mathscr{O}_X).$$

For $g \in \mathscr{O}_X^\times(X)$,

$$\delta(g) = 0 \quad \Longleftrightarrow \quad g \text{ has a global holomorphic logarithm.}$$

Proof. Apply Theorem 6.17 to the exponential sequence, regarded as a sequence of sheaves of abelian groups. Exactness at $\mathscr{O}_X^\times(X)$ says that $\delta(g) = 0$ precisely when g lies in the image of the global exponential map. This means exactly that $g = \exp h$ for some $h \in \mathscr{O}_X(X)$. □

Proposition 6.32 (A Čech representative)**.** *Let $\mathfrak{U} = \{U_i\}$ be an open cover on which $g \in \mathscr{O}_X^\times(X)$ admits holomorphic logarithms h_i. Then*

$$c_{ij} = h_j - h_i \in 2\pi\mathrm{i}\,\underline{\mathbb{Z}}_X(U_i \cap U_j)$$

defines a Čech 1-cocycle. Its image under the comparison map is the cohomology class $\delta(g)$. Changing the local logarithms changes the cocycle by a coboundary.

Proof. On an overlap,

$$\exp(h_j - h_i) = g/g = 1,$$

so $h_j - h_i$ is locally constant with values in $2\pi\mathrm{i}\mathbb{Z}$. On a triple overlap,

$$c_{ij} + c_{jk} + c_{ki} = 0,$$

so the family is a cocycle. Replacing h_i by $h_i + 2\pi\mathrm{i}n_i$ changes c_{ij} by $2\pi\mathrm{i}(n_j - n_i)$, which is a coboundary. The standard construction of the connecting homomorphism chooses local liftings and takes their differences; therefore the resulting sheaf cohomology class is $\delta(g)$. □

6.5.1 The coordinate function on $\mathbb{C}^\times$

Set

$$U = \mathbb{C}^\times \setminus \mathbb{R}_{\leq 0}, \qquad V = \mathbb{C}^\times \setminus \mathbb{R}_{\geq 0}.$$

The overlap has two connected components, the upper and lower half-planes. For the constant sheaf $\underline{\mathbb{Z}}$, the Čech complex of this two-set cover is

$$0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0, \qquad d(a, b) = (b - a, b - a).$$

Consequently,

$$\check{H}^1(\{U, V\}, \underline{\mathbb{Z}}) \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\{(n, n) : n \in \mathbb{Z}\}} \cong \mathbb{Z}.$$

Choose the branches

$$\operatorname{Log}_U z = \log|z| + \mathrm{i}\arg_U(z), \qquad -\pi < \arg_U(z) < \pi,$$

and

$$\operatorname{Log}_V z = \log|z| + \mathrm{i}\arg_V(z), \qquad 0 < \arg_V(z) < 2\pi.$$

Their difference is zero on one component of $U \cap V$ and $2\pi\mathrm{i}$ on the other. Thus the coordinate function is represented by the class of $(0, 2\pi\mathrm{i})$, a generator up to sign. Its image in sheaf cohomology is nonzero, since a zero image would give a global logarithm by Proposition 6.31.

Theorem 6.33 (Logarithms and winding numbers)**.** *Let $\Omega \subseteq \mathbb{C}$ be a connected region and let $g \in \mathscr{O}^\times(\Omega)$. The following are equivalent:*

(i) *g has a holomorphic logarithm on Ω;*

(ii) *for every closed piecewise C^1 curve γ in Ω,*

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{g'(z)}{g(z)}\, \mathrm{d}z = 0;$$

(iii) *the obstruction class*

$$\delta(g) \in H^1(\Omega, 2\pi\mathrm{i}\,\underline{\mathbb{Z}})$$

vanishes.

Proof. If $g = \exp h$, then $g'/g = h'$, and the integral of h' around every closed curve is zero. Conversely, condition (ii) is the period criterion for the holomorphic function g'/g to possess a primitive h. Then

$$(\exp(-h)g)' = 0,$$

so $\exp(-h)g = c$ for some nonzero constant c. Choosing a logarithm a of c, we obtain $g = \exp(h + a)$. Finally, the equivalence of (i) and (iii) is Proposition 6.31. □

Thus the same integer may be read in three ways: as a winding number, as the jump between local branches of the logarithm, or as a class in first sheaf cohomology.

6.6 The Mittag–Leffler obstruction

Definition 6.34. The *sheaf of principal parts* on a Riemann surface is

$$\mathscr{P}_X := \mathscr{M}_X/\mathscr{O}_X.$$

Its stalk at p is

$$\mathscr{P}_{X,p} = \mathscr{M}_{X,p}/\mathscr{O}_{X,p}.$$

In a local coordinate z, an element of this stalk is represented uniquely by a finite Laurent polynomial

$$a_{-n}z^{-n} + \cdots + a_{-1}z^{-1}.$$

A global section of $\mathscr{P}_X$ is a Mittag–Leffler distribution: it specifies a principal part at a locally finite set of points. If X is compact, that set is finite.

Proposition 6.35. *The short exact sequence*

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{M}_X \longrightarrow \mathscr{P}_X \longrightarrow 0$$

induces

$$\mathscr{M}_X(X) \longrightarrow \mathscr{P}_X(X) \xrightarrow{\delta} H^1(X, \mathscr{O}_X).$$

A Mittag–Leffler distribution $P \in \mathscr{P}_X(X)$ is the collection of principal parts of a global meromorphic function if and only if

$$\delta(P) = 0.$$

Proof. The displayed sequence is exact by the definition of the quotient sheaf. Apply the long exact sequence. Exactness at $\mathscr{P}_X(X)$ says that $\delta(P) = 0$ exactly when P lies in the image of $\mathscr{M}_X(X)$, which is precisely the existence of a global meromorphic function with the prescribed principal parts. □

The classical Mittag–Leffler theorem says that on a plane domain every locally finite distribution of principal parts is solvable. In the present language, all the corresponding obstruction classes vanish. On a compact Riemann surface they need not vanish; Serre duality will later identify the obstruction by pairing principal parts with global holomorphic differentials.

Proposition 6.36. *Fix distinct points $p_1, \dots, p_r$ and positive integers $n_1, \dots, n_r$. The sheaf*

$$\mathscr{Q} = \bigoplus_{j=1}^{r} i_{p_j,*}(\mathscr{O}_{X,p_j}/\mathfrak{m}_{p_j}^{n_j})$$

is a coherent torsion sheaf. It records principal parts whose poles at p_j have order at most n_j.

Proof. Each summand is coherent by Proposition 6.13, and a finite direct sum of coherent sheaves is coherent. Every stalk is annihilated by a power of a local coordinate, so the sheaf is torsion. In a coordinate z_j at p_j, the quotient has basis

$$1, z_j, \dots, z_j^{n_j-1}.$$

After multiplication by $z_j^{-n_j}$, this is equivalent to prescribing the coefficients of

$$z_j^{-n_j}, \dots, z_j^{-1},$$

which are exactly the principal parts of pole order at most n_j. □

The full sheaf $\mathscr{P}_X$ is generally not coherent; it is the filtered union of these finite-order torsion pieces. This distinction will become important when divisors are introduced. A divisor bounds the allowed pole orders and produces an invertible, hence coherent, sheaf.

6.7 Euler Characteristics

6.7.1 Definition and additivity

Theorem 6.37 (Finiteness theorem)**.** *Let X be a compact Riemann surface and let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Then every $H^q(X, \mathscr{F})$ is finite-dimensional and*

$$H^q(X, \mathscr{F}) = 0 \qquad (q > 1).$$

Definition 6.38. Let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module. Its *Euler characteristic* is

$$\chi(X, \mathscr{F}) := \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathscr{F}).$$

Since X has cohomological dimension one for coherent sheaves,

$$\boxed{\chi(X, \mathscr{F}) = h^0(X, \mathscr{F}) - h^1(X, \mathscr{F}).}$$

When the space X is fixed, we write simply $\chi(\mathscr{F})$.

Lemma 6.39. *Let*

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

be an exact sequence of finite-dimensional vector spaces. Then

$$\sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} V_j = 0.$$

Proof. Let $d_j : V_j \to V_{j+1}$ be the differentials, and put $B_j = \operatorname{im}(d_{j-1})$, with $B_0 = 0$ and $B_{n+1} = 0$. Exactness gives short exact sequences

$$0 \longrightarrow B_j \longrightarrow V_j \longrightarrow B_{j+1} \longrightarrow 0.$$

Hence

$$\dim V_j = \dim B_j + \dim B_{j+1}.$$

Taking the alternating sum, every $\dim B_j$ appears twice with opposite signs, so the sum is zero. □

Theorem 6.40 (Additivity of the Euler characteristic)**.** *Let*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be a short exact sequence of coherent sheaves on X. Then

$$\boxed{\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'').}$$

Proof. The cohomology long exact sequence terminates because higher coherent cohomology vanishes:

$$\begin{aligned} 0 \longrightarrow H^0(X,\mathscr{F}') \longrightarrow H^0(X,\mathscr{F}) \longrightarrow H^0(X,\mathscr{F}'') \\ \longrightarrow H^1(X,\mathscr{F}') \longrightarrow H^1(X,\mathscr{F}) \longrightarrow H^1(X,\mathscr{F}'') \longrightarrow 0. \end{aligned}$$

All terms are finite-dimensional. Applying theorem 6.39 and rearranging gives

$$\big(h^0(\mathscr{F}) - h^1(\mathscr{F})\big) = \big(h^0(\mathscr{F}') - h^1(\mathscr{F}')\big) + \big(h^0(\mathscr{F}'') - h^1(\mathscr{F}'')\big).$$

This is the required identity. □

Corollary 6.41. *For coherent sheaves $\mathscr{F}$ and $\mathscr{G}$,*

$$\chi(\mathscr{F} \oplus \mathscr{G}) = \chi(\mathscr{F}) + \chi(\mathscr{G}).$$

More generally, let $K_0(\mathrm{Coh}(X))$ be the abelian group generated by symbols $[\mathscr{F}]$, one for each coherent sheaf, modulo the relations

$$[\mathscr{F}] = [\mathscr{F}'] + [\mathscr{F}'']$$

for every short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$. Then the Euler characteristic defines a group homomorphism

$$\chi : K_0\big(\mathrm{Coh}(X)\big) \longrightarrow \mathbb{Z}.$$

Proof. Apply theorem 6.40 to the split exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F} \oplus \mathscr{G} \longrightarrow \mathscr{G} \longrightarrow 0.$$

The defining relations of $K_0(\mathrm{Coh}(X))$ are

$$[\mathscr{F}] = [\mathscr{F}'] + [\mathscr{F}'']$$

for short exact sequences. Additivity shows that $[\mathscr{F}] \mapsto \chi(\mathscr{F})$ respects these relations, so it induces the stated homomorphism. □

6.7.2 Torsion sheaves and their length

Definition 6.42. A coherent $\mathcal{O}_X$-module $\mathscr{T}$ is a *torsion sheaf* if every stalk $\mathscr{T}_p$ is a torsion $\mathcal{O}_{X,p}$-module. Equivalently, $\mathscr{T}$ has generic rank zero.

Proposition 6.43. *The support of a coherent torsion sheaf on a compact Riemann surface is finite.*

Proof. Fix $p \in X$. On a sufficiently small connected coordinate neighbourhood U of p, choose finitely many sections $s_1, \ldots, s_r$ generating $\mathscr{T}|_U$. Since $\mathscr{T}$ is torsion, the germ of each s_i at p is annihilated by a nonzero germ of a holomorphic function. After shrinking U, choose nonzero holomorphic functions $f_i \in \mathcal{O}_X(U)$ such that

$$f_i s_i = 0 \qquad \text{on } U.$$

Their product

$$f = f_1 \cdots f_r$$

annihilates $\mathscr{T}|_U$. At every point $q \in U$ with $f(q) \neq 0$, multiplication by f is invertible in $\mathcal{O}_{X,q}$, and hence $\mathscr{T}_q = 0$. Therefore

$$\operatorname{Supp}(\mathscr{T}) \cap U \subseteq \{q \in U : f(q) = 0\}.$$

The zero set of a nonzero holomorphic function on a connected Riemann surface is discrete. Thus $\operatorname{Supp}(\mathscr{T})$ is locally discrete. It is also closed, since it is the support of a coherent sheaf. A closed discrete subset of a compact space is finite. $\square$

Lemma 6.44. *Let $\mathscr{T}$ be a sheaf of $\mathcal{O}_X$-modules with finite support*

$$\operatorname{Supp}(\mathscr{T}) = \{p_1, \ldots, p_m\}.$$

Let $i_j : \{p_j\} \hookrightarrow X$ be the inclusions. Then the natural germ map gives an isomorphism

$$\boxed{\mathscr{T} \cong \bigoplus_{j=1}^{m} i_{j,*}(\mathscr{T}_{p_j}).}$$

Here $\mathscr{T}_{p_j}$ is regarded as an $\mathcal{O}_X$-module skyscraper sheaf through the homomorphisms $\mathcal{O}_X(U) \to \mathcal{O}_{X,p_j}$.

Proof. For an open set U, send a section $s \in \mathscr{T}(U)$ to the tuple of its germs $(s_{p_j})_{p_j \in U}$. This defines a morphism

$$\mathscr{T} \longrightarrow \bigoplus_j i_{j,*}(\mathscr{T}_{p_j}).$$

It is enough to check that this morphism is an isomorphism on stalks. At a point $q \notin \operatorname{Supp}(\mathscr{T})$, both stalks are zero. At $q = p_j$, all summands except the j-th have zero stalk, and the induced map is the identity

$$\mathscr{T}_{p_j} \longrightarrow \mathscr{T}_{p_j}.$$

Hence the morphism is an isomorphism on every stalk and therefore an isomorphism of sheaves. $\square$

Lemma 6.45. *For a point $p \in X$ and an $\mathcal{O}_{X,p}$-module M, the skyscraper sheaf $i_{p,*}M$ is flasque. Consequently,*

$$H^q(X, i_{p,*}M) = 0 \qquad (q > 0),$$

and

$$H^0(X, i_{p,*}M) = M.$$

Proof. For an open subset $U \subseteq X$,

$$(i_{p,*}M)(U) = \begin{cases} M, & p \in U, \\ 0, & p \notin U. \end{cases}$$

For an inclusion $V \subseteq U$, the restriction map is either the identity of M, the zero map $M \to 0$, or the identity of the zero group. In every case it is surjective, so the sheaf is flasque. Flasque sheaves are acyclic for global sections, proving the vanishing. The formula for global sections follows directly from the definition. ☐

Theorem 6.46 (Cohomology of a torsion sheaf)**.** *Let $\mathscr{T}$ be a coherent torsion sheaf on X. Then*

$$\boxed{H^q(X, \mathscr{T}) = 0 \qquad (q > 0),}$$

and the germ map induces a natural isomorphism

$$\boxed{H^0(X, \mathscr{T}) \cong \bigoplus_{p \in \operatorname{Supp}(\mathscr{T})} \mathscr{T}_p.}$$

Proof. By theorems 6.43 and 6.44, $\mathscr{T}$ is a finite direct sum of skyscraper sheaves. Each summand is flasque by theorem 6.45, and a finite direct sum of flasque sheaves is flasque. Thus all higher cohomology groups vanish. Taking global sections of the skyscraper decomposition gives the stated direct sum of stalks. ☐

Definition 6.47. Let $\mathscr{T}$ be a coherent torsion sheaf. Its *length* is

$$\operatorname{length}(\mathscr{T}) := \sum_{p \in X} \operatorname{length}_{\mathcal{O}_{X,p}}(\mathscr{T}_p).$$

The sum is finite by theorem 6.43.

Lemma 6.48. *Let $R = \mathcal{O}_{X,p}$ and let M be a finite-length R-module. Then*

$$\dim_{\mathbb{C}} M = \operatorname{length}_R(M).$$

Proof. The ring R is a local ring with residue field

$$R/\mathfrak{m}_p \cong \mathbb{C}.$$

Choose a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M.$$

Every simple R-module is isomorphic to the residue field $\mathbb{C}$, so each quotient M_j/M_{j-1} has complex dimension one. Additivity of vector-space dimension in short exact sequences gives

$$\dim_{\mathbb{C}} M = n.$$

By definition, $n = \operatorname{length}_R(M)$. ☐

Theorem 6.49 (Euler characteristic of a torsion sheaf)**.** *For every coherent torsion sheaf $\mathscr{T}$ on X,*

$$\boxed{\chi(X, \mathscr{T}) = h^0(X, \mathscr{T}) = \operatorname{length}(\mathscr{T}) = \sum_{p \in X} \operatorname{length}_{\mathcal{O}_{X,p}}(\mathscr{T}_p).}$$

Proof. By theorem 6.46, $H^1(X, \mathscr{T}) = 0$ and

$$H^0(X, \mathscr{T}) \cong \bigoplus_{p \in \mathrm{Supp}(\mathscr{T})} \mathscr{T}_p.$$

Therefore

$$\chi(X, \mathscr{T}) = h^0(X, \mathscr{T}) = \sum_p \dim_{\mathbb{C}} \mathscr{T}_p.$$

Apply theorem 6.48 to every stalk. □

Corollary 6.50. *Let $\mathbb{C}_p = i_{p,*}\mathbb{C}$ be the one-dimensional skyscraper sheaf at p. Then*

$$H^0(X, \mathbb{C}_p) = \mathbb{C}, \qquad H^q(X, \mathbb{C}_p) = 0 \quad (q > 0),$$

and

$$\boxed{\chi(X, \mathbb{C}_p) = 1.}$$

Proof. The stalk of $\mathbb{C}_p$ at p is the residue field of $\mathcal{O}_{X,p}$ and has length one. Apply theorems 6.46 and 6.49. □

Corollary 6.51 (A filtration by point sheaves)**.** *Every nonzero coherent torsion sheaf $\mathscr{T}$ admits a finite filtration*

$$0 = \mathscr{T}_0 \subset \mathscr{T}_1 \subset \cdots \subset \mathscr{T}_N = \mathscr{T}$$

whose successive quotients are skyscraper sheaves $\mathbb{C}_{p_j}$. The number of factors is

$$N = \mathrm{length}(\mathscr{T}).$$

Proof. By the skyscraper decomposition, it suffices to treat a finite-length module M over one local ring $R = \mathcal{O}_{X,p}$. Choose a composition series of M. Every simple quotient is the residue field $R/\mathfrak{m}_p \cong \mathbb{C}$. Applying the exact functor $i_{p,*}$ gives the required filtration of the corresponding skyscraper sheaf. Taking the direct sum of the filtrations at the finitely many support points produces a filtration of $\mathscr{T}$. Its number of factors is the sum of the local composition lengths, namely $\mathrm{length}(\mathscr{T})$. □

6.7.3 The cohomological and arithmetic genera

Definition 6.52. The *cohomological genus* of the compact connected Riemann surface X is

$$\boxed{g := h^1(X, \mathcal{O}_X).}$$

Proposition 6.53. *For a compact connected Riemann surface,*

$$\boxed{\chi(X, \mathcal{O}_X) = 1 - g.}$$

Proof. Every holomorphic function on a compact connected Riemann surface is constant by the maximum principle. Hence

$$H^0(X, \mathcal{O}_X) = \mathbb{C}$$

and $h^0(X, \mathcal{O}_X) = 1$. By definition, $h^1(X, \mathcal{O}_X) = g$. Therefore

$$\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) = 1 - g.$$

□

Definition 6.54. The *arithmetic genus* of X is

$$p_a(X) := 1 - \chi(X, \mathcal{O}_X).$$

Corollary 6.55. *For a compact connected Riemann surface,*

$$\boxed{p_a(X) = g.}$$

Proof. Substitute theorem 6.53 into the definition:

$$p_a(X) = 1 - (1 - g) = g.$$

□

Remark 6.56. At this stage, g is defined cohomologically. Serre duality will later identify it with

$$h^0(X, \omega_X),$$

the dimension of the space of global holomorphic differentials. The Riemann–Roch theorem will then show that the Euler characteristic of a line bundle is governed by its degree and by this single invariant of the surface.

6.8 Sheaf Hom and Ext Groups

The cohomology groups introduced above are derived functors of global sections. A second family of derived functors arises by fixing the first argument of the Hom functor. These groups are the natural language for extensions and, later, for Serre duality. Throughout this section, $(X, \mathcal{O}_X)$ is a ringed space, and all sheaves are sheaves of $\mathcal{O}_X$-modules.

6.8.1 Sheaf Hom and global Hom

Definition 6.57. Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathcal{O}_X$-modules. The *sheaf Hom* from $\mathscr{F}$ to $\mathscr{G}$ is the sheaf

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$$

whose sections over an open subset $U \subseteq X$ are

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}_{\mathcal{O}_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

The group of global morphisms is

$$\mathrm{Hom}_X(\mathscr{F}, \mathscr{G}) := \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) = \Gamma\big(X, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})\big).$$

Remark 6.58. The notation distinguishes a sheaf from its global sections:

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \quad \text{is a sheaf, whereas} \quad \mathrm{Hom}_X(\mathscr{F}, \mathscr{G}) \quad \text{is an abelian group.}$$

The stalk $\mathscr{H}om(\mathscr{F}, \mathscr{G})_x$ maps naturally to

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x).$$

This map is an isomorphism when $\mathscr{F}$ is locally finitely presented; in particular, it is an isomorphism when $\mathscr{F}$ is coherent on a Riemann surface.

Proposition 6.59. *For fixed $\mathscr{F}$, the functor*

$$\operatorname{Hom}_X(\mathscr{F}, -) : \operatorname{Mod}(\mathcal{O}_X) \longrightarrow \mathbf{Ab}$$

is left exact. For fixed $\mathscr{G}$, the contravariant functor

$$\operatorname{Hom}_X(-, \mathscr{G}) : \operatorname{Mod}(\mathcal{O}_X)^{\mathrm{op}} \longrightarrow \mathbf{Ab}$$

is also left exact.

Proof. Let

$$0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}''$$

be exact. If a morphism $u : \mathscr{F} \to \mathscr{G}$ maps to zero in $\operatorname{Hom}_X(\mathscr{F}, \mathscr{G}'')$, then its image is contained in the kernel $\mathscr{G}'$. By the universal property of the kernel, u factors uniquely through $\mathscr{G}'$. Thus

$$0 \longrightarrow \operatorname{Hom}_X(\mathscr{F}, \mathscr{G}') \longrightarrow \operatorname{Hom}_X(\mathscr{F}, \mathscr{G}) \longrightarrow \operatorname{Hom}_X(\mathscr{F}, \mathscr{G}'')$$

is exact.

Similarly, if

$$\mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

is exact, then a morphism $u : \mathscr{F} \to \mathscr{G}$ vanishes on the image of $\mathscr{F}'$ precisely when it factors uniquely through the cokernel $\mathscr{F}''$. Therefore

$$0 \longrightarrow \operatorname{Hom}_X(\mathscr{F}'', \mathscr{G}) \longrightarrow \operatorname{Hom}_X(\mathscr{F}, \mathscr{G}) \longrightarrow \operatorname{Hom}_X(\mathscr{F}', \mathscr{G})$$

is exact. □

6.8.2 Definition of the Ext groups

The category $\operatorname{Mod}(\mathcal{O}_X)$ has enough injective objects. Hence every $\mathcal{O}_X$-module admits an injective resolution, and the preceding left exact functor has right derived functors.

Definition 6.60. Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathcal{O}_X$-modules. Choose an injective resolution

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots .$$

The *global Ext groups* are

$$\operatorname{Ext}^i_X(\mathscr{F}, \mathscr{G}) := H^i\big(\operatorname{Hom}_X(\mathscr{F}, \mathcal{I}^\bullet)\big), \qquad i \geq 0.$$

Equivalently,

$$\operatorname{Ext}^i_X(\mathscr{F}, -) = R^i \operatorname{Hom}_X(\mathscr{F}, -).$$

Remark 6.61. Even when $\mathscr{F}$ and $\mathscr{G}$ are coherent, the injective sheaves appearing in a resolution need not be coherent. The derived functor is taken in the abelian category of all $\mathcal{O}_X$-modules. Coherence is a finiteness condition on the arguments, not on the chosen resolution.

Proposition 6.62. *There is a natural isomorphism*

$$\operatorname{Ext}^0_X(\mathscr{F}, \mathscr{G}) \cong \operatorname{Hom}_X(\mathscr{F}, \mathscr{G}).$$

Proof. The zeroth right derived functor of a left exact functor is naturally isomorphic to the original functor. Concretely, if $\mathscr{G} \to \mathcal{I}^\bullet$ is an injective resolution, then

$$\operatorname{Ext}^0_X(\mathscr{F}, \mathscr{G}) = \ker\big(\operatorname{Hom}_X(\mathscr{F}, \mathcal{I}^0) \to \operatorname{Hom}_X(\mathscr{F}, \mathcal{I}^1)\big).$$

Exactness of

$$0 \to \mathscr{G} \to \mathcal{I}^0 \to \mathcal{I}^1$$

identifies this kernel with the morphisms $\mathscr{F} \to \mathcal{I}^0$ whose image lies in $\mathscr{G}$, namely with $\operatorname{Hom}_X(\mathscr{F}, \mathscr{G})$. □

Theorem 6.63 (Long exact sequences of Ext)**.** *Let $\mathscr{F}$ be fixed. A short exact sequence*

$$0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$$

induces a natural long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_X(\mathscr{F},\mathscr{G}') \longrightarrow \mathrm{Hom}_X(\mathscr{F},\mathscr{G}) \longrightarrow \mathrm{Hom}_X(\mathscr{F},\mathscr{G}'') \\ &\longrightarrow \mathrm{Ext}^1_X(\mathscr{F},\mathscr{G}') \longrightarrow \mathrm{Ext}^1_X(\mathscr{F},\mathscr{G}) \longrightarrow \mathrm{Ext}^1_X(\mathscr{F},\mathscr{G}'') \longrightarrow \mathrm{Ext}^2_X(\mathscr{F},\mathscr{G}') \longrightarrow \cdots. \end{aligned}$$

Likewise, for fixed $\mathscr{G}$, a short exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

induces a natural long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_X(\mathscr{F}'',\mathscr{G}) \longrightarrow \mathrm{Hom}_X(\mathscr{F},\mathscr{G}) \longrightarrow \mathrm{Hom}_X(\mathscr{F}',\mathscr{G}) \\ &\longrightarrow \mathrm{Ext}^1_X(\mathscr{F}'',\mathscr{G}) \longrightarrow \mathrm{Ext}^1_X(\mathscr{F},\mathscr{G}) \longrightarrow \mathrm{Ext}^1_X(\mathscr{F}',\mathscr{G}) \longrightarrow \mathrm{Ext}^2_X(\mathscr{F}'',\mathscr{G}) \longrightarrow \cdots. \end{aligned}$$

Proof. For the first sequence, choose injective resolutions of $\mathscr{G}'$ and $\mathscr{G}''$. The horseshoe lemma gives an injective resolution of $\mathscr{G}$ and a short exact sequence of resolutions

$$0 \longrightarrow \mathcal{I}'^{\bullet} \longrightarrow \mathcal{I}^{\bullet} \longrightarrow \mathcal{I}''^{\bullet} \longrightarrow 0.$$

Applying $\mathrm{Hom}_X(\mathscr{F}, -)$ gives a short exact sequence of cochain complexes because every term $\mathcal{I}'^q \to \mathcal{I}^q \to \mathcal{I}''^q$ is split exact: the quotient $\mathcal{I}''^q$ is injective, so the surjection $\mathcal{I}^q \to \mathcal{I}''^q$ admits a section. The long exact cohomology sequence of these complexes is the first displayed sequence.

For the second sequence, choose one injective resolution $\mathscr{G} \to \mathcal{I}^{\bullet}$. Since every $\mathcal{I}^q$ is injective, the contravariant functor $\mathrm{Hom}_X(-, \mathcal{I}^q)$ is exact. Consequently

$$0 \longrightarrow \mathrm{Hom}_X(\mathscr{F}'',\mathcal{I}^{\bullet}) \longrightarrow \mathrm{Hom}_X(\mathscr{F},\mathcal{I}^{\bullet}) \longrightarrow \mathrm{Hom}_X(\mathscr{F}',\mathcal{I}^{\bullet}) \longrightarrow 0$$

is a short exact sequence of cochain complexes. Its long exact cohomology sequence gives the second assertion. Naturality follows from the naturality of the horseshoe construction and of connecting homomorphisms. □

6.8.3 Extensions and the Yoneda interpretation

Definition 6.64. An *extension of $\mathscr{F}$ by $\mathscr{G}$* is a short exact sequence

$$0 \longrightarrow \mathscr{G} \xrightarrow{i} \mathscr{E} \xrightarrow{p} \mathscr{F} \longrightarrow 0.$$

Two extensions are *equivalent* if there is an isomorphism between their middle terms making the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{G} & \to & \mathscr{E} & \to & \mathscr{F} & \to & 0 \\ & & \| & & \downarrow\wr & & \| & & \\ 0 & \to & \mathscr{G} & \to & \mathscr{E}' & \to & \mathscr{F} & \to & 0 \end{array}$$

commute.

Theorem 6.65 (Yoneda interpretation of Ext^1)**.** *There is a natural bijection between* $\mathrm{Ext}^1_X(\mathscr{F},\mathscr{G})$ *and equivalence classes of extensions of $\mathscr{F}$ by $\mathscr{G}$. Under this bijection, the zero element corresponds to the split extension*

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{G} \oplus \mathscr{F} \longrightarrow \mathscr{F} \longrightarrow 0.$$

Proof. Choose a monomorphism $j : \mathscr{G} \hookrightarrow \mathcal{I}$ with $\mathcal{I}$ injective, and write $\mathscr{Q} = \mathcal{I}/\mathscr{G}$. The exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathcal{I} \longrightarrow \mathscr{Q} \longrightarrow 0$$

gives

$$\mathrm{Hom}_X(\mathscr{F}, \mathcal{I}) \longrightarrow \mathrm{Hom}_X(\mathscr{F}, \mathscr{Q}) \xrightarrow{\delta} \mathrm{Ext}^1_X(\mathscr{F}, \mathscr{G}) \longrightarrow 0,$$

because $\mathrm{Ext}^1_X(\mathscr{F}, \mathcal{I}) = 0$. Thus $\mathrm{Ext}^1_X(\mathscr{F}, \mathscr{G})$ is the cokernel of

$$\mathrm{Hom}_X(\mathscr{F}, \mathcal{I}) \to \mathrm{Hom}_X(\mathscr{F}, \mathscr{Q}).$$

Given $u : \mathscr{F} \to \mathscr{Q}$, form the pullback

$$\mathscr{E}_u = \mathscr{F} \times_{\mathscr{Q}} \mathcal{I}.$$

There is a cartesian diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{G} & \to & \mathscr{E}_u & \to & \mathscr{F} & \to & 0 \\ & & \| & & \downarrow & & \downarrow u & & \\ 0 & \to & \mathscr{G} & \to & \mathcal{I} & \to & \mathscr{Q} & \to & 0. \end{array}$$

Hence u determines an extension. If u is changed by a morphism that lifts to $\mathcal{I}$, the two pullback extensions are equivalent; therefore the construction depends only on the class of u in the cokernel.

Conversely, let

$$0 \to \mathscr{G} \xrightarrow{i} \mathscr{E} \to \mathscr{F} \to 0$$

be an extension. Injectivity of $\mathcal{I}$ extends $j : \mathscr{G} \to \mathcal{I}$ across the monomorphism i, giving a morphism $v : \mathscr{E} \to \mathcal{I}$ with $v \circ i = j$. Passing to cokernels produces a morphism $u : \mathscr{F} \to \mathscr{Q}$. A different choice of v changes u by a morphism lifted from $\mathcal{I}$, so the resulting cokernel class is well-defined. The pullback associated with this class is canonically equivalent to the original extension. The two constructions are therefore inverse.

Finally, $u = 0$ gives the pullback of $\mathcal{I} \to \mathscr{Q}$ along the zero map. This pullback is $\mathscr{G} \oplus \mathscr{F}$, so the zero class corresponds to the split extension. □

Definition 6.66 (Baer sum)**.** Let ξ_1 and ξ_2 be represented by extensions $\mathscr{E}_1$ and $\mathscr{E}_2$ of $\mathscr{F}$ by $\mathscr{G}$. Their *Baer sum* is obtained by taking the direct-sum extension of $\mathscr{F} \oplus \mathscr{F}$ by $\mathscr{G} \oplus \mathscr{G}$, pulling it back along the diagonal

$$\Delta : \mathscr{F} \longrightarrow \mathscr{F} \oplus \mathscr{F},$$

and then pushing it out along addition

$$+ : \mathscr{G} \oplus \mathscr{G} \longrightarrow \mathscr{G}.$$

Proposition 6.67. *The Baer sum is well-defined on equivalence classes and makes the set of extension classes into an abelian group. Under the Yoneda bijection of theorem 6.65, this group law is the usual addition on* $\mathrm{Ext}^1_X(\mathscr{F}, \mathscr{G})$*.*

Proof. Pullback and pushout preserve equivalences of extensions, so the construction is well-defined on equivalence classes. Under the cokernel model used in the proof of theorem 6.65, extensions are represented by morphisms $u : \mathscr{F} \to \mathscr{Q}$. The direct sum is represented by $u_1 \oplus u_2$, pullback along Δ gives the map

$$\mathscr{F} \xrightarrow{\Delta} \mathscr{F} \oplus \mathscr{F} \xrightarrow{u_1 \oplus u_2} \mathscr{Q} \oplus \mathscr{Q},$$

and pushout along addition corresponds to composing with $\mathscr{Q} \oplus \mathscr{Q} \to \mathscr{Q}$. The resulting map is $u_1 + u_2$. Thus the Baer sum agrees with addition in the cokernel

$$\mathrm{coker}\big(\mathrm{Hom}_X(\mathscr{F}, \mathcal{I}) \to \mathrm{Hom}_X(\mathscr{F}, \mathscr{Q})\big).$$

Associativity, commutativity, the zero extension, and additive inverses therefore follow from the corresponding properties of this abelian group. □

6.8.4 Cohomology as an Ext group and low-degree Yoneda products

Proposition 6.68. *For every $\mathcal{O}_X$-module $\mathscr{F}$ and every $i \geq 0$, there is a natural isomorphism*

$$\boxed{H^i(X, \mathscr{F}) \cong \operatorname{Ext}^i_X(\mathcal{O}_X, \mathscr{F}).}$$

Proof. A morphism $u : \mathcal{O}_X \to \mathscr{F}$ is uniquely determined by the global section $u(1)$, and every global section defines such a morphism by multiplication. Hence there is a natural isomorphism of left exact functors

$$\operatorname{Hom}_X(\mathcal{O}_X, -) \cong \Gamma(X, -).$$

Naturally isomorphic left exact functors have naturally isomorphic right derived functors. Therefore

$$\operatorname{Ext}^i_X(\mathcal{O}_X, \mathscr{F}) = R^i \operatorname{Hom}_X(\mathcal{O}_X, -)(\mathscr{F}) \cong R^i\Gamma(X, -)(\mathscr{F}) = H^i(X, \mathscr{F}).$$

□

For Serre duality only products of total degree one are needed. They can be described entirely in terms of pullbacks and pushouts of ordinary extensions.

Definition 6.69 (Low-degree Yoneda products)**.** Let $\mathscr{E}, \mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules.

(i) If $u : \mathscr{F} \to \mathscr{G}$ and $\eta \in \operatorname{Ext}^1_X(\mathscr{E}, \mathscr{F})$, define

$$u_*\eta \in \operatorname{Ext}^1_X(\mathscr{E}, \mathscr{G})$$

to be the pushout along u of an extension representing η.

(ii) If $\xi \in \operatorname{Ext}^1_X(\mathscr{F}, \mathscr{G})$ and $v : \mathscr{E} \to \mathscr{F}$, define

$$v^*\xi \in \operatorname{Ext}^1_X(\mathscr{E}, \mathscr{G})$$

to be the pullback along v of an extension representing ξ.

These operations are the Yoneda products

$$\operatorname{Ext}^0_X(\mathscr{F}, \mathscr{G}) \times \operatorname{Ext}^1_X(\mathscr{E}, \mathscr{F}) \longrightarrow \operatorname{Ext}^1_X(\mathscr{E}, \mathscr{G})$$

and

$$\operatorname{Ext}^1_X(\mathscr{F}, \mathscr{G}) \times \operatorname{Ext}^0_X(\mathscr{E}, \mathscr{F}) \longrightarrow \operatorname{Ext}^1_X(\mathscr{E}, \mathscr{G}).$$

Proposition 6.70. *The low-degree Yoneda products are well-defined, additive in each variable, and natural. They satisfy the expected associativity relations with composition of morphisms. More precisely,*

$$(u_2 \circ u_1)_*\eta = (u_2)_*\big((u_1)_*\eta\big), \qquad (v_1 \circ v_2)^*\xi = v_2^*\big(v_1^*\xi\big),$$

and pushout commutes with pullback up to a canonical equivalence:

$$v^*(u_*\eta) \cong u_*(v^*\eta).$$

Proof. Suppose η is represented by

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{A} \longrightarrow \mathscr{E} \longrightarrow 0.$$

The pushout along $u : \mathscr{F} \to \mathscr{G}$ is characterized by its universal property. Replacing the extension by an equivalent one produces an isomorphic pushout diagram, so $u_*\eta$ is well-defined. The same argument using the universal property of a fiber product proves that pullback is well-defined.

Additivity follows from the Baer construction: direct sums, pullbacks, and pushouts commute up to canonical isomorphism, so pushing out or pulling back a Baer sum gives the Baer sum of the pushed-out or pulled-back extensions. Naturality is the same universal-property argument applied to commutative squares.

An iterated pushout along u_1 and u_2 has the universal property of the pushout along $u_2 \circ u_1$, giving the first associativity relation. Iterated pullbacks give the second. Finally, the object obtained by first pushing out the kernel and then pulling back the quotient satisfies the same universal property as the object obtained in the opposite order. The resulting canonical isomorphism is an equivalence of extensions, proving the last identity. □

Remark 6.71. By theorem 6.68, the products used in Serre duality are

$$H^{1-i}(X, \mathscr{F}) \times \operatorname{Ext}^i_X(\mathscr{F}, \omega_X) \longrightarrow H^1(X, \omega_X), \qquad i = 0, 1.$$

For $i = 0$, the product is pushout along a morphism $\mathscr{F} \to \omega_X$. For $i = 1$, it is pullback along a global section $\mathcal{O}_X \to \mathscr{F}$. Thus no theory of higher Yoneda extensions is required for the duality theorem on a Riemann surface.

6.8.5 Locally free first arguments

Proposition 6.72. *Let $\mathscr{E}$ be a finite-rank locally free $\mathcal{O}_X$-module. For every $\mathscr{G}$ there is a natural isomorphism*

$$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}) \cong \mathscr{E}^\vee \otimes_{\mathcal{O}_X} \mathscr{G}.$$

Consequently,

$$\operatorname{Hom}_X(\mathscr{E}, \mathscr{G}) \cong \Gamma\big(X, \mathscr{E}^\vee \otimes \mathscr{G}\big).$$

Proof. The assertion is local on X. On an open subset U on which $\mathscr{E}|_U \cong \mathcal{O}_U^{\oplus r}$, both sides identify naturally with $\mathscr{G}|_U^{\oplus r}$. These local identifications are defined by evaluation and are compatible with changes of frame, so they glue globally. Taking global sections gives the second isomorphism. □

Theorem 6.73 (Ext from a locally free sheaf)**.** *Let $\mathscr{E}$ be a finite-rank locally free $\mathcal{O}_X$-module. Then for every $\mathscr{G}$ and every $i \geq 0$ there is a natural isomorphism*

$$\boxed{\operatorname{Ext}^i_X(\mathscr{E}, \mathscr{G}) \cong H^i\big(X, \mathscr{E}^\vee \otimes \mathscr{G}\big).}$$

Proof. The functor $\mathscr{E}^\vee \otimes -$ is exact because $\mathscr{E}^\vee$ is locally free. It also preserves injective objects. Indeed, the tensor-Hom adjunction gives

$$\operatorname{Hom}_X(\mathscr{E} \otimes \mathscr{F}, \mathcal{I}) \cong \operatorname{Hom}_X(\mathscr{F}, \mathscr{E}^\vee \otimes \mathcal{I}).$$

Thus $\mathscr{E}^\vee \otimes -$ is right adjoint to the exact functor $\mathscr{E} \otimes -$; a right adjoint to an exact functor preserves injectives.

Choose an injective resolution $\mathscr{G} \to \mathcal{I}^\bullet$. Then

$$\mathscr{E}^\vee \otimes \mathscr{G} \longrightarrow \mathscr{E}^\vee \otimes \mathcal{I}^\bullet$$

is an injective resolution. By theorem 6.72,

$$\operatorname{Hom}_X(\mathscr{E}, \mathcal{I}^\bullet) \cong \Gamma\big(X, \mathscr{E}^\vee \otimes \mathcal{I}^\bullet\big).$$

Taking cohomology gives

$$\operatorname{Ext}^i_X(\mathscr{E}, \mathscr{G}) = H^i \operatorname{Hom}_X(\mathscr{E}, \mathcal{I}^\bullet) \cong H^i\Gamma\big(X, \mathscr{E}^\vee \otimes \mathcal{I}^\bullet\big) = H^i\big(X, \mathscr{E}^\vee \otimes \mathscr{G}\big).$$

All maps used are natural, so the resulting isomorphism is natural in both arguments. □

Corollary 6.74. *If $\mathscr{L}$ is a line bundle, then*

$$\operatorname{Ext}^i_X(\mathscr{L}, \mathscr{G}) \cong H^i\big(X, \mathscr{L}^{-1} \otimes \mathscr{G}\big).$$

In particular,

$$\operatorname{Ext}^i_X(\mathscr{L}, \omega_X) \cong H^i\big(X, \omega_X \otimes \mathscr{L}^{-1}\big).$$

Proof. For a line bundle, $\mathscr{L}^\vee = \mathscr{L}^{-1}$. Apply theorem 6.73. □

Remark 6.75. The preceding theorem is the only general Ext computation needed before duality. Local Ext groups over the discrete valuation rings $\mathcal{O}_{X,p}$ will be calculated separately when torsion sheaves are considered. No local-to-global spectral sequence is required for the arguments that follow.

Chapter 7

Divisors and Line Bundles on Riemann Surfaces

The local theory of a meromorphic function records the orders of its zeros and poles. Divisors assemble these integers into a global object. Cartier divisors express the same information by local meromorphic equations, and the corresponding transition functions produce holomorphic line bundles. The purpose of this chapter is to establish these constructions and their basic relations before the introduction of differentials, Serre duality, and the Riemann–Roch theorem.

Throughout the chapter, X denotes a connected Riemann surface, $\mathscr{O}_X$ its structure sheaf, and $\mathscr{M}_X$ its sheaf of meromorphic functions.

7.1 Orders of meromorphic functions

7.1.1 Zeros and poles in local coordinates

Definition 7.1. Let $p \in X$ and let $0 \neq f \in \mathscr{M}_{X,p}$. Choose a local coordinate z centred at p. There is a unique integer $n \in \mathbb{Z}$ and a unit $u \in \mathscr{O}_{X,p}^\times$ such that

$$f = z^n u.$$

The integer n is called the *order of f at p* and is denoted by

$$\operatorname{ord}_p(f) = n.$$

If $n > 0$, then f has a zero of order n at p; if $n < 0$, then f has a pole of order $-n$ at p.

Proposition 7.2 (Independence of the local coordinate)**.** *The integer* $\operatorname{ord}_p(f)$ *is independent of the local coordinate used in its definition.*

Proof. Let z and w be two local coordinates centred at p. The transition map is biholomorphic and has nonzero derivative at the origin, so in the local ring one may write

$$w = zv$$

for some $v \in \mathscr{O}_{X,p}^\times$. If $f = z^n u$ with $u \in \mathscr{O}_{X,p}^\times$, then

$$f = w^n v^{-n} u.$$

The factor $v^{-n}u$ is again a unit. Thus the exponent obtained from the coordinate w is also n. □

Proposition 7.3 (Valuation properties)**.** *Let* $0 \neq f, g \in \mathscr{M}_{X,p}$*. Then*

$$\operatorname{ord}_p(fg) = \operatorname{ord}_p(f) + \operatorname{ord}_p(g),$$
$$\operatorname{ord}_p(f/g) = \operatorname{ord}_p(f) - \operatorname{ord}_p(g).$$

If $f + g \neq 0$*, then*

$$\operatorname{ord}_p(f + g) \geq \min\{\operatorname{ord}_p(f), \operatorname{ord}_p(g)\},$$

with equality when $\operatorname{ord}_p(f) \neq \operatorname{ord}_p(g)$*.*

Proof. Write

$$f = z^m u, \qquad g = z^n v,$$

where u and v are units. Then

$$fg = z^{m+n}uv, \qquad \frac{f}{g} = z^{m-n}\frac{u}{v},$$

and the displayed factors are units. For the sum, assume $m \leq n$. Then

$$f + g = z^m\big(u + z^{n-m}v\big).$$

The expression in parentheses is holomorphic. Hence the order of the sum is at least m. If $m < n$, its value at p is $u(p) \neq 0$, so it is a unit and the order is exactly m. □

Proposition 7.4 (The local discrete valuation ring)**.** *For every* $p \in X$*, the local ring* $\mathscr{O}_{X,p}$ *is a discrete valuation ring with uniformizer any local coordinate* z *centred at* p*. Its fraction field is* $\mathscr{M}_{X,p}$*, and its valuation is* ord_p*.*

Proof. A coordinate identifies $\mathscr{O}_{X,p}$ with the ring $\mathbb{C}\{z\}$ of convergent power-series germs at the origin. Every nonzero element has a unique factorization $z^n u$ with $n \geq 0$ and u a unit. Consequently every nonzero ideal contains an element of least order, and such an element generates the ideal. Thus the unique nonzero prime ideal is the maximal ideal (z), and the ring is a discrete valuation ring. Allowing negative powers of z gives precisely meromorphic germs, so

$$\operatorname{Frac}(\mathscr{O}_{X,p}) = \mathscr{M}_{X,p}.$$

The exponent in the factorization is exactly ord_p. □

Proposition 7.5 (Relation with local degree)**.** *Let* $f : X \to \mathbb{P}^1(\mathbb{C})$ *be a nonconstant meromorphic function.*

(a) *If* $f(p) = 0$*, then* $\operatorname{ord}_p(f)$ *is the local degree of* f *at* p*.*

(b) *If* $f(p) = \infty$*, then* $-\operatorname{ord}_p(f)$ *is the local degree of* f *at* p*.*

Proof. If $f(p) = 0$, choose a coordinate z at p and the affine coordinate w at $0 \in \mathbb{P}^1$. The local normal form for a nonconstant holomorphic map provides coordinates in which

$$w \circ f = z^e,$$

where e is the local degree. Hence $\operatorname{ord}_p(f) = e$. If $f(p) = \infty$, use the coordinate $w = 1/\zeta$ centred at infinity. Then $1/f$ has a zero whose order is the local degree, so

$$-\operatorname{ord}_p(f) = \operatorname{ord}_p(1/f) = e.$$

□

7.1.2 Local finiteness of zeros and poles

Proposition 7.6. *Let f be a nonzero meromorphic function on an open subset $U \subseteq X$. The set*

$$\{p \in U : \operatorname{ord}_p(f) \neq 0\}$$

is locally finite in U.

Proof. Fix $p \in U$. If p is a pole, then after shrinking to a coordinate neighbourhood V of p, the function $1/f$ is holomorphic and has an isolated zero at p. Shrinking again, p is the only pole in V. Away from the poles, f is holomorphic. If it is not identically zero on a connected coordinate neighbourhood, its zeros are isolated by the identity theorem. Since the germ of f is nonzero at every point, one may shrink around each point so that only finitely many, in fact at most one after further shrinking, zeros or poles occur. Thus the set is locally finite. □

7.2 Divisors

7.2.1 Formal sums of points

Definition 7.7. A *divisor* on X is a locally finite formal sum

$$D = \sum_{p \in X} n_p[p], \qquad n_p \in \mathbb{Z}.$$

Local finiteness means that every point of X has a neighbourhood meeting only finitely many points for which $n_p \neq 0$. The coefficient n_p is also denoted by $\operatorname{ord}_p(D)$. The set

$$\operatorname{Supp}(D) = \{p \in X : n_p \neq 0\}$$

is called the *support* of D.

The sum and negative of divisors are defined coefficientwise. The resulting abelian group is denoted by

$$\operatorname{Div}(X).$$

Definition 7.8. A divisor $D = \sum n_p[p]$ is *effective*, written $D \geq 0$, if $n_p \geq 0$ for every p. For divisors D and E, write

$$D \leq E$$

when $E - D$ is effective.

Definition 7.9. Suppose that X is compact. The *degree* of a divisor is

$$\deg D = \sum_{p \in X} n_p.$$

The sum is finite because a locally finite subset of a compact space is finite.

7.2.2 Principal divisors and linear equivalence

Definition 7.10. Let $0 \neq f \in \mathscr{M}_X(X)$. The *principal divisor* of f is

$$\operatorname{div}(f) = \sum_{p \in X} \operatorname{ord}_p(f)[p].$$

The subgroup of principal divisors is denoted by

$$\operatorname{Prin}(X) = \{\operatorname{div}(f) : f \in \mathscr{M}_X(X)^\times\}.$$

This definition is legitimate by Proposition 7.6.

Proposition 7.11. *For nonzero global meromorphic functions f and g,*

$$\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g),$$
$$\operatorname{div}(f/g) = \operatorname{div}(f) - \operatorname{div}(g).$$

In particular, $f \mapsto \operatorname{div}(f)$ is a group homomorphism from $\mathscr{M}_X(X)^\times$ to $\operatorname{Div}(X)$.

Proof. Both identities follow coefficientwise from Proposition 7.3. □

Definition 7.12. Two divisors D and E are *linearly equivalent*, written $D \sim E$, if

$$D - E = \operatorname{div}(f)$$

for some $f \in \mathscr{M}_X(X)^\times$. The *divisor class group* is

$$\operatorname{Cl}(X) = \operatorname{Div}(X)/\operatorname{Prin}(X).$$

Theorem 7.13 (Degree of a principal divisor)**.** *Let X be compact and connected. For every nonzero meromorphic function f on X,*

$$\deg \operatorname{div}(f) = 0.$$

Proof. If f is constant, the assertion is immediate. Otherwise $f : X \to \mathbb{P}^1(\mathbb{C})$ is a nonconstant holomorphic map. For every $a \in \mathbb{P}^1(\mathbb{C})$, the sum of the local degrees over the fibre $f^{-1}(a)$ is independent of a and equals the degree of the map. Applying this to $a = 0$ and $a = \infty$ and using Proposition 7.5 gives

$$\sum_{f(p)=0} \operatorname{ord}_p(f) = \deg(f) = \sum_{f(p)=\infty} -\operatorname{ord}_p(f).$$

The difference of these two sums is $\deg \operatorname{div}(f)$, so it is zero. □

Corollary 7.14. *On a compact connected Riemann surface, linearly equivalent divisors have the same degree. Thus degree descends to a homomorphism*

$$\deg : \operatorname{Cl}(X) \longrightarrow \mathbb{Z}.$$

Proof. If $D - E = \operatorname{div}(f)$, then Theorem 7.13 gives

$$\deg D - \deg E = \deg \operatorname{div}(f) = 0.$$

Additivity of degree is immediate from its definition. □

7.2.3 Meromorphic functions bounded by a divisor

Definition 7.15. For a divisor D on X, define

$$L(D) = \{0\} \cup \left\{ f \in \mathscr{M}_X(X)^\times : \operatorname{div}(f) + D \geq 0 \right\}.$$

When $L(D)$ is finite dimensional, write

$$\ell(D) = \dim_{\mathbb{C}} L(D).$$

The condition means that a positive coefficient of D permits a pole of the corresponding order, while a negative coefficient forces a zero.

Proposition 7.16. *The set $L(D)$ is a complex vector space. If $D \leq E$, then*

$$L(D) \subseteq L(E).$$

Proof. Let $f, g \in L(D)$ and $a, b \in \mathbb{C}$. The assertion is clear if $af + bg = 0$. Otherwise, for every p,

$$\begin{aligned} \operatorname{ord}_p(af + bg) &\geq \min\{\operatorname{ord}_p(f), \operatorname{ord}_p(g)\} \\ &\geq -\operatorname{ord}_p(D), \end{aligned}$$

so $af + bg \in L(D)$. Thus $L(D)$ is a vector space. If $D \leq E$, then $E - D \geq 0$, and

$$\operatorname{div}(f) + E = \big(\operatorname{div}(f) + D\big) + (E - D) \geq 0$$

for every $f \in L(D)$. □

Proposition 7.17. *Suppose that*

$$E = D + \operatorname{div}(g)$$

for some $g \in \mathscr{M}_X(X)^\times$. Multiplication by g defines an isomorphism

$$L(E) \xrightarrow{\sim} L(D).$$

Equivalently, multiplication by g^{-1} defines $L(D) \cong L(E)$.

Proof. For a nonzero meromorphic function f,

$$\begin{aligned} \operatorname{div}(gf) + D &= \operatorname{div}(f) + \operatorname{div}(g) + D \\ &= \operatorname{div}(f) + E. \end{aligned}$$

Thus $f \in L(E)$ exactly when $gf \in L(D)$. The inverse map is multiplication by g^{-1}. □

Proposition 7.18. *Let X be compact and connected. If $\deg D < 0$, then*

$$L(D) = 0.$$

Proof. Suppose that $0 \neq f \in L(D)$. Then $\operatorname{div}(f) + D$ is effective, so

$$0 \leq \deg\big(\operatorname{div}(f) + D\big) = \deg D,$$

where Theorem 7.13 was used. This contradicts $\deg D < 0$. □

Theorem 7.19 (Finite dimensionality)**.** *Let X be compact and connected. For every divisor D, the vector space $L(D)$ is finite dimensional.*

Proof. Write

$$D = D_+ - D_-$$

with D_+ and D_- effective and disjointly supported. By Proposition 7.16,

$$L(D) \subseteq L(D_+),$$

so it suffices to treat an effective divisor

$$E = \sum_{j=1}^{r} n_j [p_j].$$

Choose a local coordinate z_j at p_j. For $f \in L(E)$, the Laurent expansion at p_j has no terms below degree $-n_j$. Let

$$\mathrm{pp}_{p_j}(f) = \sum_{k=1}^{n_j} a_{j,-k} z_j^{-k}$$

be its principal part. The map

$$\Phi : L(E) \longrightarrow \bigoplus_{j=1}^{r} \mathrm{span}_{\mathbb{C}}\{z_j^{-1}, \dots, z_j^{-n_j}\}, \qquad f \longmapsto \big(\mathrm{pp}_{p_j}(f)\big)_j,$$

is linear. Its kernel consists of global holomorphic functions on X. Every such function is constant by the maximum principle. Hence

$$\dim_{\mathbb{C}} \ker \Phi = 1.$$

The target has dimension $\sum_j n_j = \deg E$, so

$$\dim_{\mathbb{C}} L(E) \le 1 + \deg E.$$

Therefore $L(D)$ is finite dimensional. □

7.3 Cartier divisors

7.3.1 Local meromorphic equations

Definition 7.20. The *Cartier divisor sheaf* on X is the quotient sheaf of abelian groups

$$\mathscr{D}_X = \mathscr{M}_X^\times / \mathscr{O}_X^\times.$$

A *Cartier divisor* is a global section of this quotient sheaf:

$$D \in \Gamma(X, \mathscr{D}_X).$$

Concretely, a Cartier divisor is represented by an open cover $\mathfrak{U} = \{U_i\}$ and nonzero meromorphic functions

$$f_i \in \mathscr{M}_X^\times(U_i)$$

such that

$$\frac{f_i}{f_j} \in \mathscr{O}_X^\times(U_i \cap U_j).$$

Two such systems represent the same Cartier divisor if, after passing to a common refinement, their local equations differ by nowhere-vanishing holomorphic functions.

Definition 7.21. A Cartier divisor is *principal* if it is represented by the restrictions of one global nonzero meromorphic function $f \in \mathscr{M}_X(X)^\times$.

Proposition 7.22. *Let a Cartier divisor D be represented near p by a local meromorphic equation f. Then*

$$\mathrm{ord}_p(D) := \mathrm{ord}_p(f)$$

is independent of the representative.

Proof. Any other local representative has the form fu with $u \in \mathscr{O}_{X,p}^\times$. Since $\mathrm{ord}_p(u) = 0$, Proposition 7.3 gives

$$\mathrm{ord}_p(fu) = \mathrm{ord}_p(f).$$

□

Proposition 7.23 (The divisor sequence)**.** *There is an exact sequence of sheaves of abelian groups*

$$1 \longrightarrow \mathscr{O}_X^\times \longrightarrow \mathscr{M}_X^\times \longrightarrow \mathscr{D}_X \longrightarrow 1.$$

Proof. The first map is the inclusion. Its image is precisely the kernel of the quotient map by definition. The final map is surjective as a morphism of sheaves because $\mathscr{D}_X$ is the sheaf quotient: every germ of a Cartier divisor is locally represented by a nonzero meromorphic function. □

7.3.2 Equivalence with formal divisors

Theorem 7.24 (Formal divisors and Cartier divisors)**.** *The assignment*

$$D \longmapsto \sum_{p \in X} \operatorname{ord}_p(D)[p]$$

defines a natural isomorphism

$$\Gamma(X, \mathscr{D}_X) \xrightarrow{\sim} \operatorname{Div}(X).$$

Under this isomorphism, principal Cartier divisors correspond to principal formal divisors.

Proof. Let D be a Cartier divisor. On each member U_i of a representing cover, choose a local equation f_i. By Proposition 7.22, the integer $\operatorname{ord}_p(f_i)$ is independent of i whenever $p \in U_i$. The zeros and poles of each f_i are locally finite, so the resulting formal sum is locally finite. The construction is additive because orders are additive.

Conversely, let

$$A = \sum_{p \in X} n_p[p]$$

be a formal divisor. By local finiteness, choose an open cover such that each member contains at most one point of $\operatorname{Supp}(A)$. On a member containing no support point, take the local equation 1. On a member containing a support point p, choose a coordinate z_p centred at p and take the local equation $z_p^{n_p}$. On an overlap, the two local equations have the same order at every point. Their quotient therefore has neither zeros nor poles and is a nowhere-vanishing holomorphic function. Hence the local equations define a Cartier divisor.

Starting with a formal divisor, this construction plainly recovers its coefficients. Starting with Cartier data, replacing the local equations by coordinate powers having the same orders changes them only by local holomorphic units, so it recovers the original Cartier divisor. Thus the two constructions are inverse. If the Cartier divisor is represented by a global function f, its coefficient at p is $\operatorname{ord}_p(f)$, so principal divisors correspond. □

Remark 7.25. On a smooth complex curve, the distinction between formal divisors and Cartier divisors disappears because every local ring is a discrete valuation ring. We shall therefore use the same symbol D for either description, choosing the one most convenient for a given argument.

7.4 The invertible sheaf associated with a divisor

7.4.1 Definition and local triviality

Definition 7.26. Let $D = \sum n_p[p]$ be a divisor. Define a subsheaf of $\mathscr{M}_X$ by

$$\mathscr{O}_X(D)(U) = \{0\} \cup \left\{ f \in \mathscr{M}_X(U)^\times : \operatorname{ord}_p(f) + n_p \geq 0 \text{ for every } p \in U \right\}.$$

It is called the *invertible sheaf associated with D*.

If D is represented on U_i by a local meromorphic equation f_i, then one expects

$$\mathscr{O}_X(D)|_{U_i} = f_i^{-1}\mathscr{O}_{U_i}.$$

The next proposition makes this precise.

Proposition 7.27. *Let D be represented on U by a nonzero meromorphic function f. Then*

$$\mathscr{O}_X(D)|_U = f^{-1}\mathscr{O}_U$$

as subsheaves of $\mathscr{M}_X|_U$.

Proof. For a meromorphic function g on an open subset $V \subseteq U$, the condition $g \in \mathscr{O}_X(D)(V)$ is

$$\mathrm{ord}_p(g) + \mathrm{ord}_p(f) \geq 0$$

for every $p \in V$. By additivity of order, this is equivalent to

$$\mathrm{ord}_p(fg) \geq 0$$

for every $p \in V$, which means exactly that fg is holomorphic on V. Thus $g = f^{-1}h$ for some $h \in \mathscr{O}_X(V)$. □

Theorem 7.28. *For every divisor D, the sheaf $\mathscr{O}_X(D)$ is locally free of rank one. In particular, it is an invertible and coherent $\mathscr{O}_X$-module.*

Proof. Choose Cartier local equations f_i for D. By Proposition 7.27, multiplication by f_i gives an isomorphism

$$\mathscr{O}_X(D)|_{U_i} \xrightarrow{\sim} \mathscr{O}_{U_i}.$$

Hence $\mathscr{O}_X(D)$ is locally free of rank one. Finite-rank locally free sheaves are coherent because $\mathscr{O}_X$ is coherent. □

Example 7.29. If $p \in X$ and z is a coordinate centred at p, then near p one has

$$\mathscr{O}_X(np) = z^{-n}\mathscr{O}_X.$$

Thus a positive coefficient allows a pole, while a negative coefficient imposes a zero.

7.4.2 Tensor products, duals, and global sections

Proposition 7.30. *For divisors D and E, there are natural isomorphisms*

$$\begin{aligned} \mathscr{O}_X(0) &\cong \mathscr{O}_X, \\ \mathscr{O}_X(D+E) &\cong \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{O}_X(E), \\ \mathscr{O}_X(-D) &\cong \mathscr{O}_X(D)^\vee. \end{aligned}$$

Proof. The first isomorphism follows directly from the definition. Choose local equations f_i for D and g_i for E on a common cover. Then $f_i g_i$ is a local equation for $D+E$, and locally

$$\begin{aligned} \mathscr{O}_X(D) \otimes \mathscr{O}_X(E) &= (f_i^{-1}\mathscr{O}_{U_i}) \otimes (g_i^{-1}\mathscr{O}_{U_i}) \\ &\cong (f_i g_i)^{-1}\mathscr{O}_{U_i} = \mathscr{O}_X(D+E)|_{U_i}. \end{aligned}$$

These local multiplication maps agree on overlaps and therefore glue. Setting $E = -D$ gives

$$\mathscr{O}_X(D) \otimes \mathscr{O}_X(-D) \cong \mathscr{O}_X.$$

For an invertible sheaf, its tensor inverse is canonically its dual, yielding the final isomorphism. □

Proposition 7.31. *There is a natural identification*

$$H^0(X, \mathscr{O}_X(D)) = L(D).$$

Consequently, when X is compact,

$$h^0(X, \mathscr{O}_X(D)) = \ell(D).$$

Proof. A global section of $\mathscr{O}_X(D)$ is, by definition, a global meromorphic function f satisfying

$$\mathrm{ord}_p(f) + \mathrm{ord}_p(D) \geq 0$$

for every $p \in X$. This is exactly the defining condition for membership in $L(D)$. $\square$

Proposition 7.32. *If $E = D + \mathrm{div}(g)$, then multiplication by g gives an isomorphism*

$$\mathscr{O}_X(E) \xrightarrow{\sim} \mathscr{O}_X(D).$$

In particular, linearly equivalent divisors determine isomorphic invertible sheaves.

Proof. For every open subset U and every meromorphic section f,

$$\begin{aligned} \mathrm{div}(gf) + D &= \mathrm{div}(f) + \mathrm{div}(g) + D \\ &= \mathrm{div}(f) + E. \end{aligned}$$

Thus f is a section of $\mathscr{O}_X(E)$ exactly when gf is a section of $\mathscr{O}_X(D)$. Multiplication by g^{-1} is the inverse. $\square$

7.5 Holomorphic line bundles

7.5.1 Bundles, locally free sheaves, and transition functions

Definition 7.33. A *holomorphic line bundle* over X is a holomorphic map $\pi : L \to X$ whose fibres are one-dimensional complex vector spaces and for which every point has a neighbourhood U admitting a fibrewise linear biholomorphism

$$\pi^{-1}(U) \cong U \times \mathbb{C}.$$

A holomorphic section over U is a holomorphic map $s : U \to L$ satisfying $\pi \circ s = \mathrm{id}_U$.

Definition 7.34. An *invertible sheaf* on X is a locally free $\mathscr{O}_X$-module of rank one. A local generator of an invertible sheaf is called a *local frame.*

Theorem 7.35 (Line bundles and invertible sheaves)**.** *The construction sending a holomorphic line bundle to its sheaf of holomorphic sections gives an equivalence between the category of holomorphic line bundles on X and the category of invertible $\mathscr{O}_X$-modules.*

Proof. Let $L \to X$ be a holomorphic line bundle. Over a trivializing open set U, a holomorphic section is uniquely a holomorphic function times the standard basis vector of $\mathbb{C}$. Hence the sheaf of sections is locally isomorphic to $\mathscr{O}_U$ and is invertible.

Conversely, let $\mathscr{L}$ be invertible. Choose an open cover $\{U_i\}$ and frames $e_i \in \mathscr{L}(U_i)$. On an overlap there is a unique $g_{ij} \in \mathscr{O}_X^\times(U_i \cap U_j)$ such that

$$e_i = g_{ij} e_j.$$

The functions satisfy

$$g_{ii} = 1, \qquad g_{ij} = g_{ji}^{-1}, \qquad g_{ij} g_{jk} = g_{ik}.$$

Glue the trivial bundles $U_i \times \mathbb{C}$ by

$$(x, v)_i \sim (x, g_{ij}(x)v)_j.$$

The cocycle identities make this an equivalence relation and give a holomorphic line bundle. Its sheaf of sections has frames corresponding to the constant section 1 on each U_i, with the same transition functions as $\mathscr{L}$, and is therefore isomorphic to $\mathscr{L}$.

Morphisms are locally multiplication by holomorphic functions, so the two constructions also agree on morphisms and are mutually quasi-inverse. □

Proposition 7.36 (Change of frame). *Let e_i be local frames with transition functions g_{ij}. If*

$$e_i' = h_i e_i, \qquad h_i \in \mathscr{O}_X^\times(U_i),$$

then the new transition functions are

$$g_{ij}' = h_i g_{ij} h_j^{-1}.$$

Conversely, transition systems related by such a formula define isomorphic line bundles.

Proof. On $U_i \cap U_j$,

$$e_i' = h_i e_i = h_i g_{ij} e_j = h_i g_{ij} h_j^{-1} e_j'.$$

This gives the formula. Conversely, multiplication by h_i on each trivial piece intertwines the two gluing relations, so the local maps glue to a line bundle isomorphism. □

Definition 7.37. For line bundles $\mathscr{L}$ and $\mathscr{E}$, their tensor product and dual are the invertible sheaves

$$\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{E}, \qquad \mathscr{L}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X).$$

If g_{ij} and h_{ij} are transition functions, then those of the tensor product are $g_{ij}h_{ij}$ and those of the dual are g_{ij}^{-1}.

7.5.2 Meromorphic sections

Definition 7.38. A *meromorphic section* of an invertible sheaf $\mathscr{L}$ is a global section of

$$\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}_X.$$

If e_i is a local frame, a nonzero meromorphic section has the form

$$s = f_i e_i, \qquad f_i \in \mathscr{M}_X^\times(U_i).$$

Proposition 7.39. *For a nonzero meromorphic section s of $\mathscr{L}$, the integer*

$$\mathrm{ord}_p(s) := \mathrm{ord}_p(f_i)$$

is independent of the local frame. The sum

$$\mathrm{div}(s) = \sum_{p \in X} \mathrm{ord}_p(s)[p]$$

is a well-defined divisor.

Proof. If $e_i' = h_i e_i$ is another local frame, then

$$s = f_i e_i = f_i h_i^{-1} e_i'.$$

Since h_i is a holomorphic unit,

$$\mathrm{ord}_p(f_i h_i^{-1}) = \mathrm{ord}_p(f_i).$$

Thus the order is frame-independent. In each local frame the zeros and poles of the coefficient f_i are locally finite, so the resulting formal sum is a divisor. □

Proposition 7.40. *If an invertible sheaf $\mathscr{L}$ admits a nonzero meromorphic section s, then*

$$\mathscr{L} \cong \mathscr{O}_X(\mathrm{div}(s)).$$

Proof. Choose local frames e_i and write $s = f_i e_i$. The divisor $D = \mathrm{div}(s)$ is represented by the Cartier equations f_i, because on overlaps f_j/f_i is a holomorphic unit. The sheaf $\mathscr{O}_X(D)$ has local frame f_i^{-1}. Define a local isomorphism by

$$f_i^{-1} \longmapsto e_i.$$

If $e_i = g_{ij} e_j$, then the equality $f_i e_i = f_j e_j$ gives $f_j = f_i g_{ij}$, and hence

$$f_i^{-1} = g_{ij} f_j^{-1}.$$

Thus the local maps agree on overlaps and glue to the required isomorphism. □

7.6 The Picard group

7.6.1 Definition and cocycle classification

Definition 7.41. The *Picard group* of X is the set of isomorphism classes of holomorphic line bundles, with operation

$$[\mathscr{L}] + [\mathscr{E}] = [\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{E}].$$

It is denoted by $\mathrm{Pic}(X)$.

Proposition 7.42. *The Picard group is an abelian group. Its identity is $[\mathscr{O}_X]$, and the inverse of $[\mathscr{L}]$ is $[\mathscr{L}^\vee]$.*

Proof. Associativity and commutativity are induced by the canonical associativity and symmetry isomorphisms for tensor products. The natural map $\mathscr{O}_X \otimes \mathscr{L} \to \mathscr{L}$ gives the identity. Since $\mathscr{L}$ is locally free of rank one, evaluation induces an isomorphism

$$\mathscr{L} \otimes \mathscr{L}^\vee \xrightarrow{\sim} \mathscr{O}_X,$$

so $[\mathscr{L}^\vee]$ is the inverse class. □

For a sheaf $\mathscr{A}$ of abelian groups, let $\check{H}^1(X, \mathscr{A})$ denote the group of Čech 1-cocycles taken over all open covers, modulo coboundaries and refinement. The next lemma relates this description to derived-functor cohomology.

Lemma 7.43 (Degree-one Čech comparison)**.** *For every sheaf $\mathscr{A}$ of abelian groups on a topological space X, there is a natural isomorphism*

$$\check{H}^1(X, \mathscr{A}) \cong H^1(X, \mathscr{A}).$$

Proof. Embed $\mathscr{A}$ into an injective, hence flasque, sheaf $\mathscr{I}$, and put $\mathscr{Q} = \mathscr{I}/\mathscr{A}$. Since $\mathscr{I}$ is acyclic, the long exact sequence gives

$$H^1(X, \mathscr{A}) \cong \mathscr{Q}(X)/\operatorname{im} \mathscr{I}(X).$$

Let $q \in \mathscr{Q}(X)$. Choose an open cover $\{U_i\}$ and local lifts $i_i \in \mathscr{I}(U_i)$. On overlaps, the differences

$$a_{ij} = i_j - i_i$$

lie in $\mathscr{A}(U_i \cap U_j)$ and form a Čech 1-cocycle. Changing the local lifts changes a_{ij} by a coboundary, while changing q by the image of a global section of $\mathscr{I}$ does not change its class. This defines a map

$$\mathscr{Q}(X)/\operatorname{im}\mathscr{I}(X) \longrightarrow \check{H}^1(X, \mathscr{A}).$$

Conversely, let a_{ij} be a Čech 1-cocycle on a cover. Its image in $\mathscr{I}$ is a coboundary because a flasque sheaf has vanishing positive Čech cohomology. Hence there are sections $i_i \in \mathscr{I}(U_i)$ such that

$$a_{ij} = i_j - i_i.$$

The images of the i_i in $\mathscr{Q}$ agree on overlaps and glue to a global section $q \in \mathscr{Q}(X)$. Altering the choices changes q by the image of a global section of $\mathscr{I}$. The two constructions are inverse and natural. □

Theorem 7.44 (Cohomological classification of line bundles)**.** *There is a natural isomorphism of abelian groups*

$$\operatorname{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times).$$

Proof. Choose local frames e_i of a line bundle $\mathscr{L}$. The transition functions $g_{ij} \in \mathscr{O}_X^\times(U_i \cap U_j)$ satisfy

$$g_{ij}g_{jk} = g_{ik},$$

so they form a Čech 1-cocycle. By Proposition 7.36, changing the frames changes the cocycle by a coboundary. Isomorphic line bundles determine the same class.

Conversely, a Čech 1-cocycle g_{ij} glues the trivial line bundles $U_i \times \mathbb{C}$ by

$$(x, v)_i \sim (x, g_{ij}(x)v)_j.$$

A coboundary change produces an isomorphic bundle. These constructions are inverse, and multiplication of cocycles corresponds to tensor product of line bundles. Therefore

$$\operatorname{Pic}(X) \cong \check{H}^1(X, \mathscr{O}_X^\times).$$

Apply Lemma 7.43. □

7.6.2 Divisor classes and line bundles

Proposition 7.45. *The assignment*

$$D \longmapsto [\mathscr{O}_X(D)]$$

defines a group homomorphism

$$\operatorname{Div}(X) \longrightarrow \operatorname{Pic}(X).$$

Proof. By Proposition 7.30,

$$\mathscr{O}_X(D + E) \cong \mathscr{O}_X(D) \otimes \mathscr{O}_X(E),$$

which is exactly the homomorphism property. □

Theorem 7.46. *The kernel of the homomorphism*

$$\operatorname{Div}(X) \longrightarrow \operatorname{Pic}(X)$$

is $\operatorname{Prin}(X)$*. Consequently there is a natural injection*

$$\operatorname{Cl}(X) \hookrightarrow \operatorname{Pic}(X).$$

Proof. If $D = \operatorname{div}(f)$, multiplication by f gives an isomorphism

$$\mathscr{O}_X(D) \xrightarrow{\sim} \mathscr{O}_X$$

by Proposition 7.32. Thus every principal divisor lies in the kernel.

Conversely, suppose that $\mathscr{O}_X(D) \cong \mathscr{O}_X$. The image of the constant section 1 under an isomorphism $\mathscr{O}_X \to \mathscr{O}_X(D)$ is a global section s that generates $\mathscr{O}_X(D)$ at every point. Under the inclusion $\mathscr{O}_X(D) \subseteq \mathscr{M}_X$, regard s as a nonzero meromorphic function. If a local equation for D is f_i, then $\mathscr{O}_X(D)|_{U_i} = f_i^{-1}\mathscr{O}_{U_i}$, and the generator s has the form

$$s = f_i^{-1} u_i$$

with $u_i \in \mathscr{O}_X^{\times}(U_i)$. Hence

$$\operatorname{ord}_p(s) = -\operatorname{ord}_p(D)$$

for every p, so

$$\operatorname{div}(s) = -D.$$

Therefore $D = \operatorname{div}(s^{-1})$ is principal. The induced map on the quotient is injective. □

Proposition 7.47 (The connecting homomorphism)**.** *Under the identification*

$$\operatorname{Pic}(X) \cong H^1(X, \mathscr{O}_X^{\times}),$$

the connecting homomorphism associated with

$$1 \to \mathscr{O}_X^{\times} \to \mathscr{M}_X^{\times} \to \mathscr{D}_X \to 1$$

sends a Cartier divisor D to $[\mathscr{O}_X(D)]$.

Proof. Represent D by local meromorphic equations f_i. With the Čech sign convention

$$(df)_{ij} = f_j/f_i,$$

the connecting class is represented by the cocycle

$$g_{ij} = f_j/f_i \in \mathscr{O}_X^{\times}(U_i \cap U_j).$$

The local frame f_i^{-1} of $\mathscr{O}_X(D)$ satisfies

$$f_i^{-1} = \frac{f_j}{f_i} f_j^{-1} = g_{ij} f_j^{-1}.$$

Thus the same cocycle is the transition cocycle of $\mathscr{O}_X(D)$. □

Remark 7.48. For a compact Riemann surface, every holomorphic line bundle possesses a nonzero meromorphic section. Combined with Proposition 7.40, this yields

$$\operatorname{Cl}(X) \cong \operatorname{Pic}(X).$$

We postpone the existence theorem until the Riemann–Roch theorem is available, so no circular argument is introduced here.

7.7 Exact sequences associated with effective divisors

7.7.1 The structure sheaf of an effective divisor

Definition 7.49. For an effective divisor $E \geq 0$, define

$$\mathscr{O}_E = \mathscr{O}_X/\mathscr{O}_X(-E).$$

This is a coherent torsion sheaf supported on $\mathrm{Supp}(E)$.

Proposition 7.50. *For every effective divisor E, there is a short exact sequence*

$$0 \longrightarrow \mathscr{O}_X(-E) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_E \longrightarrow 0.$$

More generally, for every divisor D,

$$0 \longrightarrow \mathscr{O}_X(D-E) \longrightarrow \mathscr{O}_X(D) \longrightarrow \mathscr{O}_E(D) \longrightarrow 0,$$

where

$$\mathscr{O}_E(D) = \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{O}_E.$$

Proof. The first sequence is exact by the definition of $\mathscr{O}_E$; the inclusion $\mathscr{O}_X(-E) \subseteq \mathscr{O}_X$ follows from $E \geq 0$. Tensor it with the invertible sheaf $\mathscr{O}_X(D)$. Tensoring with a locally free sheaf is exact, and Proposition 7.30 identifies the first two terms with $\mathscr{O}_X(D-E)$ and $\mathscr{O}_X(D)$. $\square$

Proposition 7.51 (Local structure and length)**.** *Let*

$$E = \sum_{p \in X} n_p[p]$$

be effective. If z is a local coordinate at p, then

$$(\mathscr{O}_E)_p \cong \mathscr{O}_{X,p}/(z^{n_p}).$$

In particular,

$$\dim_{\mathbb{C}}(\mathscr{O}_E)_p = n_p.$$

If X is compact, then

$$\dim_{\mathbb{C}} H^0(X, \mathscr{O}_E) = \deg E.$$

Proof. Near p, the divisor $-E$ has local equation z^{-n_p}, and

$$\mathscr{O}_X(-E)_p = z^{n_p}\mathscr{O}_{X,p}.$$

Hence the stalk of the quotient is $\mathscr{O}_{X,p}/(z^{n_p})$. The residue classes of

$$1, z, \ldots, z^{n_p-1}$$

form a complex basis, proving the dimension statement.

When X is compact, the support of E is finite. A sheaf supported on a finite discrete set has global sections equal to the direct sum of its stalks. Therefore

$$\begin{aligned}\dim_{\mathbb{C}} H^0(X, \mathscr{O}_E) &= \sum_{p \in \mathrm{Supp}(E)} \dim_{\mathbb{C}}(\mathscr{O}_E)_p \\ &= \sum_p n_p = \deg E.\end{aligned}$$

$\square$

Corollary 7.52 (Adding one point). *For every divisor D and every point $p \in X$, there is a short exact sequence*
$$0 \longrightarrow \mathscr{O}_X(D-p) \longrightarrow \mathscr{O}_X(D) \longrightarrow \mathbb{C}_p \longrightarrow 0,$$
where $\mathbb{C}_p$ is the skyscraper sheaf with value $\mathbb{C}$ at p.

Proof. Apply Proposition 7.50 with $E = [p]$. The quotient is supported at p, and its stalk there is
$$\mathscr{O}_{X,p}/(z) \cong \mathbb{C}.$$
Tensoring a one-dimensional skyscraper stalk with the one-dimensional fibre of $\mathscr{O}_X(D)$ again gives a one-dimensional skyscraper sheaf. After a choice of local frame it is identified with $\mathbb{C}_p$. □

Corollary 7.53. *If X is compact, then*
$$0 \le \ell(D) - \ell(D-p) \le 1.$$
More generally, for every effective divisor E,
$$0 \le \ell(D) - \ell(D-E) \le \deg E.$$

Proof. Taking global sections of the twisted exact sequence gives an exact sequence
$$0 \longrightarrow H^0(X, \mathscr{O}_X(D-E)) \longrightarrow H^0(X, \mathscr{O}_X(D)) \longrightarrow H^0(X, \mathscr{O}_E(D)).$$
The quotient of the first two spaces therefore embeds into the last. Since twisting by a line bundle does not change the local lengths of a finite support sheaf,
$$\dim_{\mathbb{C}} H^0(X, \mathscr{O}_E(D)) = \deg E.$$
Use Proposition 7.31. The case $E = p$ gives the first inequality. □

Corollary 7.54 (Cohomology sequence). *For every divisor D and point $p \in X$, the sequence of Corollary 7.52 induces*
$$\begin{aligned} 0 \longrightarrow & H^0(X, \mathscr{O}_X(D-p)) \longrightarrow H^0(X, \mathscr{O}_X(D)) \longrightarrow \mathbb{C} \\ \longrightarrow & H^1(X, \mathscr{O}_X(D-p)) \longrightarrow H^1(X, \mathscr{O}_X(D)) \longrightarrow \cdots . \end{aligned}$$

Proof. Apply the long exact cohomology sequence to Corollary 7.52. The skyscraper sheaf is flasque, since all of its restriction maps are surjective, and therefore it has no higher cohomology. Hence the displayed portion has the stated form; the remaining terms continue in the usual way. □

Chapter 8

Differentials and the Canonical Sheaf

The derivative of a holomorphic function is initially defined in a local coordinate, but its transformation law shows that the expression $f'(z)\,dz$ is intrinsic. The purpose of this chapter is to organize that observation in the language developed in the preceding chapters. The sheaf of holomorphic differentials will be characterized by a universal property, identified as an invertible sheaf, and studied functorially under holomorphic maps. Meromorphic sections of this sheaf lead to canonical divisors, while their residues provide the local algebra underlying Serre duality.

Throughout the chapter, X denotes a connected Riemann surface, $\mathscr{O}_X$ its structure sheaf, and $\mathscr{M}_X$ its sheaf of meromorphic functions.

8.1 Analytic derivations and holomorphic differentials

Purely algebraic Kahler differentials are defined by the Leibniz rule alone. In the analytic category one also wants differentiation to respect holomorphic functional calculus. This additional requirement excludes pathological discontinuous derivations of rings of convergent power series and produces the finite-rank cotangent sheaf of a complex manifold.

Definition 8.1. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. An *analytic* $\mathbb{C}$-*derivation*

$$D : \mathscr{O}_X \longrightarrow \mathscr{F}$$

is a morphism of sheaves of complex vector spaces with the following property. If $U \subseteq X$ is open, $f_1, \ldots, f_n \in \mathscr{O}_X(U)$, and Φ is holomorphic on a neighborhood of the image of

$$(f_1, \ldots, f_n) : U \longrightarrow \mathbb{C}^n,$$

then

$$D\big(\Phi(f_1, \ldots, f_n)\big) = \sum_{j=1}^{n} \frac{\partial \Phi}{\partial t_j}(f_1, \ldots, f_n) D(f_j).$$

The sheaf of analytic derivations is denoted by

$$\mathrm{Der}^{\mathrm{an}}_{\mathbb{C}}(\mathscr{O}_X, \mathscr{F}).$$

Taking $\Phi(t_1, t_2) = t_1 t_2$ shows that every analytic derivation satisfies the Leibniz rule

$$D(fg) = fD(g) + gD(f),$$

and taking Φ constant shows that $D(c) = 0$ for $c \in \mathbb{C}$.

Proposition 8.2. *For every $\mathscr{O}_X$-module $\mathscr{F}$, the sheaf*

$$\mathrm{Der}^{\mathrm{an}}_{\mathbb{C}}(\mathscr{O}_X, \mathscr{F})$$

is naturally an $\mathscr{O}_X$-module.

Proof. Let $a \in \mathscr{O}_X(U)$ and let D be an analytic derivation on U. Define

$$(aD)(f) = aD(f).$$

This is complex linear and compatible with restriction. Moreover,

$$\begin{aligned}(aD)\big(\Phi(f_1, \dots, f_n)\big) &= a \sum_j \frac{\partial \Phi}{\partial t_j}(f_1, \dots, f_n) D(f_j) \\ &= \sum_j \frac{\partial \Phi}{\partial t_j}(f_1, \dots, f_n)(aD)(f_j).\end{aligned}$$

Thus aD is again an analytic derivation. The remaining module axioms follow pointwise from those of $\mathscr{F}$. □

Definition 8.3. A *holomorphic differential* on an open set $U \subseteq X$ is a collection of expressions

$$\omega|_{U_i} = a_i(z_i)\, dz_i$$

on coordinate neighborhoods (U_i, z_i) covering U, where $a_i \in \mathscr{O}_X(U_i)$ and, on every overlap,

$$a_i(z_i)\, dz_i = a_j(z_j)\, dz_j.$$

Equivalently, if $z_j = z_j(z_i)$, then

$$a_i(z_i) = a_j(z_j(z_i)) \frac{dz_j}{dz_i}.$$

The resulting sheaf of holomorphic differentials is denoted by Ω^1_X.

Addition and multiplication by holomorphic functions are defined in coordinates. The transformation law ensures that these operations are independent of the chosen coordinate.

Proposition 8.4 (Local form of the differential sheaf)**.** *If (U, z) is a coordinate neighborhood, then*

$$\Omega^1_X|_U \cong \mathscr{O}_U\, dz.$$

In particular, Ω^1_X is a locally free $\mathscr{O}_X$-module of rank one and hence is coherent and invertible.

Proof. On U, every differential has a unique expression $a(z)\, dz$. The map

$$\mathscr{O}_U \longrightarrow \Omega^1_X|_U, \qquad a \longmapsto a\, dz,$$

is therefore an isomorphism of $\mathscr{O}_U$-modules. Local freeness of rank one follows. Since $\mathscr{O}_X$ is coherent and finite-rank locally free modules over a coherent sheaf of rings are coherent, Ω^1_X is coherent. □

Definition 8.5. For $f \in \mathscr{O}_X(U)$, define its differential locally by

$$df = f'(z)\, dz.$$

This gives a morphism of sheaves of complex vector spaces

$$d : \mathscr{O}_X \longrightarrow \Omega^1_X.$$

Proposition 8.6. *The differential df is independent of the local coordinate. It satisfies*

$$d(f+g) = df + dg, \qquad d(fg) = f\, dg + g\, df, \qquad dc = 0$$

for $c \in \mathbb{C}$.

Proof. Let w be a second coordinate. The chain rule gives

$$\frac{df}{dw} = \frac{df}{dz}\frac{dz}{dw},$$

and hence

$$\frac{df}{dw}\,dw = \frac{df}{dz}\,dz.$$

Thus the local expressions glue. The three displayed identities follow from the ordinary sum, product, and constant rules in any coordinate. □

Theorem 8.7 (Universal property of analytic differentials)**.** *For every $\mathscr{O}_X$-module $\mathscr{F}$, composition with d induces a natural isomorphism of sheaves*

$$\mathscr{H}om_{\mathscr{O}_X}(\Omega^1_X, \mathscr{F}) \xrightarrow{\sim} \mathrm{Der}^{\mathrm{an}}_{\mathbb{C}}(\mathscr{O}_X, \mathscr{F}).$$

Consequently, the pair (Ω^1_X, d) is determined up to a unique isomorphism by this universal property.

Proof. An $\mathscr{O}_X$-linear morphism $\varphi : \Omega^1_X \to \mathscr{F}$ gives an analytic derivation $\varphi \circ d$, because the ordinary chain rule holds for d and φ is $\mathscr{O}_X$-linear.

Conversely, let $D : \mathscr{O}_X \to \mathscr{F}$ be an analytic derivation. On a coordinate neighborhood (U, z) define

$$\varphi_U : \Omega^1_X|_U \longrightarrow \mathscr{F}|_U, \qquad \varphi_U(a\,dz) = aD(z).$$

If $f \in \mathscr{O}_X(U)$, then locally $f = \Phi(z)$ for a holomorphic function Φ, so the analytic chain rule gives

$$D(f) = \Phi'(z)D(z) = \varphi_U(df).$$

If w is another coordinate, then

$$D(w) = \frac{dw}{dz}D(z), \qquad dw = \frac{dw}{dz}dz,$$

so the maps φ_U agree on overlaps and glue to an $\mathscr{O}_X$-linear morphism $\varphi : \Omega^1_X \to \mathscr{F}$. By construction, $D = \varphi \circ d$.

Uniqueness follows because dz locally generates Ω^1_X and any factorization must send dz to $D(z)$. The two constructions are inverse and are natural in $\mathscr{F}$. □

Remark 8.8. For a smooth algebraic curve over $\mathbb{C}$, the usual sheaf of Kahler differentials is already locally free of rank one. In the analytic category, the preceding universal property is formulated for analytic derivations, so that differentiation respects convergent holomorphic substitutions. This is the analytic counterpart of the algebraic construction used for smooth curves.

Example 8.9 (A plane domain)**.** If $U \subseteq \mathbb{C}$ is open with its standard coordinate z, then

$$\Omega^1_U = \mathscr{O}_U\,dz, \qquad df = f'(z)\,dz.$$

Thus the abstract differential sheaf recovers the ordinary differential from one-variable complex analysis.

8.2 Cotangent spaces and the tangent sheaf

Definition 8.10. Let $p \in X$, let $\mathfrak{m}_p \subseteq \mathscr{O}_{X,p}$ be the maximal ideal, and let

$$\kappa(p) = \mathscr{O}_{X,p}/\mathfrak{m}_p \cong \mathbb{C}.$$

The *cotangent space* and *tangent space* at p are

$$T_p^* X = \mathfrak{m}_p/\mathfrak{m}_p^2, \qquad T_p X = \mathrm{Hom}_{\mathbb{C}}(T_p^* X, \mathbb{C}).$$

Proposition 8.11. *The map $f \mapsto df$ induces a natural isomorphism*

$$\mathfrak{m}_p/\mathfrak{m}_p^2 \xrightarrow{\sim} \Omega^1_{X,p} \otimes_{\mathscr{O}_{X,p}} \kappa(p).$$

In particular, both $T_p^* X$ and $T_p X$ are one-dimensional complex vector spaces.

Proof. Choose a coordinate z centered at p. Then

$$\mathscr{O}_{X,p} \cong \mathbb{C}\{z\}, \qquad \mathfrak{m}_p = (z), \qquad \Omega^1_{X,p} = \mathscr{O}_{X,p}\, dz.$$

Every $f \in \mathfrak{m}_p$ has an expansion

$$f = a_1 z + a_2 z^2 + \cdots ,$$

so modulo $\mathfrak{m}_p \Omega^1_{X,p}$ one has

$$df \equiv a_1\, dz.$$

The induced map is surjective because the class of z maps to the class of dz. Its kernel consists exactly of those germs with $a_1 = 0$, namely $\mathfrak{m}_p^2$. This proves the isomorphism and the dimension statement. ☐

Definition 8.12. The *holomorphic tangent sheaf* is the dual invertible sheaf

$$\mathscr{T}_X = \mathscr{H}om_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X).$$

Proposition 8.13. *There is a natural isomorphism*

$$\mathscr{T}_X \cong \mathrm{Der}^{\mathrm{an}}_{\mathbb{C}}(\mathscr{O}_X, \mathscr{O}_X).$$

On a coordinate neighborhood (U, z),

$$\mathscr{T}_X|_U = \mathscr{O}_U \frac{\partial}{\partial z}.$$

Proof. The first assertion is the universal property of Ω^1_X with $\mathscr{F} = \mathscr{O}_X$. On (U, z), the dual of the free module $\mathscr{O}_U\, dz$ is free with dual generator $\partial/\partial z$, characterized by

$$\frac{\partial}{\partial z}(dz) = 1.$$

Under the universal-property identification, the section $a\, \partial/\partial z$ acts on a holomorphic function f by $af'(z)$. ☐

8.3 Functoriality and ramification

Definition 8.14. Let $f : X \to Y$ be a holomorphic map and let $\mathscr{F}$ be an $\mathscr{O}_Y$-module. Its pullback is

$$f^*\mathscr{F} = f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X.$$

The composite

$$f^{-1}\mathscr{O}_Y \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega^1_X$$

is an analytic derivation and therefore induces a natural $\mathscr{O}_X$-linear morphism

$$df : f^*\Omega^1_Y \longrightarrow \Omega^1_X.$$

The *relative differential sheaf* is

$$\Omega^1_{X/Y} = \operatorname{coker}(df).$$

Proposition 8.15 (Local expression and chain rule)**.** *Let z be a coordinate on X and w a coordinate on Y. Then*

$$df(dw) = d(w \circ f) = \frac{d(w \circ f)}{dz}\, dz.$$

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are holomorphic maps, then

$$d(g \circ f) = df \circ f^*(dg).$$

Proof. The local formula follows from the definition of the induced derivation. For a local holomorphic function h on Z,

$$\begin{aligned}\big(df \circ f^*(dg)\big)(dh) &= df\big(d(h \circ g)\big)\\ &= d(h \circ g \circ f),\end{aligned}$$

which is precisely $d(g \circ f)(dh)$. Since local differentials dh generate Ω^1_Z, the two morphisms agree. ☐

Let $f : X \to Y$ now be nonconstant. The local mapping theorem gives coordinates centered at p and $f(p)$ in which

$$w \circ f = z^{e_p}$$

for a uniquely determined integer $e_p \geq 1$, the local degree of f at p.

Proposition 8.16 (Relative differentials at a ramification point)**.** *For a nonconstant holomorphic map $f : X \to Y$, the sequence*

$$0 \longrightarrow f^*\Omega^1_Y \xrightarrow{df} \Omega^1_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is exact. In local coordinates satisfying $w \circ f = z^{e_p}$,

$$(\Omega^1_{X/Y})_p \cong \mathscr{O}_{X,p}/(z^{e_p-1}).$$

Consequently,

$$\operatorname{length}_{\mathscr{O}_{X,p}}(\Omega^1_{X/Y,p}) = e_p - 1.$$

Proof. In the chosen coordinates,

$$df(dw) = e_p z^{e_p-1} dz.$$

After identifying both source and target with free rank-one $\mathscr{O}_{X,p}$-modules, df_p is multiplication by $e_p z^{e_p-1}$. The local ring $\mathscr{O}_{X,p}$ is an integral domain and $e_p z^{e_p-1} \neq 0$, so this map is injective. Its cokernel is

$$\frac{\mathscr{O}_{X,p}\, dz}{e_p z^{e_p-1}\mathscr{O}_{X,p}\, dz} \cong \mathscr{O}_{X,p}/(z^{e_p-1}),$$

since e_p is a unit. The classes of $1, z, \ldots, z^{e_p-2}$ form a composition basis when $e_p > 1$, giving length $e_p - 1$; for $e_p = 1$ the cokernel is zero. ☐

Definition 8.17. A point p is a *ramification point* of f if $e_p > 1$. The locally finite effective divisor

$$R_f = \sum_{p \in X} (e_p - 1)[p]$$

is the *ramification divisor* of f.

Proposition 8.18. *The support of R_f is discrete. If X is compact, then R_f has finite support and $\Omega^1_{X/Y}$ is a coherent torsion sheaf of finite length*

$$\sum_{p \in X} \operatorname{length}(\Omega^1_{X/Y,p}) = \deg R_f.$$

Proof. In local coordinates, ramification points are zeros of the nonzero holomorphic function $d(w \circ f)/dz$. These zeros are isolated, so the support is discrete. A discrete subset of a compact space is finite. The preceding proposition shows that the cokernel is locally a finite-length quotient of $\mathscr{O}_{X,p}$ at ramification points and is zero elsewhere, so it is coherent torsion. Summing the local lengths gives the stated equality. □

Definition 8.19. The invertible sheaf

$$\omega_X := \Omega^1_X$$

is called the *canonical sheaf* or *canonical line bundle* of X.

Theorem 8.20 (Canonical bundle formula)**.** *Let $f : X \to Y$ be a nonconstant holomorphic map of connected Riemann surfaces. Then there is a natural isomorphism*

$$\boxed{\omega_X \cong f^*\omega_Y \otimes_{\mathscr{O}_X} \mathscr{O}_X(R_f).}$$

Proof. The nonzero morphism of line bundles

$$df : f^*\omega_Y \longrightarrow \omega_X$$

may be viewed as a nonzero holomorphic section of

$$\mathscr{H}om_{\mathscr{O}_X}(f^*\omega_Y, \omega_X) \cong \omega_X \otimes (f^*\omega_Y)^\vee.$$

In local coordinates with $w \circ f = z^{e_p}$, this section is represented by

$$e_p z^{e_p - 1}.$$

Its zero divisor is therefore exactly R_f. The divisor-line-bundle construction from the preceding chapter gives

$$\omega_X \otimes (f^*\omega_Y)^\vee \cong \mathscr{O}_X(R_f).$$

Tensoring by $f^*\omega_Y$ yields the formula. □

Example 8.21 (The power map)**.** For

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad f(z) = z^n,$$

the only ramification points are 0 and ∞, each with local degree n. Hence

$$R_f = (n-1)[0] + (n-1)[\infty].$$

The canonical bundle formula becomes

$$\omega_{\mathbb{P}^1} \cong f^*\omega_{\mathbb{P}^1} \otimes \mathscr{O}_{\mathbb{P}^1}\big((n-1)[0] + (n-1)[\infty]\big).$$

8.4 An algebraic interlude: the conormal sequence

For this section, let S be a complex surface and let $i : X \hookrightarrow S$ be a closed holomorphic embedding of a Riemann surface. Write $\mathscr{I} \subseteq \mathscr{O}_S$ for its ideal sheaf. The sheaf Ω^1_S is defined in local holomorphic coordinates exactly as above and is locally free of rank two.

Definition 8.22. The *conormal sheaf* and *normal bundle* of X in S are

$$\mathscr{N}^\vee_{X/S} = \mathscr{I}/\mathscr{I}^2, \qquad \mathscr{N}_{X/S} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X).$$

Theorem 8.23 (Conormal exact sequence)**.** *There is a natural short exact sequence of locally free $\mathscr{O}_X$-modules*

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \xrightarrow{d} i^*\Omega^1_S \longrightarrow \Omega^1_X \longrightarrow 0.$$

Proof. The assertion is local. Near any $p \in X$, choose holomorphic coordinates (z, w) on S such that X is defined by $w = 0$. Then $\mathscr{I}/\mathscr{I}^2$ is freely generated by the class of w, while

$$i^*\Omega^1_S = \mathscr{O}_X\, dz \oplus \mathscr{O}_X\, dw.$$

The first map sends the class of w to dw. Restricting differentials to X sends dz to the local generator dz of Ω^1_X and sends dw to zero. Thus the sequence is locally

$$0 \longrightarrow \mathscr{O}_X\, dw \longrightarrow \mathscr{O}_X\, dz \oplus \mathscr{O}_X\, dw \longrightarrow \mathscr{O}_X\, dz \longrightarrow 0,$$

which is exact. □

Corollary 8.24 (Adjunction in conormal form)**.** *Let*

$$\omega_S = \det \Omega^1_S.$$

Then

$$\boxed{\omega_X \cong i^*\omega_S \otimes \mathscr{N}_{X/S}.}$$

Proof. Taking determinants in the conormal exact sequence gives

$$\det(i^*\Omega^1_S) \cong \det(\mathscr{I}/\mathscr{I}^2) \otimes \det(\Omega^1_X).$$

All three sheaves have ranks $2, 1, 1$ as appropriate, so

$$i^*\omega_S \cong \mathscr{N}^\vee_{X/S} \otimes \omega_X.$$

Tensoring with $\mathscr{N}_{X/S}$ proves the formula. □

Example 8.25 (A regular plane curve)**.** Let

$$X = \{F(x, y) = 0\} \subseteq U \subseteq \mathbb{C}^2$$

be a regular plane curve, so F_x and F_y do not vanish simultaneously on X. The conormal sequence gives

$$\Omega^1_X \cong \frac{\mathscr{O}_X\, dx \oplus \mathscr{O}_X\, dy}{\mathscr{O}_X(F_x\, dx + F_y\, dy)}.$$

Indeed, the ideal is locally generated by F, and $dF = F_x\, dx + F_y\, dy$. On the open set where $F_y \neq 0$, the relation expresses dy in terms of dx, so dx generates Ω^1_X; similarly, dy generates it where $F_x \neq 0$.

8.5 Meromorphic differentials and canonical divisors

Definition 8.26. A *meromorphic differential* on an open set $U \subseteq X$ is a section of

$$\omega_X \otimes_{\mathscr{O}_X} \mathscr{M}_X.$$

In a coordinate z it has a unique expression

$$\eta = f(z)\,dz, \qquad f \in \mathscr{M}_X(U).$$

If $\eta \neq 0$, define

$$\mathrm{ord}_p(\eta) = \mathrm{ord}_p(f).$$

Proposition 8.27. *The integer* $\mathrm{ord}_p(\eta)$ *is independent of the chosen local coordinate. The formal sum*

$$\mathrm{div}(\eta) = \sum_{p \in X} \mathrm{ord}_p(\eta)[p]$$

is therefore a well-defined locally finite divisor.

Proof. Let w be another coordinate centered at p. Since

$$\eta = f(z)\,dz = f(z(w))\frac{dz}{dw}\,dw,$$

the coefficient in the w-coordinate differs from f by composition with a local biholomorphism and multiplication by dz/dw. The derivative dz/dw is a holomorphic unit, so it has order zero. Composition with a biholomorphic coordinate change preserves the order of a meromorphic germ. Hence the order is coordinate-independent. Local finiteness follows from the isolatedness of zeros and poles of the local coefficient f. □

Proposition 8.28. *If g is a nonzero meromorphic function and η is a nonzero meromorphic differential, then*

$$\mathrm{div}(g\eta) = \mathrm{div}(g) + \mathrm{div}(\eta).$$

Proof. In a local coordinate, write $\eta = f\,dz$. Then $g\eta = (gf)\,dz$, so

$$\mathrm{ord}_p(g\eta) = \mathrm{ord}_p(gf) = \mathrm{ord}_p(g) + \mathrm{ord}_p(f).$$

Summing over all points gives the formula. □

Definition 8.29. A divisor of the form

$$K = \mathrm{div}(\eta)$$

for a nonzero global meromorphic section η of ω_X is called a *canonical divisor*. The sheaf ω_X is the primary object; the existence of a canonical divisor amounts to the existence of a nonzero meromorphic section of this line bundle and will be established later in the general theory of line bundles.

Proposition 8.30. *If $K = \mathrm{div}(\eta)$ is a canonical divisor, then*

$$\omega_X \cong \mathscr{O}_X(K).$$

Any two canonical divisors are linearly equivalent.

Proof. The first assertion is the general result from the preceding chapter: a line bundle admitting a nonzero meromorphic section s is isomorphic to $\mathscr{O}_X(\operatorname{div}(s))$.

If η' is another nonzero meromorphic differential, then the quotient $g = \eta'/\eta$ is a global nonzero meromorphic function. The preceding proposition gives

$$\operatorname{div}(\eta') = \operatorname{div}(g) + \operatorname{div}(\eta),$$

so the two divisors are linearly equivalent. □

Example 8.31 (The Riemann sphere and a complex torus)**.** On $\mathbb{P}^1$, use the coordinate z on $\mathbb{C}$ and $w = 1/z$ near ∞. Since

$$dz = -w^{-2}\, dw,$$

the differential dz has a pole of order two at ∞ and no other zero or pole. Thus

$$\operatorname{div}(dz) = -2[\infty], \qquad \omega_{\mathbb{P}^1} \cong \mathscr{O}_{\mathbb{P}^1}(-2[\infty]).$$

If $X = \mathbb{C}/\Lambda$ is a complex torus, the differential dz is invariant under every translation $z \mapsto z + \lambda$ and therefore descends to a nowhere-vanishing holomorphic differential on X. Hence

$$\omega_X \cong \mathscr{O}_X.$$

8.6 Local primitives and the analytic de Rham sequence

Theorem 8.32 (Local primitive theorem)**.** *Every holomorphic differential is locally exact. More precisely, if $\omega \in \Omega^1_X(U)$ and $p \in U$, then there is a neighborhood $V \subseteq U$ of p and a function $F \in \mathscr{O}_X(V)$ such that*

$$\omega|_V = dF.$$

Proof. Choose a coordinate disc (V, z) centered at p and write

$$\omega|_V = f(z)\, dz.$$

After shrinking V to a disc, the holomorphic function f has a holomorphic primitive F. Then

$$dF = F'(z)\, dz = f(z)\, dz = \omega|_V.$$

□

Corollary 8.33 (Analytic de Rham sequence in dimension one)**.** *There is a short exact sequence of sheaves of complex vector spaces*

$$0 \longrightarrow \underline{\mathbb{C}}_X \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega^1_X \longrightarrow 0,$$

where $\underline{\mathbb{C}}_X$ is the locally constant sheaf.

Proof. A holomorphic function lies in the kernel of d exactly when its derivative vanishes, hence exactly when it is locally constant. The local primitive theorem says that d is surjective as a morphism of sheaves. Exactness at the remaining terms is immediate. □

Remark 8.34. This is not a sequence of $\mathscr{O}_X$-modules, because d is not $\mathscr{O}_X$-linear. It is a sequence of sheaves of complex vector spaces. On a compact complex torus, the descended differential dz is locally exact but not globally exact: every global holomorphic function is constant, so its differential is zero.

8.7 Residues and local duality

Definition 8.35. Let η be meromorphic near p, and let z be a coordinate centered at p. Write

$$\eta = \left(\sum_{n=-N}^{\infty} a_n z^n\right) dz.$$

The *residue* of η at p is

$$\mathrm{Res}_p(\eta) = a_{-1}.$$

Lemma 8.36. *If f is meromorphic near p, then*

$$\mathrm{Res}_p(df) = 0.$$

Proof. In a local coordinate,

$$f = \sum_{n=-N}^{\infty} a_n z^n, \qquad df = \sum_{n=-N}^{\infty} n a_n z^{n-1}\, dz.$$

A term in $z^{-1}\,dz$ would require $n = 0$, but its coefficient is then $na_n = 0$. Hence the residue vanishes. □

Proposition 8.37 (Coordinate invariance of the residue)**.** *The number* $\mathrm{Res}_p(\eta)$ *is independent of the local coordinate.*

Proof. Let z and w be coordinates centered at p. Then

$$z = w\,u(w)$$

for a holomorphic unit u. The residue is linear, and every Laurent monomial $z^n\,dz$ with $n \neq -1$ is exact:

$$z^n\,dz = d\left(\frac{z^{n+1}}{n+1}\right).$$

By the lemma, it has residue zero in every coordinate. For the remaining monomial,

$$\frac{dz}{z} = \frac{dw}{w} + \frac{du}{u}.$$

The second term is holomorphic because u is a unit, so the residue of dz/z in the w-coordinate is 1. Therefore the coefficient of the unique nonexact Laurent monomial is unchanged by coordinate transformation. □

Proposition 8.38 (Logarithmic differentials)**.** *For every nonzero meromorphic function f,*

$$\boxed{\mathrm{Res}_p\left(\frac{df}{f}\right) = \mathrm{ord}_p(f).}$$

Proof. Choose a coordinate z centered at p and write

$$f = z^m u, \qquad m = \mathrm{ord}_p(f),$$

where u is a holomorphic unit. Then

$$\frac{df}{f} = m\frac{dz}{z} + \frac{du}{u}.$$

The second term is holomorphic, so its residue is zero, while dz/z has residue one. The result follows. □

Definition 8.39. For $p \in X$ and $n \geq 1$, define the finite-dimensional principal-parts space

$$P_{p,n} = \mathfrak{m}_p^{-n}/\mathscr{O}_{X,p} \cong z^{-n}\mathscr{O}_{X,p}/\mathscr{O}_{X,p}.$$

Also set

$$J_{p,n} = \Omega^1_{X,p}/\mathfrak{m}_p^n\Omega^1_{X,p}.$$

Theorem 8.40 (Local residue pairing)**.** *The bilinear map*

$$P_{p,n} \times J_{p,n} \longrightarrow \mathbb{C}, \qquad (P,\omega) \longmapsto \mathrm{Res}_p(P\omega),$$

is a perfect pairing.

Proof. Choose a coordinate z at p. The classes

$$z^{-1}, z^{-2}, \ldots, z^{-n}$$

form a basis of $P_{p,n}$, while

$$dz, z\,dz, \ldots, z^{n-1}\,dz$$

form a basis of $J_{p,n}$. Direct calculation gives

$$\mathrm{Res}_p\left(z^{-i}z^{j-1}\,dz\right) = \delta_{ij}.$$

Thus the displayed bases are dual, proving nondegeneracy in both variables. Since residue is coordinate-independent, the pairing itself is intrinsic. □

Corollary 8.41 (Residue pairing for an effective divisor)**.** *Let*

$$D = \sum_{p \in X} n_p[p]$$

be an effective divisor with finite support. Set

$$\mathscr{P}_D = \mathscr{O}_X(D)/\mathscr{O}_X, \qquad \mathscr{J}_D = \omega_X/\omega_X(-D).$$

Then

$$H^0(X,\mathscr{P}_D) \times H^0(X,\mathscr{J}_D) \longrightarrow \mathbb{C}, \qquad (P,\omega) \longmapsto \sum_{p \in \mathrm{Supp}(D)} \mathrm{Res}_p(P_p\omega_p),$$

is a perfect pairing.

Proof. Both sheaves are supported on the finite set $\mathrm{Supp}(D)$, so their global sections are direct sums of their stalks. At p, the two stalks are precisely P_{p,n_p} and J_{p,n_p}. The pairing is therefore the direct sum of the perfect local residue pairings, and is perfect. □

Proposition 8.42 (The simple-pole residue sequence)**.** *For every point $p \in X$, there is a short exact sequence*

$$0 \longrightarrow \omega_X \longrightarrow \omega_X(p) \xrightarrow{\mathrm{Res}_p} \mathbb{C}_p \longrightarrow 0,$$

where $\mathbb{C}_p$ is the skyscraper sheaf at p.

Proof. Away from p, the first two sheaves agree and the skyscraper stalk is zero. At p, choose a coordinate z. Then

$$\omega_{X,p} = \mathscr{O}_{X,p}\,dz, \qquad \omega_X(p)_p = z^{-1}\mathscr{O}_{X,p}\,dz.$$

The quotient is one-dimensional, represented by the class of dz/z, and the residue map sends this class to 1. Its kernel consists exactly of differentials with no $z^{-1}dz$ term, namely holomorphic differentials. Thus the stalk sequence is exact at every point. □

8.8 Integration and the global residue theorem

Definition 8.43. Let $\gamma : [a,b] \to X$ be a piecewise C^1 curve avoiding the poles of a meromorphic differential η. Choose a subdivision

$$a = t_0 < t_1 < \cdots < t_r = b$$

such that each segment of γ lies in a coordinate neighborhood and write $\eta = f_j(z_j)\, dz_j$ there. Define

$$\int_\gamma \eta = \sum_{j=1}^{r} \int_{t_{j-1}}^{t_j} f_j(z_j(\gamma(t))) \frac{d(z_j \circ \gamma)}{dt}(t)\, dt.$$

Proposition 8.44. *The integral of a differential is independent of the chosen coordinates and subdivision. It is additive under concatenation, changes sign under reversal, and is invariant under orientation-preserving piecewise C^1 reparametrization.*

Proof. On an overlap, the identity

$$f(z)\, dz = g(w)\, dw$$

and the real chain rule show that the two coordinate integrands agree. Refining a subdivision only splits ordinary integrals into sums over smaller intervals. The remaining properties follow from the corresponding properties of ordinary line integrals on each coordinate segment. □

Proposition 8.45 (Local residue integral)**.** *Let z be a coordinate centered at p and let γ_r be the positively oriented circle $|z| = r$, small enough to contain no other pole. Then*

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \eta = \operatorname{Res}_p(\eta).$$

Proof. Integrate the Laurent expansion of the coefficient of η term by term on the circle. Every monomial $z^n\, dz$ integrates to zero except $z^{-1}\, dz$, whose integral is $2\pi\mathrm{i}$. □

Theorem 8.46 (Global residue theorem)**.** *Let X be a compact Riemann surface and let η be a meromorphic differential on X. Then*

$$\boxed{\sum_{p \in X} \operatorname{Res}_p(\eta) = 0.}$$

Proof. The set of poles is finite. Choose pairwise disjoint coordinate discs around them and remove smaller closed discs, obtaining a compact surface with boundary X° on which η is holomorphic. Choose a finite triangulation of X° subordinate to coordinate neighborhoods. On each triangle, the ordinary Cauchy theorem gives zero integral around its oriented boundary. Summing over all triangles, the integrals along interior edges cancel in opposite pairs, leaving only the integral along ∂X°. Hence

$$\int_{\partial X^\circ} \eta = 0.$$

The boundary orientation of X° is clockwise around each removed disc. Reversing orientation and applying the local residue integral therefore gives

$$0 = -2\pi\mathrm{i} \sum_{p \in X} \operatorname{Res}_p(\eta).$$

Dividing by $-2\pi\mathrm{i}$ proves the theorem. □

Corollary 8.47. *A meromorphic differential on a compact Riemann surface cannot have exactly one simple pole with nonzero residue.*

Proof. If such a differential existed, the sum of all residues would equal the nonzero residue at that pole, contradicting the global residue theorem. □

Example 8.48 (A rational differential)**.** On $\mathbb{P}^1$, consider

$$\eta = \frac{dz}{z(z-1)}.$$

At 0 and 1 one finds

$$\mathrm{Res}_0(\eta) = -1, \qquad \mathrm{Res}_1(\eta) = 1.$$

With $w = 1/z$ near ∞,

$$\eta = -\frac{dw}{1-w},$$

which is holomorphic at $w = 0$, so $\mathrm{Res}_\infty(\eta) = 0$. The residues sum to zero, as predicted.

Chapter 9

Serre Duality and the Riemann–Roch Theorem

This chapter completes the passage from local complex analysis to global geometry. The principal parts introduced in the cohomology chapter provide a concrete model for the obstruction group H^1. The residue theorem turns meromorphic differentials into linear functionals on this obstruction space. Serre duality asserts that every functional arises in this way. Combining this perfect pairing with the additivity of the Euler characteristic produces the Riemann–Roch theorem.

Throughout, X denotes a compact connected Riemann surface and

$$\omega_X = \Omega^1_X$$

denotes its canonical sheaf. We freely use the following results from Chapter 6:

(i) coherent sheaves on X have finite-dimensional cohomology and $H^q(X, \mathscr{F}) = 0$ for $q > 1$;

(ii) the functors $\mathrm{Ext}^i_X(-,-)$, their long exact sequences, and the Yoneda interpretation of Ext^1_X;

(iii) the Euler characteristic

$$\chi(X, \mathscr{F}) = h^0(X, \mathscr{F}) - h^1(X, \mathscr{F})$$

is additive in short exact sequences;

(iv) a coherent torsion sheaf $\mathscr{T}$ satisfies $H^1(X, \mathscr{T}) = 0$ and

$$h^0(X, \mathscr{T}) = \sum_{p \in X} \mathrm{length}_{\mathcal{O}_{X,p}} \mathscr{T}_p;$$

(v) the meromorphic Cousin complex computes the derived-functor group H^1.

The theory of divisors and line bundles is taken from Chapter 7, while the canonical sheaf, meromorphic differentials, local residue pairing, and global residue theorem are taken from Chapter 8.

9.1 Laurent tails associated with a divisor

Let

$$D = \sum_{p \in X} n_p[p]$$

be a divisor. At each point choose a local parameter z_p. Although the following vector spaces are written in coordinates, the quotient spaces and pairings constructed from them will be intrinsic.

Definition 9.1. The *Laurent-tail space bounded by D* is

$$T[D] = \bigoplus_{p \in X} \left\{ \sum_{k < -n_p} a_{p,k} z_p^k \;\middle|\; \text{only finitely many coefficients are nonzero} \right\}.$$

Thus an element of $T[D]$ is a finite distribution of local Laurent tails, retaining at p exactly the terms whose exponent is strictly smaller than $-n_p$.

Definition 9.2. For a global meromorphic function $f \in \mathscr{M}_X(X)$, let

$$\alpha_D(f) \in T[D]$$

be the collection of the terms of the Laurent expansion of f at p whose exponent is strictly smaller than $-n_p$. Define the *Mittag–Leffler obstruction space*

$$H^1_{\mathrm{ML}}(D) = \operatorname{coker}\left(\alpha_D : \mathscr{M}_X(X) \longrightarrow T[D]\right).$$

Proposition 9.3 (Principal-parts comparison)**.** *There is a natural isomorphism*

$$\boxed{H^1_{\mathrm{ML}}(D) \cong H^1\big(X, \mathcal{O}_X(D)\big).}$$

Under this isomorphism, the class of a Laurent-tail distribution is its Mittag–Leffler obstruction class.

Proof. Apply the meromorphic Cousin resolution from Chapter 6 to the line bundle $\mathcal{O}_X(D)$. In a local trivialization determined by a local equation of D, the quotient of meromorphic sections by holomorphic sections is precisely the space of Laurent terms of exponent $< -n_p$. The degree-one cohomology of the Cousin resolution is therefore the cokernel of

$$\mathscr{M}_X(X) \xrightarrow{\alpha_D} T[D].$$

The comparison theorem for the Cousin resolution identifies this cohomology with the derived-functor group $H^1(X, \mathcal{O}_X(D))$. All identifications commute with refinement and changes of local equation, so the resulting isomorphism is natural. $\square$

From now on we write

$$H^1(D) = H^1\big(X, \mathcal{O}_X(D)\big)$$

and identify it with $H^1_{\mathrm{ML}}(D)$. The defining exact sequence is

$$0 \longrightarrow L(D) \longrightarrow \mathscr{M}_X(X) \xrightarrow{\alpha_D} T[D] \longrightarrow H^1(D) \longrightarrow 0.$$

Remark 9.4 (Coordinate independence)*.* A change of local parameter replaces the ordered monomials

$$z_p^{-1}, z_p^{-2}, \ldots$$

by another basis related by an invertible triangular transformation. Hence $T[D]$ itself depends on the chosen coordinates only up to a canonical change of presentation. More conceptually, it is the finite-support part of the sheaf quotient

$$\mathscr{M}_X / \mathcal{O}_X(D),$$

which is intrinsic.

Definition 9.5. Suppose $A \leq D$. The *truncation map*

$$t_{A,D} : T[A] \longrightarrow T[D]$$

forgets all Laurent terms whose exponents are at least $-\operatorname{ord}_p(D)$. It is compatible with the maps α_A and α_D and therefore induces a map

$$t_{A,D} : H^1(A) \longrightarrow H^1(D).$$

Proposition 9.6 (Kernel of truncation)**.** *If $A \leq D$, then $t_{A,D} : T[A] \to T[D]$ is surjective and*

$$\boxed{\dim_{\mathbb{C}} \ker(t_{A,D}) = \deg(D - A).}$$

Proof. Write

$$A = \sum_p a_p[p], \qquad D = \sum_p d_p[p].$$

At p, the kernel is spanned by the monomials

$$z_p^{-d_p}, z_p^{-d_p+1}, \dots, z_p^{-a_p-1}.$$

There are $d_p - a_p$ such monomials. Only finitely many of these integers are nonzero, so

$$\dim_{\mathbb{C}} \ker(t_{A,D}) = \sum_p (d_p - a_p) = \deg(D - A).$$

Surjectivity is immediate: a tail satisfying the stronger truncation bound for D is already a tail satisfying the weaker bound for A. ☐

Proposition 9.7 (Comparison exact sequence)**.** *If $A \leq D$, there is a natural exact sequence*

$$0 \longrightarrow L(A) \longrightarrow L(D) \longrightarrow \ker(t_{A,D}) \longrightarrow H^1(A) \longrightarrow H^1(D) \longrightarrow 0.$$

Proof. The maps α_A and α_D form the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & L(A) & \to & \mathscr{M}_X(X) & \xrightarrow{\alpha_A} & T[A] & \to & H^1(A) & \to 0 \\
 & & \downarrow & & \| & & \downarrow t_{A,D} & & \downarrow & \\
0 & \to & L(D) & \to & \mathscr{M}_X(X) & \xrightarrow{\alpha_D} & T[D] & \to & H^1(D) & \to 0.
\end{array}$$

The middle vertical map is the identity and the tail map is surjective. The snake lemma gives precisely the displayed sequence. ☐

9.2 The first Riemann–Roch formula

The general definition and additivity of the Euler characteristic were established in Chapter 6. We now combine those results with the divisor exact sequences of Chapter 7.

Proposition 9.8. *If $A \leq D$, then*

$$\boxed{\chi\big(X, \mathcal{O}_X(D)\big) - \chi\big(X, \mathcal{O}_X(A)\big) = \deg(D - A).}$$

Proof. Taking alternating dimensions in the exact sequence of theorem 9.7 gives

$$0 = h^0(A) - h^0(D) + \dim \ker(t_{A,D}) - h^1(A) + h^1(D).$$

Rearranging and applying theorem 9.6, we obtain

$$\chi(D) - \chi(A) = \deg(D - A).$$

Equivalently, one may apply additivity repeatedly to

$$0 \longrightarrow \mathcal{O}_X(E - p) \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathbb{C}_p \longrightarrow 0,$$

using $\chi(\mathbb{C}_p) = 1$. ☐

Proposition 9.9 (Euler characteristic of a divisor)**.** *For every divisor D,*

$$\boxed{\chi\big(X, \mathcal{O}_X(D)\big) = \deg D + \chi(X, \mathcal{O}_X).}$$

Proof. Choose an effective divisor E such that

$$-E \leq D \qquad \text{and} \qquad -E \leq 0.$$

Applying theorem 9.8 to the pairs $(-E, D)$ and $(-E, 0)$ gives

$$\chi(D) - \chi(-E) = \deg(D + E)$$

and

$$\chi(0) - \chi(-E) = \deg E.$$

Subtracting yields

$$\chi(D) - \chi(0) = \deg D.$$

□

Recall that the cohomological genus is

$$g = h^1(X, \mathcal{O}_X).$$

Since every global holomorphic function on a compact connected Riemann surface is constant,

$$h^0(X, \mathcal{O}_X) = 1 \qquad \text{and hence} \qquad \chi(X, \mathcal{O}_X) = 1 - g.$$

Theorem 9.10 (First Riemann–Roch formula)**.** *For every divisor D,*

$$\boxed{h^0\big(X, \mathcal{O}_X(D)\big) - h^1\big(X, \mathcal{O}_X(D)\big) = \deg D + 1 - g.}$$

Proof. By theorem 9.9,

$$\chi\big(X, \mathcal{O}_X(D)\big) = \deg D + \chi(X, \mathcal{O}_X) = \deg D + 1 - g.$$

Expanding the definition of χ gives the formula. □

Remark 9.11. This is the one-dimensional analogue of the Hilbert-polynomial theorem, but it is not yet the classical Riemann–Roch theorem. The unknown term

$$h^1\big(X, \mathcal{O}_X(D)\big)$$

has not yet been identified geometrically. Serre duality will identify its dual with a space of holomorphic differentials.

9.3 Meromorphic sections and degrees

The next results complete the relation between divisors and line bundles and provide a filtration of vector bundles by line bundles. This filtration will later allow us to extend duality from rank one to arbitrary rank.

Proposition 9.12 (Existence of meromorphic sections)**.** *Every holomorphic line bundle $\mathscr{L}$ on X admits a nonzero meromorphic section.*

Proof. Fix $p \in X$. For each integer n there is a short exact sequence

$$0 \longrightarrow \mathscr{L}((n-1)p) \longrightarrow \mathscr{L}(np) \longrightarrow \mathscr{L}(np)|_p \longrightarrow 0.$$

The quotient is a skyscraper sheaf of length one, so additivity of the Euler characteristic gives

$$\chi\big(\mathscr{L}(np)\big) = \chi(\mathscr{L}) + n.$$

For $n \gg 0$, this integer is positive. Since

$$\chi\big(\mathscr{L}(np)\big) = h^0\big(X, \mathscr{L}(np)\big) - h^1\big(X, \mathscr{L}(np)\big)$$

and $h^1 \geq 0$, one has

$$H^0\big(X, \mathscr{L}(np)\big) \neq 0.$$

A nonzero holomorphic section of $\mathscr{L}(np)$ is a meromorphic section of $\mathscr{L}$ with a pole of order at most n at p. $\square$

Corollary 9.13 (Divisor classes and the Picard group)**.** *The homomorphism constructed in Chapter 7 induces an isomorphism*

$$\boxed{\operatorname{Div}(X)/\operatorname{Prin}(X) \xrightarrow{\sim} \operatorname{Pic}(X).}$$

Proof. Chapter 7 proved that the kernel of

$$D \longmapsto \mathcal{O}_X(D)$$

is the group of principal divisors. Conversely, let $\mathscr{L}$ be a line bundle. By theorem 9.12, choose a nonzero meromorphic section s. The divisor-line-bundle construction gives

$$\mathscr{L} \cong \mathcal{O}_X\big(\operatorname{div}(s)\big),$$

so the map is surjective. $\square$

Definition 9.14. If $\mathscr{L} \cong \mathcal{O}_X(D)$, define the *degree* of $\mathscr{L}$ by

$$\deg \mathscr{L} = \deg D.$$

Proposition 9.15 (Basic properties of degree)**.** *The degree of a line bundle is well-defined. Moreover,*

$$\deg(\mathscr{L} \otimes \mathscr{M}) = \deg \mathscr{L} + \deg \mathscr{M},$$

$$\deg(\mathscr{L}^\vee) = -\deg \mathscr{L},$$

and

$$\deg \mathcal{O}_X(D) = \deg D.$$

Proof. If $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$, then $D - E$ is principal by theorem 9.13. Principal divisors have degree zero, hence $\deg D = \deg E$, proving well-definedness.

Choose divisor representatives $\mathscr{L} \cong \mathcal{O}_X(D)$ and $\mathscr{M} \cong \mathcal{O}_X(E)$. The tensor-product and duality formulas from Chapter 7 give

$$\mathscr{L} \otimes \mathscr{M} \cong \mathcal{O}_X(D+E), \qquad \mathscr{L}^\vee \cong \mathcal{O}_X(-D).$$

Taking degrees proves all assertions. $\square$

Proposition 9.16 (Degree under pullback)**.** *Let*

$$f : X \longrightarrow Y$$

be a nonconstant holomorphic map of compact connected Riemann surfaces of degree d. For every line bundle $\mathscr{L}$ on Y,

$$\boxed{\deg(f^*\mathscr{L}) = d \deg \mathscr{L}.}$$

Proof. Choose a divisor $D = \sum_q n_q[q]$ on Y with $\mathscr{L} \cong \mathcal{O}_Y(D)$. Then

$$f^*\mathscr{L} \cong \mathcal{O}_X(f^*D),$$

where

$$f^*D = \sum_{p \in X} n_{f(p)} e_p[p] = \sum_{q \in Y} n_q \sum_{p \in f^{-1}(q)} e_p[p].$$

For every $q \in Y$, the sum of the local degrees over the fiber is d:

$$\sum_{p \in f^{-1}(q)} e_p = d.$$

Therefore

$$\deg(f^*D) = \sum_q n_q d = d \deg D.$$

□

Corollary 9.17 (Existence of canonical divisors)**.** *The canonical sheaf ω_X admits a nonzero meromorphic section. If η is such a section and*

$$K = \operatorname{div}(\eta),$$

then

$$\omega_X \cong \mathcal{O}_X(K).$$

Any two such canonical divisors are linearly equivalent.

Proof. Apply theorem 9.12 to ω_X. The remaining assertions were proved for meromorphic sections of line bundles in Chapter 7. □

9.3.1 Filtrations of vector bundles

Proposition 9.18 (Meromorphic sections of vector bundles)**.** *Every nonzero holomorphic vector bundle $\mathscr{E}$ on X admits a nonzero meromorphic section.*

Proof. Let $r = \operatorname{rk} \mathscr{E}$ and fix $p \in X$. For every integer n, the quotient in

$$0 \longrightarrow \mathscr{E}((n-1)p) \longrightarrow \mathscr{E}(np) \longrightarrow \mathscr{E}(np)|_p \longrightarrow 0$$

is a skyscraper sheaf whose fiber is an r-dimensional complex vector space. Hence its length is r, and additivity gives

$$\chi\big(\mathscr{E}(np)\big) = \chi(\mathscr{E}) + nr.$$

For $n \gg 0$ the right-hand side is positive, so

$$H^0\big(X, \mathscr{E}(np)\big) \neq 0.$$

A nonzero holomorphic section of $\mathscr{E}(np)$ is a nonzero meromorphic section of $\mathscr{E}$. □

Definition 9.19. Let $\mathscr{F} \subseteq \mathscr{E}$ be a coherent subsheaf of a vector bundle. Its *saturation* $\mathscr{F}^{\mathrm{sat}}$ is the inverse image in $\mathscr{E}$ of the torsion subsheaf of $\mathscr{E}/\mathscr{F}$. Equivalently, it is the smallest coherent subsheaf containing $\mathscr{F}$ such that

$$\mathscr{E}/\mathscr{F}^{\mathrm{sat}}$$

is torsion-free.

Lemma 9.20 (Saturation of a rank-one subsheaf). *Let $\mathscr{E}$ be a vector bundle and let*

$$\mathcal{O}_X \longrightarrow \mathscr{E}$$

be a nonzero morphism. The saturation of its image is a line subbundle $\mathscr{L} \subseteq \mathscr{E}$, and the quotient $\mathscr{E}/\mathscr{L}$ is locally free.

Proof. The assertion is local. At p, write

$$R = \mathcal{O}_{X,p}, \qquad \mathscr{E}_p \cong R^r.$$

The local ring R is a discrete valuation ring with uniformizer z. The image is generated by a nonzero vector

$$v = (v_1, \ldots, v_r) \in R^r.$$

Let m be the minimum of the valuations of the components and write

$$v = z^m v',$$

where at least one component of v' is a unit. Then v' is a primitive vector. Over a principal ideal domain, a primitive vector can be completed to a basis of R^r. Hence

$$Rv' \subseteq R^r$$

is a free direct summand of rank one and the quotient is free of rank $r-1$.

The module Rv' is exactly the saturation of Rv: an element $w \in R^r$ has a nonzero multiple in Rv if and only if $w \in Rv'$. These local saturated submodules are uniquely characterized and therefore glue. Thus the saturation is a line bundle and the quotient is locally free. □

Theorem 9.21 (Line-bundle filtration). *Every holomorphic vector bundle $\mathscr{E}$ of rank r admits a filtration by subbundles*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_r = \mathscr{E}$$

such that each quotient

$$\mathscr{E}_j/\mathscr{E}_{j-1}$$

is a line bundle.

Proof. We argue by induction on r. The case $r = 1$ is immediate. Suppose $r > 1$. By theorem 9.18, after twisting by a divisor we obtain a nonzero holomorphic morphism from a line bundle to $\mathscr{E}$. More explicitly, a nonzero section of $\mathscr{E}(np)$ gives a morphism

$$\mathcal{O}_X(-np) \longrightarrow \mathscr{E}.$$

It is injective because its kernel is a torsion subsheaf of the torsion-free sheaf $\mathcal{O}_X(-np)$. By theorem 9.20, the saturation of its image is a line subbundle $\mathscr{L} \subseteq \mathscr{E}$ and

$$\mathscr{E}' = \mathscr{E}/\mathscr{L}$$

is a vector bundle of rank $r-1$. Apply the induction hypothesis to $\mathscr{E}'$ and pull the resulting filtration back along $\mathscr{E} \to \mathscr{E}'$. □

9.4 The trace map and the duality morphisms

9.4.1 The trace on $H^1(X, \omega_X)$

By theorem 9.3, a class in $H^1(X, \omega_X)$ may be represented by a finite distribution of meromorphic differential tails.

Definition 9.22. The *trace map* is

$$\mathrm{tr}_X : H^1(X, \omega_X) \longrightarrow \mathbb{C}, \qquad [\xi] \longmapsto \sum_{p \in X} \mathrm{Res}_p(\xi_p).$$

Proposition 9.23. *The trace map is well-defined and independent of local coordinates, representatives, and refinements.*

Proof. Only finitely many tails occur, so the sum is finite. Coordinate independence follows from the coordinate invariance of the residue. If a representative is changed by the principal parts of a global meromorphic differential θ, the trace changes by

$$\sum_{p \in X} \mathrm{Res}_p(\theta),$$

which is zero by the global residue theorem. Refinement merely rewrites the same germs on smaller neighborhoods and does not change their residues. □

9.4.2 Natural duality morphisms

The trace defines duality maps for all coherent sheaves, even before we know that they are isomorphisms.

Definition 9.24. Let $\mathscr{F}$ be a coherent sheaf.

For $u \in \mathrm{Hom}_X(\mathscr{F}, \omega_X)$ and $\xi \in H^1(X, \mathscr{F})$, define

$$\langle \xi, u \rangle = \mathrm{tr}_X\big(H^1(u)(\xi)\big).$$

This gives a natural map

$$\mathrm{SD}^0_{\mathscr{F}} : \mathrm{Hom}_X(\mathscr{F}, \omega_X) \longrightarrow H^1(X, \mathscr{F})^\vee.$$

For an extension class

$$e \in \mathrm{Ext}^1_X(\mathscr{F}, \omega_X)$$

and a section

$$s \in H^0(X, \mathscr{F}) = \mathrm{Hom}_X(\mathcal{O}_X, \mathscr{F}),$$

pull back the extension along s. This gives

$$s^*e \in \mathrm{Ext}^1_X(\mathcal{O}_X, \omega_X) \cong H^1(X, \omega_X).$$

Define

$$\langle s, e \rangle = \mathrm{tr}_X(s^*e).$$

This gives a natural map

$$\mathrm{SD}^1_{\mathscr{F}} : \mathrm{Ext}^1_X(\mathscr{F}, \omega_X) \longrightarrow H^0(X, \mathscr{F})^\vee.$$

Proposition 9.25 (Naturality). *For every morphism $v : \mathscr{F}' \to \mathscr{F}$, the diagrams*

$$\begin{array}{ccc} \mathrm{Hom}_X(\mathscr{F}, \omega_X) & \xrightarrow{\mathrm{SD}^0_{\mathscr{F}}} & H^1(X, \mathscr{F})^\vee \\ \downarrow v^* & & \downarrow H^1(v)^\vee \\ \mathrm{Hom}_X(\mathscr{F}', \omega_X) & \xrightarrow{\mathrm{SD}^0_{\mathscr{F}'}} & H^1(X, \mathscr{F}')^\vee \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Ext}^1_X(\mathscr{F}, \omega_X) & \xrightarrow{\mathrm{SD}^1_{\mathscr{F}}} & H^0(X, \mathscr{F})^\vee \\ \downarrow v^* & & \downarrow H^0(v)^\vee \\ \mathrm{Ext}^1_X(\mathscr{F}', \omega_X) & \xrightarrow{\mathrm{SD}^1_{\mathscr{F}'}} & H^0(X, \mathscr{F}')^\vee \end{array}$$

commute.

Proof. For $i = 0$, one has
$$H^1(u \circ v) = H^1(u) \circ H^1(v),$$
so the first diagram follows immediately from the definition.

For $i = 1$, let e be an extension of $\mathscr{F}$ by ω_X and let s' be a section of $\mathscr{F}'$. Pullback is associative:
$$(s')^*(v^*e) = (v \circ s')^*e.$$
Applying the trace proves the second commutative diagram. □

Lemma 9.26 (Compatibility with long exact sequences). *Let*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

be a short exact sequence of coherent sheaves. With the standard signs on the dual of the cohomology long exact sequence, the maps SD^0 *and* SD^1 *form a morphism from the contravariant Ext long exact sequence to the dual cohomology long exact sequence.*

Proof. All squares not involving a connecting morphism follow from theorem 9.25. For the remaining square, let
$$\delta : H^0(X, \mathscr{F}'') \longrightarrow H^1(X, \mathscr{F}')$$
be the cohomological connecting homomorphism, represented by the extension class of the given short exact sequence. The Ext connecting homomorphism is Yoneda composition with the same extension class. Associativity of pullback and Yoneda composition shows that the two ways of pairing an element with the connecting class agree. The standard sign in the dual exact sequence accounts for the reversal of arrows. Thus the complete diagram commutes. □

9.4.3 The residue pairing for a divisor

Definition 9.27. For a divisor D, define

$$\langle -, - \rangle_D : H^1(D) \times H^0\big(X, \omega_X(-D)\big) \longrightarrow \mathbb{C}$$

by

$$\langle [P], \eta \rangle_D = \sum_{p \in X} \mathrm{Res}_p(P_p \eta).$$

Proposition 9.28 (Well-definedness of the residue pairing). *The residue pairing is independent of local parameters and tail representatives, vanishes on principal tails, and is natural under truncation of divisors.*

Proof. Coordinate independence follows from the coordinate invariance of residues. If two representatives differ by terms of exponent at least $-\operatorname{ord}_p(D)$, multiplying by a differential of order at least $\operatorname{ord}_p(D)$ produces a holomorphic differential, whose residue is zero.

If P is changed by $\alpha_D(f)$ for a global meromorphic function f, the value changes by

$$\sum_p \operatorname{Res}_p(f\eta),$$

which vanishes by the global residue theorem. Thus the pairing descends to $H^1(D)$. If $A \leq D$, the terms discarded by $t_{A,D}$ cannot contribute against a section of $\omega_X(-D)$, proving naturality. □

Proposition 9.29. *Under the natural identification*

$$\operatorname{Hom}_X\big(\mathcal{O}_X(D), \omega_X\big) \cong H^0\big(X, \omega_X(-D)\big),$$

the residue pairing is the pairing defining

$$\mathrm{SD}^0_{\mathcal{O}_X(D)}.$$

Proof. A morphism $u : \mathcal{O}_X(D) \to \omega_X$ is multiplication by a global section η of $\omega_X(-D)$. If $[P] \in H^1(D)$ is represented by Laurent tails, then $H^1(u)([P])$ is represented by the differential tails $P_p\eta$. Applying the trace gives

$$\operatorname{tr}_X(H^1(u)([P])) = \sum_p \operatorname{Res}_p(P_p\eta),$$

which is the residue pairing. □

The pairing induces a map

$$\rho_D : H^0\big(X, \omega_X(-D)\big) \longrightarrow H^1(D)^\vee.$$

Proposition 9.30 (Injectivity)**.** *The map ρ_D is injective.*

Proof. Let $0 \neq \eta \in H^0(X, \omega_X(-D))$. Choose p and a local coordinate z such that

$$\eta = (a_m z^m + a_{m+1} z^{m+1} + \cdots)\, dz, \qquad a_m \neq 0.$$

Since $m \geq \operatorname{ord}_p(D)$, one has

$$-m - 1 < -\operatorname{ord}_p(D),$$

so the tail z^{-m-1} supported at p belongs to $T[D]$. Its pairing with η is

$$\operatorname{Res}_p(z^{-m-1}\eta) = a_m \neq 0.$$

Therefore the functional defined by η is nonzero. □

9.5 Serre duality for line bundles

The difficult point is surjectivity of ρ_D. We follow the finite-dimensional growth argument behind the adelic proof, expressed entirely through Laurent tails and multiplication operators.

9.5.1 Multiplication and truncation

Let f be a nonzero meromorphic function. Multiplication by f induces an isomorphism

$$\mu_f^E : T[E + \operatorname{div}(f)] \xrightarrow{\sim} T[E].$$

Indeed, multiplication sends local functions holomorphic with respect to the bound $E + \operatorname{div}(f)$ to functions holomorphic with respect to the bound E, and multiplication by $1/f$ is the inverse.

Definition 9.31. Let A be a divisor, let $C \geq 0$ be effective, and let $f \in L(C)$. Since

$$A - C \leq A + \operatorname{div}(f),$$

define the *multiplication–truncation operator*

$$M_f^{A,C} = \mu_f^A \circ t_{A-C,A+\operatorname{div}(f)} : T[A - C] \longrightarrow T[A].$$

Proposition 9.32. *For every global meromorphic function g,*

$$\alpha_A(fg) = M_f^{A,C}\big(\alpha_{A-C}(g)\big).$$

Consequently, $M_f^{A,C}$ induces a linear map

$$M_f^{A,C} : H^1(A - C) \longrightarrow H^1(A).$$

If $f \neq 0$, this induced map is surjective.

Proof. The first identity is a direct comparison of Laurent expansions: truncate g at the bound $A - C$, discard terms which become irrelevant after multiplication by f, and then multiply by f. This produces exactly the tail of fg at the bound A.

Therefore principal tails are sent to principal tails, so the map descends to the cokernels. The tail map is surjective because the truncation is surjective and μ_f^A is an isomorphism. In the commutative diagram defining the cokernels, multiplication by f on the field $\mathscr{M}_X(X)$ is an isomorphism. Hence the induced map on cokernels is surjective. ☐

Lemma 9.33 (Growth lemma)**.** *Let A be a divisor and let*

$$0 \neq \varphi_1, \varphi_2 \in H^1(A)^\vee.$$

There exist an effective divisor C and nonzero functions

$$f_1, f_2 \in L(C)$$

such that

$$\varphi_1 \circ M_{f_1}^{A,C} = \varphi_2 \circ M_{f_2}^{A,C}$$

as functionals on $H^1(A - C)$.

Proof. Fix $p \in X$ and take $C = np$. Define

$$\Phi_n : L(np) \oplus L(np) \longrightarrow H^1(A - np)^\vee$$

by

$$\Phi_n(f_1, f_2) = \varphi_1 \circ M_{f_1}^{A,np} - \varphi_2 \circ M_{f_2}^{A,np}.$$

The map is linear because multiplication of tails is linear in f_i.

Suppose no desired relation exists. Then every Φ_n is injective. Indeed, if a nonzero pair lies in the kernel and one component is zero, say $f_1 = 0$, then

$$\varphi_2 \circ M_{f_2}^{A,np} = 0.$$

When $f_2 \neq 0$, theorem 9.32 says that $M_{f_2}^{A,np}$ is surjective, contradicting $\varphi_2 \neq 0$. Thus every nonzero kernel element would already have both components nonzero and would provide the required relation.

Injectivity gives

$$2h^0(np) \leq h^1(A - np).$$

By the first Riemann–Roch formula,

$$h^1(A - np) = h^0(A - np) - \deg A + n - 1 + g.$$

Since $L(A - np) \subseteq L(A)$, the right-hand side is bounded above by $n + c$ for a constant c independent of n. On the other hand,

$$h^0(np) = n + 1 - g + h^1(np) \geq n + 1 - g.$$

Thus the left-hand side grows at least as $2n + 2 - 2g$, whereas the right-hand side grows at most as $n + c$. This is impossible for $n \gg 0$. $\square$

Lemma 9.34 (Descent lemma)**.** *Let $B \leq A$ and let*

$$\theta \in H^0\big(X, \omega_X(-B)\big).$$

If the residue functional defined by θ on $T[B]$ vanishes on

$$\ker(t_{B,A}),$$

then

$$\theta \in H^0\big(X, \omega_X(-A)\big).$$

Proof. Fix p and write

$$\theta = (a_m z^m + a_{m+1} z^{m+1} + \cdots)\, dz, \qquad a_m \neq 0.$$

Because $\theta \in \omega_X(-B)$, one has $m \geq \operatorname{ord}_p(B)$. Suppose that

$$m < \operatorname{ord}_p(A).$$

Then the monomial z^{-m-1} belongs to $T[B]$ but is killed by truncation to $T[A]$. Yet

$$\operatorname{Res}_p(z^{-m-1}\theta) = a_m \neq 0,$$

contradicting the hypothesis. Therefore $m \geq \operatorname{ord}_p(A)$ at every point, which is exactly the condition $\theta \in \omega_X(-A)$. $\square$

Lemma 9.35 (Compatibility with multiplication)**.** *Let f be a nonzero meromorphic function and let*

$$\eta \in H^0\big(X, \omega_X(-E)\big).$$

Then

$$f\eta \in H^0\big(X, \omega_X(-E - \operatorname{div}(f))\big),$$

and under the multiplication isomorphism

$$\mu_f^E : T[E + \operatorname{div}(f)] \to T[E]$$

one has

$$\rho_{E+\operatorname{div}(f)}(f\eta) = \rho_E(\eta) \circ \mu_f^E.$$

Proof. The divisor assertion follows from

$$\operatorname{div}(f\eta) = \operatorname{div}(f) + \operatorname{div}(\eta).$$

For a tail P in the source,

$$\operatorname{Res}_p\big((fP)\eta\big) = \operatorname{Res}_p\big(P(f\eta)\big).$$

Summing over p gives the functional identity. $\square$

Theorem 9.36 (Serre duality for divisors)**.** *For every divisor D, the residue map is an isomorphism*

$$\boxed{\rho_D : H^0\big(X, \omega_X(-D)\big) \xrightarrow{\sim} H^1\big(X, \mathcal{O}_X(D)\big)^\vee.}$$

Equivalently, the residue pairing

$$H^1\big(X, \mathcal{O}_X(D)\big) \times H^0\big(X, \omega_X(-D)\big) \longrightarrow \mathbb{C}$$

is perfect.

Proof. Injectivity is theorem 9.30. Let

$$0 \neq \varphi \in H^1(D)^\vee.$$

Choose a nonzero meromorphic differential η and put

$$K = \operatorname{div}(\eta).$$

Choose a divisor A with

$$A \leq D \qquad \text{and} \qquad A \leq K.$$

Truncation gives a surjection

$$t_{A,D} : H^1(A) \longrightarrow H^1(D),$$

so

$$\varphi_A = \varphi \circ t_{A,D}$$

is a nonzero functional on $H^1(A)$. Since $A \leq K$, the differential η belongs to $H^0(X, \omega_X(-A))$, and its residue functional

$$\psi_A = \rho_A(\eta)$$

is nonzero by injectivity.

Apply the growth lemma to φ_A and ψ_A. There exist an effective divisor C and nonzero $f_1, f_2 \in L(C)$ such that

$$\varphi_A \circ M_{f_1}^{A,C} = \psi_A \circ M_{f_2}^{A,C}$$

on $H^1(A - C)$, hence on $T[A - C]$ modulo principal tails.

Set

$$B = A - C - \operatorname{div}(f_1)$$

and

$$\theta = \frac{f_2}{f_1}\eta.$$

The map

$$\mu_{1/f_1}^{A-C} : T[B] \xrightarrow{\sim} T[A - C]$$

is the inverse multiplication isomorphism corresponding to f_1. Precompose the equality of functionals with this map. By the definition of $M_{f_1}^{A,C}$, the left-hand side becomes

$$\varphi_A \circ t_{B,A}.$$

By theorem 9.35, the right-hand side becomes the residue functional of θ on $T[B]$. Therefore

$$\varphi_A \circ t_{B,A} = \rho_B(\theta).$$

We first verify that θ lies in $\omega_X(-B)$. Since $f_2 \in L(C)$ and $\eta \in \omega_X(-A)$,

$$\operatorname{div}(f_2) + C \geq 0, \qquad \operatorname{div}(\eta) - A \geq 0.$$

Hence

$$\begin{aligned}\operatorname{div}(\theta) &= \operatorname{div}(f_2) - \operatorname{div}(f_1) + \operatorname{div}(\eta) \\ &\geq -C - \operatorname{div}(f_1) + A = B.\end{aligned}$$

Thus $\rho_B(\theta)$ is defined. Since the left-hand side vanishes on $\ker(t_{B,A})$, so does $\rho_B(\theta)$. The descent lemma implies

$$\theta \in H^0(X, \omega_X(-A)).$$

Because $t_{B,A}$ is surjective, the equality above now gives

$$\varphi_A = \rho_A(\theta).$$

Finally, φ_A is pulled back from $H^1(D)$ and therefore vanishes on $\ker(t_{A,D})$. Hence $\rho_A(\theta)$ vanishes on this kernel. A second application of the descent lemma yields

$$\theta \in H^0(X, \omega_X(-D)).$$

The functional induced by θ on $H^1(D)$ is precisely φ. Thus ρ_D is surjective. □

Corollary 9.37 (Line-bundle form)**.** *For every holomorphic line bundle $\mathscr{L}$ on X, there is a natural perfect pairing*

$$H^1(X, \mathscr{L}) \times H^0\big(X, \omega_X \otimes \mathscr{L}^{-1}\big) \longrightarrow \mathbb{C},$$

and therefore a natural isomorphism

$$\boxed{H^1(X, \mathscr{L})^\vee \cong H^0\big(X, \omega_X \otimes \mathscr{L}^{-1}\big).}$$

Proof. Choose D with $\mathscr{L} \cong \mathcal{O}_X(D)$. Then

$$\omega_X \otimes \mathscr{L}^{-1} \cong \omega_X(-D),$$

and the result follows from theorem 9.36. Replacing D by a linearly equivalent divisor changes both factors by the same multiplication isomorphism, so the pairing is independent of the choice. □

Theorem 9.38 (Ext form for line bundles)**.** *For every line bundle $\mathscr{L}$ and $i = 0, 1$, the natural duality morphism is an isomorphism*

$$\boxed{\mathrm{SD}^i_{\mathscr{L}} : \operatorname{Ext}^i_X(\mathscr{L}, \omega_X) \xrightarrow{\sim} H^{1-i}(X, \mathscr{L})^\vee.}$$

Proof. For $i = 0$,

$$\operatorname{Hom}_X(\mathscr{L}, \omega_X) = H^0(X, \omega_X \otimes \mathscr{L}^{-1}),$$

and theorems 9.29 and 9.37 identify $\mathrm{SD}^0_{\mathscr{L}}$ with the line-bundle residue isomorphism.

For $i = 1$, Chapter 6 gives

$$\operatorname{Ext}^1_X(\mathscr{L}, \omega_X) \cong H^1(X, \omega_X \otimes \mathscr{L}^{-1}).$$

Apply theorem 9.37 to the line bundle $\omega_X \otimes \mathscr{L}^{-1}$:

$$H^1(X, \omega_X \otimes \mathscr{L}^{-1})^\vee \cong H^0(X, \mathscr{L}).$$

All spaces are finite-dimensional, so dualizing yields

$$\operatorname{Ext}^1_X(\mathscr{L}, \omega_X) \cong H^0(X, \mathscr{L})^\vee.$$

Naturality and the definition by pullback of extensions show that this is precisely $\mathrm{SD}^1_{\mathscr{L}}$. □

9.6 Local duality for torsion sheaves

We next prove duality for sheaves supported at finitely many points. The local calculation is the one-dimensional form of Matlis duality and explains why the canonical sheaf is the correct dualizing object.

9.6.1 Finite-length modules over a discrete valuation ring

Let

$$R = \mathcal{O}_{X,p}, \qquad K = \mathrm{Frac}(R), \qquad \omega_R = \omega_{X,p}.$$

After choosing a local parameter z, one has

$$R \cong \mathbb{C}\{z\}, \qquad \omega_R = R\,dz.$$

Proposition 9.39. *If M is a finite-length R-module, then*

$$\mathrm{Hom}_R(M, \omega_R) = 0.$$

Proof. Every element of M is annihilated by a nonzero power of z. If

$$\varphi : M \to \omega_R$$

is R-linear and $z^n m = 0$, then

$$z^n \varphi(m) = \varphi(z^n m) = 0.$$

The module ω_R is torsion-free, so $\varphi(m) = 0$. Hence $\varphi = 0$. □

Theorem 9.40 (Cyclic local duality)**.** *For every $n \geq 1$, there is a natural perfect pairing*

$$R/(z^n) \times \mathrm{Ext}^1_R\big(R/(z^n), \omega_R\big) \longrightarrow \mathbb{C}.$$

Consequently,

$$\boxed{\mathrm{Ext}^1_R\big(R/(z^n), \omega_R\big) \cong \mathrm{Hom}_{\mathbb{C}}\big(R/(z^n), \mathbb{C}\big).}$$

Proof. Apply $\mathrm{Hom}_R(-, \omega_R)$ to the free resolution

$$0 \longrightarrow R \xrightarrow{\cdot z^n} R \longrightarrow R/(z^n) \longrightarrow 0.$$

Using theorem 9.39, one obtains

$$\mathrm{Ext}^1_R\big(R/(z^n), \omega_R\big) \cong \omega_R / z^n \omega_R.$$

Multiplication by z^{-n} identifies this quotient with

$$z^{-n}\omega_R/\omega_R,$$

the space of meromorphic differential principal parts of order at most n. Define

$$\langle \bar{a}, \bar{\eta} \rangle = \mathrm{Res}_p(a\eta),$$

where $a \in R$ represents $\bar{a} \in R/(z^n)$ and $\eta \in z^{-n}\omega_R$ represents the corresponding Ext class. Changing a by $z^n b$ changes $a\eta$ by a holomorphic differential, and changing η by a holomorphic differential also leaves the residue unchanged. Thus the pairing is well-defined.

The classes

$$1, z, \ldots, z^{n-1}$$

form a basis of $R/(z^n)$, while

$$z^{-1}dz, z^{-2}dz, \ldots, z^{-n}dz$$

form a basis of $z^{-n}\omega_R/\omega_R$. One has

$$\mathrm{Res}_p\big(z^i z^{-j-1} dz\big) = \delta_{ij}$$

for $0 \leq i, j \leq n-1$. Hence the pairing is perfect. □

Proposition 9.41 (Structure of finite-length modules)**.** *Every finite-length R-module is isomorphic to a finite direct sum*

$$M \cong \bigoplus_{j=1}^{m} R/(z^{n_j}).$$

Proof. The ring R is a principal ideal domain. The structure theorem for finitely generated modules over a principal ideal domain decomposes M as a direct sum of a free part and cyclic torsion modules. Finite length excludes a nonzero free part, and every nonzero ideal is generated by a power of the uniformizer, giving the stated form. ☐

Theorem 9.42 (Finite-length local duality)**.** *For every finite-length R-module M, residue induces a natural isomorphism*

$$\boxed{\operatorname{Ext}^1_R(M, \omega_R) \cong \operatorname{Hom}_{\mathbb{C}}(M, \mathbb{C}).}$$

Moreover,

$$\operatorname{Ext}^i_R(M, \omega_R) = 0 \qquad (i \neq 1).$$

Proof. Consider the exact sequence

$$0 \longrightarrow \omega_R \longrightarrow K \otimes_R \omega_R \longrightarrow (K \otimes_R \omega_R)/\omega_R \longrightarrow 0.$$

The middle module is divisible over the principal ideal domain R and is therefore injective. Since M is torsion while $K \otimes_R \omega_R$ is torsion-free,

$$\operatorname{Hom}_R(M, K \otimes_R \omega_R) = 0.$$

The associated long exact sequence gives a natural isomorphism

$$\operatorname{Ext}^1_R(M, \omega_R) \cong \operatorname{Hom}_R\big(M, (K \otimes_R \omega_R)/\omega_R\big).$$

Composition with residue defines a natural map

$$\operatorname{Hom}_R\big(M, (K \otimes_R \omega_R)/\omega_R\big) \longrightarrow \operatorname{Hom}_{\mathbb{C}}(M, \mathbb{C}).$$

By theorem 9.41, it is enough to check that this map is an isomorphism for $M = R/(z^n)$, and that is exactly the perfect pairing of theorem 9.40. Hence the map is an isomorphism for every finite-length module.

The group $\operatorname{Ext}^0_R(M, \omega_R)$ vanishes by theorem 9.39. The structure theorem expresses M as a finite direct sum of modules with free resolutions of length one, so all higher Ext groups vanish. ☐

Remark 9.43. The decomposition in theorem 9.41 is useful for the proof, but the resulting pairing is intrinsic. It is obtained by representing an Ext class by a meromorphic differential principal part and applying the coordinate-independent residue.

9.6.2 Torsion sheaves on a compact Riemann surface

Proposition 9.44 (Finite filtration by skyscraper sheaves)**.** *Every coherent torsion sheaf $\mathscr{T}$ on X has finite support and admits a finite filtration*

$$0 = \mathscr{T}_0 \subset \mathscr{T}_1 \subset \cdots \subset \mathscr{T}_N = \mathscr{T}$$

whose successive quotients are skyscraper sheaves $\mathbb{C}_{p_j}$.

Proof. The support of a coherent torsion sheaf on a Riemann surface is discrete; compactness makes it finite. At each point p of the support, the stalk $\mathscr{T}_p$ is a finite-length module over the local ring $\mathcal{O}_{X,p}$. Such a module has a composition series with successive quotients equal to the residue field $\mathbb{C}$. A sheaf supported at finitely many points is the direct sum of its subsheaves supported at the individual points, and the stalk composition series therefore globalize to a finite filtration with skyscraper quotients. □

Proposition 9.45 (Duality for a skyscraper sheaf)**.** *For every $p \in X$, the natural duality morphisms are isomorphisms*

$$\mathrm{SD}^0_{\mathbb{C}_p} : \mathrm{Hom}_X(\mathbb{C}_p, \omega_X) \xrightarrow{\sim} H^1(X, \mathbb{C}_p)^\vee = 0$$

and

$$\boxed{\mathrm{SD}^1_{\mathbb{C}_p} : \mathrm{Ext}^1_X(\mathbb{C}_p, \omega_X) \xrightarrow{\sim} H^0(X, \mathbb{C}_p)^\vee \cong \mathbb{C}.}$$

Furthermore,

$$\mathrm{Ext}^i_X(\mathbb{C}_p, \omega_X) = 0 \qquad (i > 1).$$

Proof. The group $\mathrm{Hom}_X(\mathbb{C}_p, \omega_X)$ vanishes because ω_X is torsion-free, while $H^1(X, \mathbb{C}_p) = 0$ by Chapter 6. Hence the degree-zero map is an isomorphism.

Consider the residue sequence from Chapter 8:

$$0 \longrightarrow \omega_X \longrightarrow \omega_X(p) \xrightarrow{\mathrm{Res}_p} \mathbb{C}_p \longrightarrow 0.$$

Its extension class

$$e_p \in \mathrm{Ext}^1_X(\mathbb{C}_p, \omega_X)$$

pairs with the section $1 \in H^0(X, \mathbb{C}_p)$ as follows. Pulling the extension back along $1 : \mathcal{O}_X \to \mathbb{C}_p$ produces the principal-parts class of dz/z at p, whose trace is

$$\mathrm{Res}_p(dz/z) = 1.$$

Thus $\mathrm{SD}^1_{\mathbb{C}_p}(e_p)$ is the nonzero functional sending 1 to 1.

It remains to see that the source is one-dimensional. Apply $\mathrm{Hom}_X(-, \omega_X)$ to

$$0 \longrightarrow \mathcal{O}_X(-p) \longrightarrow \mathcal{O}_X \longrightarrow \mathbb{C}_p \longrightarrow 0.$$

The inclusion

$$H^0(X, \omega_X) \longrightarrow H^0(X, \omega_X(p))$$

is an isomorphism: a meromorphic differential with at most one simple pole has total residue zero, so its residue at p is zero and the apparent pole is removable. Furthermore, line-bundle duality gives

$$H^1(X, \omega_X) \cong H^0(X, \mathcal{O}_X)^\vee \cong \mathbb{C}$$

and

$$H^1(X, \omega_X(p)) \cong H^0(X, \mathcal{O}_X(-p))^\vee = 0.$$

The Ext long exact sequence therefore yields

$$\mathrm{Ext}^1_X(\mathbb{C}_p, \omega_X) \cong H^1(X, \omega_X) \cong \mathbb{C}.$$

The same sequence, together with the vanishing of higher Ext for line bundles, gives

$$\mathrm{Ext}^i_X(\mathbb{C}_p, \omega_X) = 0 \qquad (i > 1).$$

□

Theorem 9.46 (Serre duality for torsion sheaves). *Let $\mathscr{T}$ be a coherent torsion sheaf. For $i = 0, 1$, the natural duality morphisms are isomorphisms*

$$\boxed{\mathrm{SD}^i_{\mathscr{T}} : \mathrm{Ext}^i_X(\mathscr{T}, \omega_X) \xrightarrow{\sim} H^{1-i}(X, \mathscr{T})^\vee.}$$

Moreover,

$$\mathrm{Ext}^i_X(\mathscr{T}, \omega_X) = 0 \qquad (i > 1).$$

In particular,

$$\mathrm{Hom}_X(\mathscr{T}, \omega_X) = 0$$

and

$$\boxed{\mathrm{Ext}^1_X(\mathscr{T}, \omega_X) \cong H^0(X, \mathscr{T})^\vee.}$$

Proof. Use induction on the length of $\mathscr{T}$. The zero sheaf is trivial, and the length-one case is theorem 9.45. For positive length, choose a short exact sequence from the filtration of theorem 9.44:

$$0 \longrightarrow \mathscr{T}' \longrightarrow \mathscr{T} \longrightarrow \mathbb{C}_p \longrightarrow 0$$

with $\mathscr{T}'$ of smaller length.

By the induction hypothesis and theorem 9.45, duality is known for the outer terms. The compatibility lemma theorem 9.26 gives a commutative diagram between the Ext long exact sequence and the dual cohomology long exact sequence. Since H^1 vanishes for torsion sheaves, these sequences are finite. The five lemma, or a direct diagram chase, shows that both duality maps for $\mathscr{T}$ are isomorphisms. The same Ext long exact sequence and induction give the vanishing in degrees greater than one. ☐

9.7 Serre duality for coherent sheaves

We now pass from rank one and finite support to arbitrary coherent sheaves. The two structural inputs are the filtration of a vector bundle by line bundles and the decomposition of a coherent sheaf into its torsion subsheaf and a locally free quotient.

Proposition 9.47 (Extension step for vector bundles). *Let*

$$0 \longrightarrow \mathscr{E}' \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}'' \longrightarrow 0$$

be a short exact sequence of vector bundles. If the duality morphisms are isomorphisms for $\mathscr{E}'$ and $\mathscr{E}''$, then they are isomorphisms for $\mathscr{E}$.

Proof. Because the first arguments are locally free,

$$\mathrm{Ext}^i_X(\mathscr{E}^{(\cdot)}, \omega_X) \cong H^i\big(X, (\mathscr{E}^{(\cdot)})^\vee \otimes \omega_X\big)$$

and vanish for $i > 1$. Apply $\mathrm{Hom}_X(-, \omega_X)$ to the short exact sequence and compare the resulting Ext long exact sequence with the dual of the cohomology long exact sequence. By theorem 9.26, the comparison diagram commutes. All vertical arrows except those involving $\mathscr{E}$ are isomorphisms by assumption. The five lemma applied to the six-term exact sequences proves that the two arrows for $\mathscr{E}$ are isomorphisms. ☐

Theorem 9.48 (Serre duality for vector bundles). *Let $\mathscr{E}$ be a finite-rank holomorphic vector bundle on X. For $i = 0, 1$,*

$$\boxed{\mathrm{SD}^i_{\mathscr{E}} : \mathrm{Ext}^i_X(\mathscr{E}, \omega_X) \xrightarrow{\sim} H^{1-i}(X, \mathscr{E})^\vee.}$$

Equivalently,

$$H^1(X, \mathscr{E})^\vee \cong H^0(X, \mathscr{E}^\vee \otimes \omega_X)$$

and

$$H^0(X, \mathscr{E})^\vee \cong H^1(X, \mathscr{E}^\vee \otimes \omega_X).$$

Proof. Choose a line-bundle filtration of $\mathscr{E}$ as in theorem 9.21. Duality holds for each successive line-bundle quotient by theorem 9.38. Apply theorem 9.47 inductively along the filtration. $\square$

Proposition 9.49 (Torsion-free coherent sheaves)**.** *A torsion-free coherent sheaf on a Riemann surface is locally free.*

Proof. At p, the stalk is a finitely generated torsion-free module over the discrete valuation ring $\mathscr{O}_{X,p}$. Every finitely generated torsion-free module over a principal ideal domain is free. The rank is locally constant, and local bases of the stalks lift to bases on neighborhoods by Nakayama's lemma. Hence the sheaf is locally free. $\square$

Theorem 9.50 (Serre duality for coherent sheaves)**.** *Let $\mathscr{F}$ be a coherent analytic sheaf on X. For $i = 0, 1$, the natural duality morphism is an isomorphism*

$$\boxed{\mathrm{SD}^i_{\mathscr{F}} : \mathrm{Ext}^i_X(\mathscr{F}, \omega_X) \xrightarrow{\sim} H^{1-i}(X, \mathscr{F})^\vee.}$$

Furthermore,

$$\mathrm{Ext}^i_X(\mathscr{F}, \omega_X) = 0 \qquad (i > 1).$$

Proof. Let $\mathscr{T} \subseteq \mathscr{F}$ be the torsion subsheaf. The quotient

$$\mathscr{E} = \mathscr{F}/\mathscr{T}$$

is coherent and torsion-free, hence locally free by theorem 9.49. We have a short exact sequence

$$0 \longrightarrow \mathscr{T} \longrightarrow \mathscr{F} \longrightarrow \mathscr{E} \longrightarrow 0.$$

Duality holds for $\mathscr{T}$ by theorem 9.46 and for $\mathscr{E}$ by theorem 9.48. The compatibility lemma theorem 9.26 gives a commutative diagram from the Ext long exact sequence to the dual cohomology long exact sequence. Applying the five lemma proves that the duality morphisms for $\mathscr{F}$ are isomorphisms.

For the higher Ext groups, the same long exact sequence contains

$$\mathrm{Ext}^i_X(\mathscr{E}, \omega_X) \longrightarrow \mathrm{Ext}^i_X(\mathscr{F}, \omega_X) \longrightarrow \mathrm{Ext}^i_X(\mathscr{T}, \omega_X).$$

Both outer groups vanish for $i > 1$, by the vector-bundle and torsion results, so the middle group vanishes as well. $\square$

Remark 9.51 (The one-dimensional dualizing sheaf)*.* The theorem shows that ω_X is a dualizing sheaf for the compact Riemann surface X. In higher dimensions the corresponding statement involves the top exterior power of differentials together with a cohomological shift; on a smooth curve the shift has length one, which is why only H^0, H^1, Hom, and Ext^1 appear.

9.8 The Riemann–Roch theorem

Serre duality identifies the unknown H^1 term in the first Riemann–Roch formula. The final numerical theorem is therefore an immediate consequence of the structural results already established.

Definition 9.52. For a divisor D, the *index of speciality* is

$$i(D) = h^1\big(X, \mathscr{O}_X(D)\big).$$

A divisor is called *special* if $i(D) > 0$.

If K is a canonical divisor, then $\omega_X \cong \mathcal{O}_X(K)$, and Serre duality gives

$$i(D) = h^0\big(X, \omega_X(-D)\big) = \ell(K - D).$$

Theorem 9.53 (Riemann–Roch for line bundles)**.** *For every holomorphic line bundle $\mathscr{L}$ on X,*

$$\boxed{h^0(X, \mathscr{L}) - h^0\big(X, \omega_X \otimes \mathscr{L}^{-1}\big) = \deg \mathscr{L} + 1 - g.}$$

Proof. Choose a divisor D with $\mathscr{L} \cong \mathcal{O}_X(D)$. The first Riemann–Roch formula gives

$$h^0(X, \mathscr{L}) - h^1(X, \mathscr{L}) = \deg \mathscr{L} + 1 - g.$$

Line-bundle Serre duality gives

$$h^1(X, \mathscr{L}) = h^0\big(X, \omega_X \otimes \mathscr{L}^{-1}\big).$$

Substitution proves the formula. ◻

Theorem 9.54 (Riemann–Roch for divisors)**.** *Let K be a canonical divisor. For every divisor D,*

$$\boxed{\ell(D) - \ell(K - D) = \deg D + 1 - g.}$$

Proof. Apply theorem 9.53 to $\mathscr{L} = \mathcal{O}_X(D)$. Since

$$\omega_X \otimes \mathcal{O}_X(-D) \cong \mathcal{O}_X(K - D),$$

the two spaces of sections are $L(D)$ and $L(K - D)$. ◻

9.9 Consequences and applications

Corollary 9.55 (Holomorphic differentials)**.** *The space of global holomorphic differentials has dimension g:*

$$\boxed{h^0(X, \omega_X) = g.}$$

Proof. Apply line-bundle Serre duality to $\mathcal{O}_X$:

$$H^0(X, \omega_X) \cong H^1(X, \mathcal{O}_X)^\vee.$$

Taking dimensions gives the result. ◻

Corollary 9.56 (Degree of the canonical bundle)**.** *For every canonical divisor K,*

$$\boxed{\deg K = \deg \omega_X = 2g - 2.}$$

Proof. Apply Riemann–Roch to $D = K$:

$$\ell(K) - \ell(0) = \deg K + 1 - g.$$

By theorem 9.55, $\ell(K) = g$, while $\ell(0) = 1$. Hence

$$g - 1 = \deg K + 1 - g,$$

which gives $\deg K = 2g - 2$. ◻

Corollary 9.57 (Large-degree vanishing). *If*

$$\deg D > 2g - 2,$$

then

$$H^1\big(X, \mathcal{O}_X(D)\big) = 0$$

and

$$\boxed{\ell(D) = \deg D + 1 - g.}$$

Proof. By theorem 9.56,

$$\deg(K - D) < 0.$$

A divisor of negative degree has no nonzero section, so $L(K - D) = 0$. Serre duality gives

$$H^1\big(X, \mathcal{O}_X(D)\big)^\vee \cong L(K - D) = 0.$$

The Riemann–Roch formula then reduces to the stated equality. □

Corollary 9.58 (Riemann's inequality). *For every divisor D,*

$$\boxed{\ell(D) \geq \deg D + 1 - g.}$$

Proof. Riemann–Roch gives

$$\ell(D) = \deg D + 1 - g + \ell(K - D),$$

and the last term is nonnegative. □

Corollary 9.59 (Riemann–Hurwitz). *Let*

$$f : X \longrightarrow Y$$

be a nonconstant holomorphic map of degree d, and let R_f be its ramification divisor. Then

$$\boxed{2g_X - 2 = d(2g_Y - 2) + \deg R_f.}$$

Proof. The canonical bundle formula of Chapter 8 gives

$$\omega_X \cong f^*\omega_Y \otimes \mathcal{O}_X(R_f).$$

Taking degrees and using theorems 9.15 and 9.16 yields

$$\deg \omega_X = d \deg \omega_Y + \deg R_f.$$

Apply theorem 9.56 to X and Y. □

Corollary 9.60 (Smooth plane curves). *Let*

$$C \subseteq \mathbb{P}^2$$

be a smooth plane curve of degree d. Then

$$\boxed{g(C) = \frac{(d-1)(d-2)}{2}.}$$

Proof. The adjunction formula of Chapter 8 gives

$$\omega_C \cong \mathcal{O}_C(d-3).$$

A hyperplane meets C in a divisor of degree d, so

$$\deg \mathcal{O}_C(1) = d.$$

Consequently,

$$\deg \omega_C = d(d-3).$$

Using $\deg \omega_C = 2g(C) - 2$ gives

$$2g(C) - 2 = d(d-3),$$

and solving for $g(C)$ proves the formula. □

Proposition 9.61 (The linear Hilbert polynomial on a curve)**.** *Let $\mathscr{L}$ and $\mathscr{A}$ be line bundles on X. For every $n \in \mathbb{Z}$,*

$$\boxed{\chi\big(X, \mathscr{L} \otimes \mathscr{A}^{\otimes n}\big) = \deg \mathscr{L} + n \deg \mathscr{A} + 1 - g.}$$

If $\deg \mathscr{A} > 0$*, this is a linear polynomial in n with leading coefficient* $\deg \mathscr{A}$*.*

Proof. The first Riemann–Roch formula for line bundles is

$$\chi(X, \mathscr{E}) = \deg \mathscr{E} + 1 - g.$$

By theorem 9.15,

$$\deg(\mathscr{L} \otimes \mathscr{A}^{\otimes n}) = \deg \mathscr{L} + n \deg \mathscr{A}.$$

Substitution gives the formula. □

Example 9.62 (The Riemann sphere)**.** On $X = \mathbb{P}^1$, one has $g = 0$ and

$$\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2).$$

Riemann–Roch gives

$$h^0\big(\mathbb{P}^1, \mathcal{O}(n)\big) - h^0\big(\mathbb{P}^1, \mathcal{O}(-n-2)\big) = n + 1.$$

Since a line bundle of negative degree has no nonzero global section,

$$h^0\big(\mathbb{P}^1, \mathcal{O}(n)\big) = \begin{cases} n+1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Thus the abstract theorem recovers the elementary homogeneous-polynomial computation.

The logical chain is now complete:

$$\text{principal parts} \longrightarrow H^1 \longrightarrow \text{residue pairing} \longrightarrow \text{Serre duality} \longrightarrow \text{Riemann–Roch.}$$

The numerical theorem is the visible consequence of a deeper perfect pairing between local-to-global obstructions and holomorphic differentials.